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γ-GRAPHS OF GRAPHS

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Abstract

A set $S \subseteq V$ is a *dominating set* of a graph $G = (V, E)$ if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of G equals the minimum cardinality of a dominating set S in G; we say that such a set S is a *γ-set*. In this paper we consider the family of all γ-sets in a graph G and we define the γ-graph $G(\gamma) = (V(\gamma), E(\gamma))$ of G to be the graph whose vertices $V(\gamma)$ correspond 1-to-1 with the γ-sets of G, and two γ-sets, say D_1 and D_2, are adjacent in $E(\gamma)$ if there exists a vertex $v \in D_1$ and a vertex $w \in D_2$ such that v is adjacent to w and $D_1 = D_2 - \{w\} \cup \{v\}$, or equivalently, $D_2 = D_1 - \{v\} \cup \{w\}$. In this paper we initiate the study of γ-graphs of graphs.

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1. Introduction

Let $G = (V, E) = (V(G), E(G))$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and *order* $n = |V|$. The *open neighborhood* of a vertex v is the set $N(v) = \{u | uv \in E\}$ of vertices u that are adjacent to v; the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly we define the *closed neighborhood of a set* S to be the set $N[S] = \bigcup_{v \in S} N[v]$. A set $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex in S, or equivalently if $N[S] = V$. The *domination number* $\gamma(G)$ of G equals the minimum cardinality of a dominating set S in G; we say that such a set S is a γ*-set*.

Given a set of vertices $S \subseteq V$, by the *subgraph of* G *induced by* S we mean the subgraph $G[S] = (S, E \cap (S \times S))$. We say that a vertex $v \in S$ has a *private neighbor with respect to* S if $N[v] - N[S - \{v\}] \neq \emptyset$, in which case every vertex in $N[v] - N[S - \{v\}]$ is called a *private neighbor* of v with respect to S. A vertex $w \in V - S$ is said to be an *external private neighbor*, or *epn*, of a vertex $v \in S$ if $N(w) \cap S = \{v\}$. If vertex $v \in S$ is not adjacent to any vertex in S it is called *its own private neighbor, or self-private neighbor, spn*.

It is well known that a graph G can have many γ-sets, even exponentially many in some cases. We raise the general question: what can you say about the class of all γ-sets of a graph G? Are some of them somehow better than others, and for what reasons? What criteria would you use to prefer one γ-set over another? Here are some possibilities; among all γ-sets S you might want to either minimize or maximize the following (for elaboration of the following terms, see [4, 5, 2, 6]):

1. The number of isolated vertices in $G[S]$; the smaller this number, the closer S is to being a *total dominating set*, that is a dominating set for which $G[S]$ has no isolated vertices; whereas the larger this number is the closer S is to being an *independent dominating set*, that is, a dominating set in which all vertices in $G[S]$ are isolated vertices.
2. The number of isolated vertices in $G[V - S]$; the smaller this number, the closer S is to being a *restrained dominating set*, that is a set for which

$G[V - S]$ has no isolated vertices. On the other hand, the larger this number, the closer S is to being a *vertex cover*, that is a set S of vertices such that every edge in E contains a vertex in S.

3. The number of edges in $G[S]$.
4. The number of edges in $G[V - S]$; again, the smaller this number, the closer S is to being a vertex cover.
5. The number of connected components in $G[S]$; the smaller this number, the closer S is to being a *connected dominating set*, that is, a set for which $G[S]$ is a connected subgraph.
6. The number of connected components in $G[V - S]$.
7. The number of vertices in $V - S$ that are dominated more than once by vertices in S; if every vertex in $V - S$ is dominated at least twice then S is called a 2-*dominating set*.
8. The number of vertices in S having an external private neighbor in $V-S$; the larger this number, the closer S is to being an *open irredundant dominating set*, that is, a set S in which every vertex has an external private neighbor in $V - S$.
9. The number of vertices in $V - S$ that are private neighbors of vertices in S; the larger this number, the closer S is to being an *open efficient dominating set*, that is, every vertex in $V - S$ is dominated by only one vertex in S.
10. The number of edges between vertices in S and vertices in $V - S$.
11. The sum of the degrees $deg(v)$ of all vertices in S.

With this in mind, if one had a way of listing all γ-sets of a graph G, then one could easily determine any of the above maximum or minimal values over all γ-sets of G. One way of listing all γ-sets of a graph G is the following.

Consider the family of all γ-sets of a graph G and define the *γ-graph* $G(\gamma) = (V(\gamma), E(\gamma))$ of G to be the graph whose vertices $V(\gamma)$ correspond 1-to-1 with the γ-sets of G, and two γ-sets, say S_1 and S_2, form an edge in $E(\gamma)$ if there exists a vertex $v \in S_1$ and a vertex $w \in S_2$ such that (i) v is adjacent to w and (ii) $S_1 = S_2 - \{w\} \cup \{v\}$ and $S_2 = S_1 - \{v\} \cup \{w\}$. With this definition, two γ-sets are said to be *adjacent* if they differ by one vertex, and the two vertices defining this difference are adjacent in G. We can consider therefore placing tokens on the vertices of any γ-set S, and moving any one token in S to an adjacent vertex if the resulting set S' of vertices is another γ-set.

In the following sections we study properties of γ-graphs, and raise quite a number of open questions.

2. Preliminary Observations

Our first observation has to do with the possibility of removing a vertex v in a γ-set S and replacing it with an adjacent vertex $w \in V - S$ in such a way that the resulting set $S' = S - \{v\} \cup \{w\}$ is also a γ-set. This can be done in the following ways.

If a vertex $v \in S$ has two or more non-adjacent external private neighbors, then it cannot be replaced with any vertex in $V - S$ and still produce a γ-set. However, if a vertex $v \in S$ has exactly one external private neighbor $w \in V - S$ then v can be replaced by w to produce another γ-set. Finally, if vertex v has no external private neighbors, then it must be its own private neighbor, and therefore it can be replaced by any neighbor in $V - S$ to produce another γ-set. From this it follows that if every vertex v in a γ-set S has two or more non-adjacent external private neighbors, then the vertex corresponding to S in the γ-graph $G(\gamma)$ must be an isolated vertex.

Let $\overline{K_n}$ denote the graph consisting of n isolated vertices, and K_n denote the complete graph of order n. Let $K_{1,n}$ denote the tree having n leaves, each of which is joined to the same, central vertex.

Proposition 1. *If a graph $G = (V, E)$ has a unique γ-set, then $G(\gamma) \simeq K_1$, and conversely.*

Corollary 2. $K_{1,n}(\gamma) \simeq K_1$.

Proposition 3. $\overline{K_n}(\gamma) \simeq K_1$, *whereas* $K_n(\gamma) \simeq K_n$.

Proposition 4. *For* $n \geq 2$, $K_{2,n}(\gamma) \simeq K_{1,2n}$.

Proposition 5. *For* $m, n \geq 3$, $K_{m,n}(\gamma) \simeq \overline{K_{mn}}$.

It is interesting to observe that the γ-graph of the complete graph K_n is isomorphic to itself. Other examples of graphs having this property are the cycles of order $3k + 2$.

Proposition 6. $C_{3k+2}(\gamma) \simeq C_{3k+2}$.

Cycles of order $3k$ have three γ-sets, but no two of them are adjacent in $C_{3k}(\gamma)$.

Proposition 7. *For $k \geq 2$, $C_{3k}(\gamma) \simeq \overline{K_3}$.*

Since paths P_{3k} of order $3k$ have a unique γ-set, we have the following.

Corollary 8. $P_{3k}(\gamma) \simeq K_1$.

The following result can be proved; we omit the details.

Proposition 9. $P_{3k+2}(\gamma) \simeq P_{k+2}$.

Let $G \Box H$ denote the *Cartesian product* of two graphs G and H, where $G \Box H = (V(G) \times V(H), E(G) \Box E(H))$, where two vertices $(u, v), (x, w)$ are adjacent in $G \Box H$ if and only if either $u = x$ and v is adjacent to w in H, or u is adjacent to x in G and $v = w$.

The $m \times n$ *grid graph* is the Cartesian product graph $P_m \Box P_n$. One can observe that $2 \times (2k + 1)$ grid graphs have only two γ-sets and they are not adjacent.

Proposition 10. *For $k \geq 2$, $(P_2 \Box P_{2k+1})(\gamma) \simeq \overline{K_2}$.*

The structures of the γ-graphs of paths and cycles of order $3k + 1$ are more interesting. Assume that the vertices in each of these graphs have been labeled $1, 2, \ldots, 3k + 1$. Note for $G = P_{3k+1}$ or $G = C_{3k+1}$ that $S = \{1, 4, 7, \ldots, 3k+1\}$ is a γ-set of size $k+1$. In Figure 1, we show P_{10} and C_{10} with S highlighted. In each case, vertices 1 and $3k + 1$ have one external private neighbor, while the other members of S have two non-adjacent external private neighbors. So $S - \{1\} \cup \{2\}$ and $S - \{3k + 1\} \cup \{3k\}$ are γ-sets. Further, if S' is a γ-set for $G = P_{3k+1}$ or $G = C_{3k+1}$ and vertex i has exactly one external private neighbor, $j = i + 1$ or $j = i - 1$, then $S' - \{i\} \cup \{j\}$ is a γ-set. Let us refer to the process of changing from a γ-set S' to a γ-set $S' - \{i\} \cup \{j\}$ as a *swap*. Notice that each swap defines an edge in $G(\gamma)$.

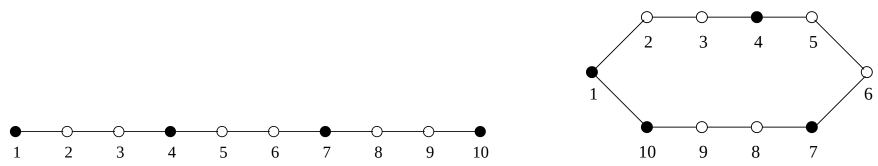

Figure 1. P_{10} and C_{10}

We claim that every γ-set of $G = P_{3k+1}$ or $G = C_{3k+1}$ is some number of swaps from the γ-set $S = \{1, 4, 7, \ldots, 3k+1\}$, thus showing that $G(\gamma)$ is connected for these graphs. To this end, let $G = P_{3k+1}$, and let $X = \{x_0, x_1, \ldots, x_k\}$, where $x_0 < x_1 < \cdots < x_k$, be a γ-set for G. Consider the vector $D = [D_0, D_1, \ldots, D_k] = [x_0 - 1, x_1 - 4, \ldots, x_{k+1} - (3k+1)]$. If $G = P_{3k+1}$, then $-1 \leq D[i] \leq 1$, for $0 \leq i \leq k$. To see that $D[i] \leq 1$, $0 \leq i \leq k$, suppose to the contrary that j is the first position where $D[j] > 1$. Note that $j > 0$ since otherwise $x_0 \geq 3$, and no vertex in X dominates vertex 1. Thus, for $j > 0$, $x_j - (3j+1) > 1$ but $x_{j-1} - (3(j-1)+1) \leq 1$. However, this implies that no vertex in X dominates vertex $3j+1$, a contradiction. A similar argument shows $D[i] \geq -1$, $0 \leq i \leq k$. Further, if j is the first position where $D[j] < 0$ then for all $\ell > j$, $D[\ell] = -1$. To see this, suppose to the contrary that $D[\ell] \geq 0$ but $D[\ell - 1] = -1$ for some $\ell > j$. Thus $x_\ell \geq 3\ell + 1$ and $x_{\ell-1} < 3(\ell-1)+1$, and this leaves vertex $3(\ell-1)+2 = 3\ell-1$ undominated in G, a contradiction. A similar argument shows that if j is the last occurence such that $D[j] = 1$, then for all $\ell < j$, $D[\ell] = 1$. This implies that the vector D consists of a run of 1's followed by a run of 0's and then a run of -1's, where each of these runs is of possibly length 0. To find a path from the vertex corresponding to X in $G(\gamma)$ to the vertex corresponding to S, find the last occurence of a 1 in D, call this position j. Note that $j < k$, since $x_k \leq 3k+1$. Since $D[j+1] \leq 0$, x_j has one external private neighbor, namely $x_j - 1$. The set $X' = X - \{x_j\} \cup \{x_j - 1\}$ is then a γ-set of G. Since $D[j-1] = 1$, this swap decreases the number of external private neighbors of x_{j-1} to one. Hence we can perform the swap $X' - \{x_{j-1}\} \cup \{x_{j-1} - 1\}$ and produce a γ-set. This process continues until the swap of x_0 for 1 occurs. Then starting with the earliest occurence of -1 in D, say at vertex x_ℓ, we perform the swap of x_ℓ for $x_\ell + 1$. Note that $\ell > 0$ since $x_0 \geq 1$. Thus this swap can occur since $D[\ell - 1] \geq 0$ leaving x_ℓ with only one external private neighbor, namely $x_\ell + 1$. We continue this second swapping process until x_k swaps with $x_k + 1 = 3k+1$. Thus each dominating set X is some number of swaps away from S, and each swap under the above process produces a γ-set.

For $G = C_{3k+1}$, a similar argument, accounting for the cyclic nature of the graph, holds. We leave the details of this argument to the reader. As a consequence, we have the following.

Theorem 11. *$G(\gamma)$, where $G = P_{3k+1}$ or $G = C_{3k+1}$, is a connected graph.*

We define a *stepgrid* $SG(k)$ to be the induced subgraph of the $k \times k$ grid graph $P_k \Box P_k$ defined as follows: $SG(k) = (V(k), E(k))$, where

$$V(k) = \{(i,j) : 1 \le i, j \le k, i + j \le k + 2\}, \text{ and}$$
$$E(k) = \{((i,j),(i',j')) : i = i', j' = j + 1; i' = i + 1, j = j'\}.$$

Each γ-set X of P_{3k+1} is some number of swaps of say type-1 ($X - \{i\} \cup \{i+1\}$) and type-2 ($X - \{i\} \cup \{i-1\}$) from S. Alternatively, we can perform swaps from S to X. Thus each vertex in $P_{3k+1}(\gamma)$ can be associated with an ordered pair (i,j) where i is the number of swaps of type-1 and j is the number of swaps of type-2 needed to convert S to X. Note that vertices 1 and $3k+1$ in P_{3k+1} can swap with an external private neighbor at most once freeing other vertices to swap with newly created external private neighbors. However, each vertex can be freed to swap at most once in either direction. Thus, the conditions on the ordered pairs (i,j) are $1 \le i \le k$, $1 \le j \le k$, $i + j \le k + 2$. If $q = i + 1$ and $r = j + 1$, we then have $1 \le q \le k + 1$, $1 \le r \le k + 1$, and $q + r \le (k + 1) + 2$. We have shown the following.

Theorem 12. *$P_{3k+1}(\gamma) \simeq SG(k + 1)$.*

The γ-graphs of cycles of orders $3k+1$ are much more complex; the γ-graph $C_{3k+1}(\gamma)$ is connected and has some of the same structural properties as $P_{3k+1}(\gamma)$ as can be seen in Figures 2 and 3 comparing the graphs of $P_{10}(\gamma)$ and $C_{10}(\gamma)$.

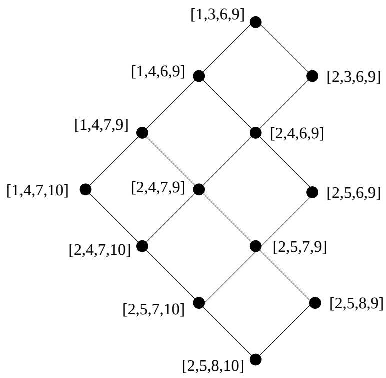

Figure 2. $P_{10}(\gamma) \simeq SG(4)$

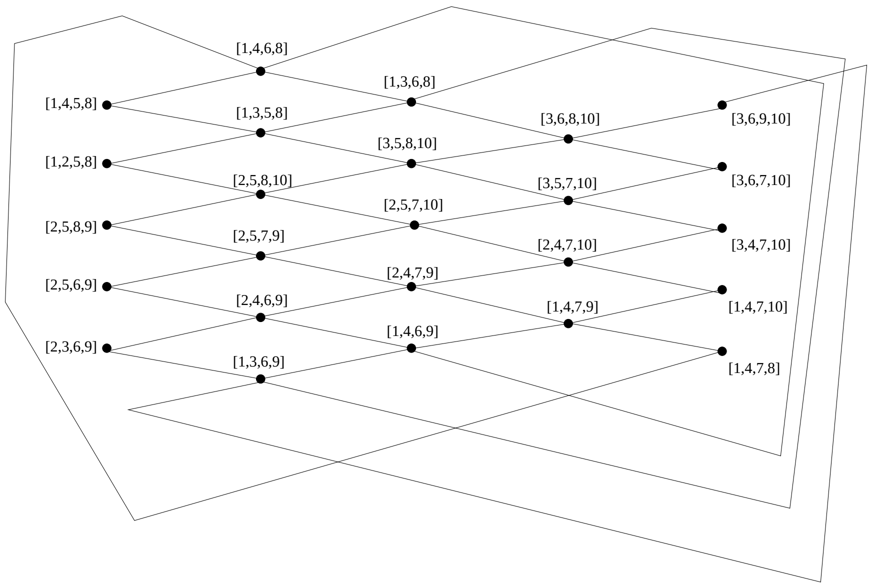

Figure 3. $C_{10}(\gamma)$

3. General Properties of γ-Graphs

It is clear from the definition of the γ-graph $G(\gamma)$ that the number of vertices in $G(\gamma)$ is at most the number of ways of choosing $\gamma(G)$ vertices from n. It is also clear that the order of $G(\gamma)$ can be exponential in the order n of G. Consider, for example, the *corona* $G \circ K_1$ of a graph G, which is the graph obtained from G by attaching a leaf to each vertex in G. If G has order n, then the γ-graph of the corona $G \circ K_1$ has order 2^n. In fact, we have the following result, where Q_n denotes the n-cube, which is the Cartesian product graph $Q_n = Q_{n-1} \Box K_2$, whose vertex set corresponds to the 2^n n-tuples of 0's and 1's, where two n-tuples are adjacent if and only if they differ in exactly one position.

Proposition 13. *For any graph G of order n, $(G \circ K_1)(\gamma) \simeq Q_n$.*

We can bound the maximum degree $\Delta(G(\gamma))$ of a vertex in $G(\gamma)$ as follows:

Proposition 14. *For any graph G of order n and having maximum degree $\Delta(G)$, the maximum degree of a vertex in $G(\gamma)$ satisfies $\Delta(G(\gamma)) \leq \min\{\gamma(G)(n - \gamma(G)), \gamma(G)\Delta(G)\}$.*

It is easy to see that this bound is sharp for complete graphs K_n and for complete bipartite graphs of the form $K_{2,n}$.

Proposition 15. *If $G \cup H$ denotes the disjoint union of two graphs G and H, then $(G \cup H)(\gamma) \simeq G(\gamma)\square H(\gamma)$.*

4. γ-Graphs of Trees

Let T be a tree, and let $x \in V(T)$ be a vertex that does not appear in any γ-set of T. Let $T_1, T_2, \ldots, T_k$ be the disjoint subtrees created by deleting x from T, and let $x_i \in T_i$ be the vertex in subtree T_i adjacent to vertex x. Let D_i be the set of γ-sets of subtree T_i, $1 \le i \le k$.

Lemma 16. *Let $S_1 \in D_1$, $S_2 \in D_2, \ldots, S_k \in D_k$ such that $x_i \in S_i$ for at least one value of $1 \le i \le k$. Then D is a γ-set of T if and only if $D = S_1 \cup S_2 \cup \cdots \cup S_k$.*

Proof. Let $S_1 \in D_1, S_2 \in D_2, \ldots, S_k \in D_k$. Consider $D = S_1 \cup S_2 \cup \cdots \cup S_k$. If $x_1 \notin S_1$, $x_2 \notin S_2, \ldots,$ and $x_k \notin S_k$, then the set D is not a dominating set of T since x is not in D and thus x is not dominated. So suppose, that $x_i \in S_i$ for some i, $1 \le i \le k$. We wish to show that D is a γ-set of T. Since S_i dominates T_i, $1 \le i \le k$, and x is dominated, D is a dominating set. Suppose there exists a dominating set D' such that $|D'| < |D|$. Note that $x \notin D'$ since x appears in no γ-set of T. Partition D' into sets $S'_1, S'_2, \ldots, S'_k$, where $S'_i \subseteq V(T_i)$. Since $\sum_{i=1}^{k} |S'_k| = |D'| < |D| = \sum_{i=1}^{k} |S_k|$, there exists S'_i such that $|S'_i| < |S_i|$, which is a contradiction since S_i is a γ-set of T_i. Thus, D is a γ-set of T.

Now, let D be a γ-set of T. Since by assumption $x \notin D$, there must be a vertex, call it $y \in D$ that dominates x. Partition D into sets $S_i = D \cap V(T_i)$, $1 \le i \le k$. Note that $y \in S_j$ for some $1 \le j \le k$. We need to show that S_i is a γ-set of T_i. If S_i does not dominate T_i, then since $x \notin D$ and T is a tree, then D is not a dominating set, which is a contradiction. So S_i dominates T_i for $1 \le i \le k$. Suppose that there exists a set S^* which dominates T_i such that if $y \in S_i$ then $y \in S^*$ and $|S^*| < |S_i|$. Consider the set $D^* = S_1 \cup S_2 \cup \cdots \cup S_{i-1} \cup S^* \cup S_{i+1} \cup \cdots \cup S_k$. Then $|D| = |S_i| + \cdots + |S_i| + \cdots + |S_k| > |S_1| + \cdots + |S^*| + \cdots + |S_k| = |D^*|$, which contradicts the minimality of D. So D can be written as $D = S_1 \cup S_2 \cup \cdots \cup S_k$, where $S_1 \in D_1, S_2 \in D_2, \ldots,$ and $S_k \in D_k$. ∎

Now, let $T_i(\gamma)$ be the γ-graph for the γ-sets of subtree T_i. Let $T_i^{x_i}(\gamma)$ be the γ-graph for the subtree T_i using only those dominating sets of D_i that do not contain x_i. Under these conditions, the preceding lemma can be used to show that $T(\gamma) = T_1(\gamma)\Box T_2(\gamma)\Box\cdots\Box T_k(\gamma) - (T_1^{x_1}(\gamma)\Box T_2^{x_2}(\gamma)\Box\cdots\Box T_k^{x_k}(\gamma))$.

Theorem 17. *Let T be a tree, and let $x \in V(T)$ be a vertex that does not appear in any γ-set of T. Let $T_1, T_2, \ldots, T_k$ be the disjoint subtrees created by deleting x from T, and let $x_i \in T_i$ be the vertex in subtree T_i adjacent to vertex x. Let D_i be the set of minimum dominating sets of subtree T_i, $1 \le i \le k$, and let $T_i(\gamma)$ be the γ-graph of subtree T_i. Let $T_i^{x_i}(\gamma)$ be the γ-graph of subtree T_i using only those γ-sets of D_i that do not contain x_i. Then $T(\gamma) = T_1(\gamma)\Box T_2(\gamma)\Box\cdots\Box T_k(\gamma) - (T_1^{x_1}(\gamma)\Box T_2^{x_2}(\gamma)\Box\cdots\Box T_k^{x_k}(\gamma))$.*

Consider the tree in Figure 4.

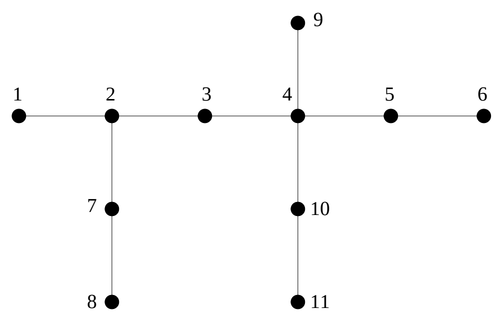

Figure 4. Tree T

Vertex 3 appears in no γ-set of T. It is adjacent to vertices 2 and 4, at least one of which must be included in any γ-set S of T. Let T_1 and T_2 be the two subtrees which result from removing vertex 3 from T. Consider the γ-sets of T_1 and T_2. Note that

$$\begin{aligned} D_1 &= \{(2,8),(2,7),(1,7),(1,8)\} \text{ and} \\ D_2 &= \{(4,6,11),(4,6,10),(4,5,11),(4,5,10),(9,6,11), \\ &\quad (9,6,10),(9,5,11),(9,5,10)\}. \end{aligned}$$

Now consider the γ-sets of T_1 and T_2 that do not contain either vertex 2 or vertex 4. Note that

$$\begin{aligned} D_1^2 &= \{(1,8),(1,7)\} \text{ and} \\ D_2^4 &= \{(9,6,11),(9,6,10),(9,5,11),(9,5,10)\}. \end{aligned}$$

In Figure 5 we show the graph of $T(\gamma)$. Note that $T(\gamma) = T_1(\gamma)\Box T_2(\gamma) - (T_1^2(\gamma)\Box T_2^4(\gamma))$.

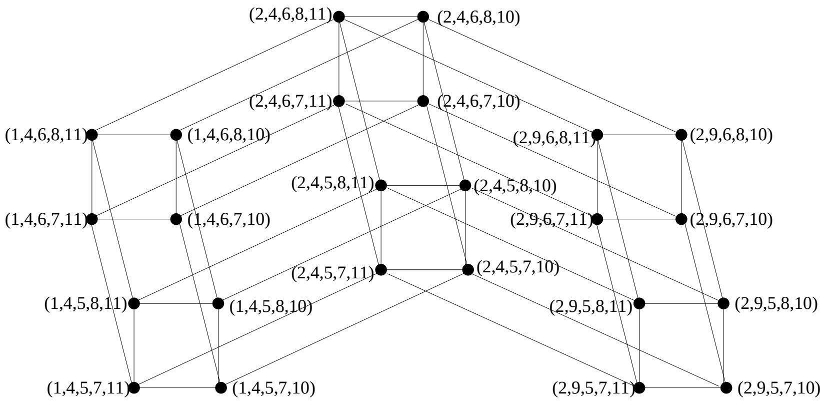

Figure 5. $T[\gamma]$

Theorem 18. *The γ-graph $T(\gamma)$ of every tree T is a connected graph.*

Proof. It is easy to see that the γ-graphs of the trees $T = K_1$ and $T = K_2$ are connected, since $K_1(\gamma) \simeq K_1$ and $K_2(\gamma) \simeq K_2$. Therefore, let T be any tree of order $n \geq 3$, and let us assume that we root T at a non-leaf vertex r that has an adjacent leaf s. To this rooted tree T_r we can apply the linear algorithm of Cockayne, Goodman, and Hedetniemi [1] for computing the value of $\gamma(T)$ and finding a particular γ-set S.

The set S found by this algorithm has the property that every vertex $u \in S$ has an external private neighbor $v \in V - S$, that is, a child of u in T_r. In particular, this set S contains the root vertex r, and r has the leaf vertex s as a private neighbor.

To each vertex $w \in T_r$ we can associate a level number $l(w) = d(w, r)$ that equals the distance from w to the root vertex r. For each γ-set $S' = \{v_1, v_2, \ldots, v_k\}$ of T, where $k = \gamma(T)$, we can associate a corresponding vector $L(S') = \{l(v_1), l(v_2), \ldots, l(v_k)\}$, where the vertices in S' are ordered so that the corresponding level numbers are sorted in ascending order. Given this, the level vectors of all γ-sets can be sorted lexicographically.

It can be seen that the vector $L(S)$, corresponding to the γ-set S found by the algorithm, is the lexicographically smallest of all of the level vectors corresponding to all γ-sets in T. It can also be seen that for every γ-set $S' \neq S$, there exists a γ-set S'' where S' is adjacent to S'' and $L(S'') < L(S')$ in lexicographic order. This γ-set S'' can be found as follows. Compare the level vector $L(S')$ with the level vector $L(S)$ for S. Proceeding from the vertices in S' and in S with the highest level numbers to the lowest, find the first vertex, say x, in S' that is not the same as a vertex in S. It must be the case that the parent, call it y, of x is in S, and that the set $S'' = S' - \{x\} \cup \{y\}$ is a γ-set of T that is closer lexicographically to $L(S)$ than is $L(S')$.

From this it follows that the γ-graph $T(\gamma)$ of T is connected. ■

Theorem 19. *For any triangle-free graph G, $G(\gamma)$ is triangle-free.*

Proof. Suppose $G(\gamma)$ contains a triangle of 3 vertices corresponding to γ-sets S_1, S_2, and S_3. Since (S_1, S_2) corresponds to an edge in $G(\gamma)$, $S_2 = S_1 - \{x\} \cup \{y\}$ for some $x, y \in V(G)$ such that $(x, y) \in E(G)$. Further, since (S_2, S_3) corresponds to an edge in $G(\gamma)$, $S_3 = S_2 - \{c\} \cup \{d\}$ for some $c, d \in V(G)$ such that $(c, d) \in E(G)$. However, $S_3 = S_2 - \{c\} \cup \{d\} = S_1 - \{x, c\} \cup \{y, d\}$. But since (S_1, S_3) corresponds to an edge in $G(\gamma)$, $S_3 = S_1 - \{a\} \cup \{b\}$ for some $a, b \in V(G)$ such that $(a, b) \in E(G)$. Since S_3 is not two swaps away from S_1, it must be the case that $x = a$, $c = y$, and $b = d$. But this implies that (x, y), (x, b), and (y, b) are edges in $E(G)$, a contradiction since G is triangle-free. Thus for any triangle-free graph G, there is no K_3 induced subgraph in $G(\gamma)$. ■

Corollary 20. *For any tree T, $T(\gamma)$ is triangle-free.*

Theorem 21. *For any tree T, $T(\gamma)$ is C_n-free, for any odd $n \geq 3$.*

Proof. Suppose $T(\gamma)$ contains a cycle, C, of $k \geq 3$, k odd, vertices. Let x be a vertex in C, and let S be the γ-set corresponding to vertex x. Let y and z be the two vertices on C of distance $m = \frac{k-1}{2}$ swaps away from x with corresponding γ-sets S_1 and S_2. That is, there is a path P_1 corresponding to a series of vertex swaps, say x_1 for y_1, x_2 for $y_2, \ldots, x_m$ for y_m, so that $S_1 = S - X \cup Y$, where $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. Likewise, there is a path P_2 corresponding to a series of vertex swaps, say w_1 for z_1, w_2 for $z_2, \ldots, w_m$ for z_m, so that $S_2 = S - W \cup Z$, where $W = \{w_1, w_2, \ldots, w_m\}$ and $Z = \{z_1, z_2, \ldots, z_m\}$. However, since $(y, z) \in E(T(\gamma))$, $S_2 = S_1 - a \cup b$ for some $a, b \in V(T)$. Thus, it must be the case that the set $X = W - w_j \cup x_i$ and $Y = Z - z_j \cup y_i$. This implies that $S_2 = S_1 - \{y_j\} \cup \{x_i\}$ and $(x_i, y_j) \in E(T)$. Since x_i was swapped for y_i and x_j was swapped for y_j in P_1, we also know that $(x_i, y_i) \in E(T)$ and $(x_j, y_j) \in E(T)$. Now both x_i and y_i are in S_2, so there exists a swap x_l for y_i in P_2 which implies $(x_l, y_i) \in E(T)$. However, in path P_1, x_l was swapped for y_l, and thus $(x_l, y_l) \in E(T)$. Similiarly, $y_l \in S_2$, so there exists some x_s so that in path P_2, x_s was swapped for y_l. We can continue to find the these alternating P_1 and P_2 swaps, but, since m is finite, we must reach a vertex y_q which swapped with x_j in P_2, thus creating a cycle in T and contradicting the fact that T is cycle-free. Hence, $T(\gamma)$ is free of odd cycles. ∎

Theorem 22. *Every tree T is the γ-graph of some graph.*

Proof. We proceed by induction on the order n of a tree T. It is easy to see that the trees $T = K_1$ and $T = K_2$ are the γ-graphs of K_1 and K_2, respectively.

Assume that the theorem is true for all trees T of order at most n, and let T' be any tree of order $n + 1$. Assume that T' is obtained by attaching a leaf v to a vertex u in a tree T of order n. By induction we know that the tree T is the γ-graph of some graph, say G. Let $\gamma(G) = k$ and let $S_u = \{u_1, u_2, \ldots, u_k\}$ be the γ-set of G corresponding to the vertex u in T.

Construct a new graph G' by attaching k leaves to the vertices in S_u, say $S'_u = \{u'_1, u'_2, \ldots, u'_k\}$. Now add a new vertex x and join it to each of the k vertices in S'_u. Finally, attach a leaf y adjacent to x. Note that every γ-set of the new graph G' must either be of the form $S \cup \{x\}$, for any γ-set S in G, or the one new γ-set $S_u \cup \{y\}$.

Note that $S_u \cup \{x\}$ is adjacent to $S_u \cup \{y\}$ in the γ-graph of G'. Note also that the vertex corresponding to the γ-set $S_u \cup \{y\}$ is only adjacent to the vertex corresponding to the γ-set $S_u \cup \{x\}$, and the γ-set $S_u \cup \{y\}$ corresponds to the vertex v in T'. Thus, the γ-graph of the graph G' is isomorphic to the tree T'. ∎

5. γ-Graph Sequences

It is interesting to consider applying the γ-graph construction repeatedly, starting from a given graph, that is, $G \xrightarrow{\gamma} G(\gamma) \xrightarrow{\gamma} G(\gamma)(\gamma)$, etc. Although we do not know much about the nature of these sequences, we have noticed that often the sequence ends with K_1. We can list several examples of this phenomenon.

1. $K_{1,n} \xrightarrow{\gamma} K_1$.

2. $C_{3k} \xrightarrow{\gamma} \overline{K_3} \xrightarrow{\gamma} K_1$.
3. $K_n \xrightarrow{\gamma} K_n$.
4. $C_{3k+2} \xrightarrow{\gamma} C_{3k+2}$.
5. $P_4 \xrightarrow{\gamma} C_4 \simeq P_2 \Box P_2 \xrightarrow{\gamma} K_{2,4} \xrightarrow{\gamma} K_{1,8} \xrightarrow{\gamma} K_1$.
6. $P_2 \Box P_3 \xrightarrow{\gamma} \overline{K_3} \xrightarrow{\gamma} K_1$.
7. $P_3 \Box P_3 \xrightarrow{\gamma} C_8 \cup 2K_1 \xrightarrow{\gamma} C_8 \xrightarrow{\gamma} C_8$.
8. Let $C_8(1,1,0,0,1,1,0,0)$ denote the graph obtained from a cycle C_8 of order 8 by attaching to each vertex, in order, the number of leaves indicated in the parenthesis. Thus, this graph has two adjacent vertices on the cycle with one attached leaf each, the next two vertices have no attached leaf, the next two vertices each have an attached leaf, and the final two vertices have no attached leaf.
   $P_2 \Box P_4 \xrightarrow{\gamma} C_8(1,1,0,0,1,1,0,0) \xrightarrow{\gamma} K_1$.
9. $P_2 \Box P_{2k+1} \xrightarrow{\gamma} \overline{K_2} \xrightarrow{\gamma} K_1$.
10. $P_2 \Box P_6 \xrightarrow{\gamma} 4P_3 \cup 5K_1 \xrightarrow{\gamma} K_1$.
11. $P_{3k+2} \xrightarrow{\gamma} P_{3k} \xrightarrow{\gamma} K_1$.

Although all of the γ-graph sequences so far have terminated after a small number of steps, for some graphs this sequence can be infinite. Here is an example.

Proposition 23. $C_3 \Box P_2 \xrightarrow{\gamma} C_3 \Box C_3 \xrightarrow{\gamma} C_3 \Box C_3 \Box C_3 \xrightarrow{\gamma} \ldots$.

6. Open Questions

We conclude with a series of questions that we have not been able to answer.

1. Is $\Delta(T(\gamma)) = O(n)$ for any tree T?
2. Is $diam(T(\gamma)) = O(n)$ for any tree T?
3. Is $|V(T(\gamma))| \leq 2^{\gamma(T)}$?
4. Which graphs are γ-graphs of trees?
5. Which graphs are γ-graphs? Can you construct a graph H that is not a γ-graph of any graph G? We believe that for any graph H, there exists a graph G such that $G(\gamma) \simeq H$. Recently, in [3], it has been shown that for any graph H, there is a simple construction of a graph G, using H, that gives $G(\gamma) \simeq H$.

6. For which graphs G is $G(\gamma) \simeq G$? This is true for complete graphs K_n and cycles of order $3k+2$.
7. Under what conditions is $G(\gamma)$ a disconnected graph?

7. Added In Proof

The authors have recently become aware of the existence of the 2008 paper "γ-graph of a graph" by K. Subramanaian and N. Sridharan appearing in *Bull. Kerala Math. Assoc.* **5**(1), pp. 17–34. Two other papers on this topic also exist: N. Sridharan and K. Subramanian, Trees and unicyclic graphs are γ-graphs, *J. Combin. Math. Combin. Comput.*, **69** (2009), 231–236, and S. A. Lakshmanan and A. Vijayakumar, The Gamma Graph of a Graph, *AKCE J. Graphs. Combin.*, **7**(1), 2010, pp. 53–59. It is important to note that in these papers the definition of γ-graphs is different from ours, and thus these are two different classes of graphs.

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