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THE HULL NUMBER OF STRONG PRODUCT GRAPHS

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Abstract

For a connected graph G with at least two vertices and S a subset of vertices, the convex hull $[S]_G$ is the smallest convex set containing S. The hull number $h(G)$ is the minimum cardinality among the subsets S of $V(G)$ with $[S]_G = V(G)$. Upper bound for the hull number of strong product $G \boxtimes H$ of two graphs G and H is obtainted. Improved upper bounds are obtained for some class of strong product graphs. Exact values for the hull number of some special classes of strong product graphs are obtained. Graphs G and H for which $h(G \boxtimes H) = h(G)h(H)$ are characterized.

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1. Introduction

By a graph $G = (V(G), E(G))$ we mean a finite undirected connected graph without loops or multiple edges. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G.

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An $u-v$ path of length $d_G(u,v)$ is called an $u-v$ *geodesic*. It is known that the distance is a metric on the vertex set $V(G)$. The set $I_G[u,v]$ consists of all vertices lying on some $u-v$ geodesic of G, while for $S \subseteq V(G)$, $I_G[S] = \bigcup_{u,v \in S} I_G[u,v]$. The set S is *convex* if $I_G[S] = S$. The *convex hull* $[S]_G$ is the smallest convex containing S. The convex hull $[S]_G$ can also be formed from the sequence $\{I_G^k[S]\}$, $k \geq 0$, where $I_G^0[S] = S$, $I_G^1[S] = I_G[S]$ and $I_G^k[S] = I_G[I_G^{k-1}[S]]$ for $k \geq 2$. From some term on, this sequence must be constant. Let p be the smallest number such that $I_G^p[S] = I_G^{p+1}[S]$. Then $I_G^p[S]$ is the convex hull $[S]_G$. A set S of vertices of G is a *hull set* of G if $[S]_G = V(G)$, and a hull set of minimum cardinality is a *minimum hull set* of G. The cardinality of a minimum hull set of G is the *hull number* $h(G)$ of G. A set S of vertices of G is a *geodetic set* if $I_G[S] = V(G)$, and a geodetic set of minimum cardinality is a *minimum geodetic set* of G. The cardinality of a minimum geodetic set of G is the *geodetic number* $g(G)$. The length of a shortest cycle in G is the *girth* of G. A vertex x is an *extreme vertex* of G if the induced subgraph of the neighbors of x is complete or equivalently, $V(G) - \{x\}$ is convex in G. The set of all extreme vertices is denoted by $Ext(G)$ and $e(G) = |Ext(G)|$. A graph G is an *extreme geodesic graph* if the set of all extreme vertices forms a geodetic set. Extreme geodesic graphs were introduced and studied in [4].

The *strong product* of graphs G and H, denoted by $G \boxtimes H$, has vertex set $V(G) \times V(H)$, where two distinct vertices (x_1,y_1) and (x_2,y_2) are adjacent with respect to the strong product if, (a) $x_1 = x_2$ and $y_1y_2 \in E(H)$, or (b) $y_1 = y_2$ and $x_1x_2 \in E(G)$, or (c) $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. The mappings $\pi_G : (x,y) \mapsto x$ and $\pi_H : (x,y) \mapsto y$ from $V(G \boxtimes H)$ onto G and H respectively are called *projections*. For a set $S \subseteq V(G \boxtimes H)$, we define the *G-projection* on G as $\pi_G(S) = \{x \in V(G) : (x,y) \in S \text{ for some } y \in V(H)\}$, and the *H-projection* $\pi_H(S) = \{y \in V(H) : (x,y) \in S \text{ for some } x \in V(G)\}$. For a walk $P : (x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$ in $G \boxtimes H$, we define the *G-projection* $\pi_G(P)$ of P as a sequence that is obtained from $(x_1,x_2,\ldots,x_n)$ by changing each constant subsequence with its unique element. For example, if $P : (x_2,y_3),(x_2,y_4),(x_2,y_5),(x_4,y_5),(x_4,y_2),(x_3,y_2),(x_2,y_2)$ is a walk, then $\pi_G(P)$ is x_2,x_4,x_3,x_2 (it is obtained from the sequence $(x_2,x_2,x_2,$ $x_4,x_4,x_3,x_2)$). The H-projection $\pi_H(P)$ is defined similarly. It is clear from the defintion of strong product that for any walk P in $G \boxtimes H$, both $\pi_G(P)$ and $\pi_H(P)$ are walks in the factor graphs G and H respectively.

The hull number of a graph was introduced in [8] and further studied in [3, 6]. The hull number of composition and Cartesian product of graphs were studied in [2, 10]. In this paper we study the hull number of strong product of two graphs. In Section 2, we obtain upper bounds for the hull number of strong product of two graphs. Improved upper bounds are also obtained for a class of strong product graphs. In Section 3, the exact value of $h(G \boxtimes H)$ is obtained for several classes of graphs. In particular, it is proved that for any connected graph G, $h(G \boxtimes K_{r_1,r_2,\ldots,r_n}) = 2$ and $h(G \boxtimes C_{2n}) = 2$ for all $n, r_i \geq 2$. It is shown that $h(G \boxtimes K_m) = h(G) + e(G)(m-1)$ for any connected graph G. Graphs G and H for which $h(G \boxtimes H) = h(G)h(H)$ are characterized.

For basic graph theoretic terminology, we refer to [7]. We also refer to [1] for results on distance in graphs and to [9] for metric structures in strong product of graphs. Throughout the following G denotes a connected graph with at least two vertices. For a vertex x in G and a subset S of vertices in G, we mean by $x \times S$, the Cartesian product $\{x\} \times S$. The following theorems will be used in the sequel.

Theorem 1.1 [9]**.** *Let G and H be connected graphs with (u, v) and (x, y) arbitrary vertices of the strong product $G \boxtimes H$ of G and H. Then $d_{G \boxtimes H}((u, v), (x, y)) = \max\{d_G(u, x), d_H(v, y)\}$.*

Theorem 1.2 [3]**.** *Each extreme vertex of a connected graph G belongs to every hull set of G.*

Theorem 1.3 [5]**.** *Each extreme vertex of a connected graph G belongs to every geodetic set of G.*

2. Bounds for the Hull Number

In this section we determine possible bounds for the hull number of the strong product of two connected graphs. And improved upper bounds are obtained for some classes strong product graphs.

Proposition 2.1. *Let G and H be connected graphs and P a $(u, v) - (u', v')$ geodesic in $G \boxtimes H$ of length n. If $d_G(u, u') \geq d_H(v, v')$, then $\pi_G(P)$ is a $u - u'$ geodesic in G of length n, and if $d_G(u, u') \leq d_H(v, v')$, then $\pi_H(P)$ is a $v - v'$ geodesic in H of length n.*

Proof. Let $P : (u, v) = (u_0, v_0), (u_1, v_1), \ldots, (u_n, v_n) = (u', v')$ be a $(u, v) - (u', v')$ geodesic of length n in $G \boxtimes H$. If $d_G(u, u') \geq d_H(v, v')$, then by Theorem 1.1, $d_G(u, u') = \max\{d_G(u, u'), d_H(v, v')\} = d_{G \boxtimes H}((u, v), (u', v')) = n$.

Hence it follows that $\pi_G(P) : u = u_0, u_1, \ldots, u_n = u'$ must be an $u - u'$ geodesic in G. The other case follows similarly. ■

Theorem 2.2. *Let G and H be connected graphs. Then $Ext(G \boxtimes H) = Ext(G) \times Ext(H)$.*

Proof. Let $(g, h) \in Ext(G \boxtimes H)$. If g is an pendant vertex of G, then $g \in Ext(G)$. So, let $x_1, x_2 \in N_G(g)$ be such that $x_1 \neq x_2$. Then $(x_1, h), (x_2, h) \in N_{G\boxtimes H}((g, h))$. Since the induced subgraph of $N_{G\boxtimes H}(g, h)$ is complete, it follows that $x_1x_2 \in E(G)$ and so the induced subgraph of $N_G(g)$ is complete. Similarly, we can prove that $\langle N_H(h)\rangle$ is complete. Thus, $(g, h) \in Ext(G) \times Ext(H)$. Conversely, let $(g, h) \in Ext(G) \times Ext(H)$. Let (x_1, y_1), (x_2, y_2) be distinct vertices in $N_{G\boxtimes H}(g, h)$. Then $(x_1, y_1)(g, h) \in E(G \boxtimes H)$ and exactly one of the following three conditions holds.

(1) $x_1 = g$ and $y_1 \in N_H(h)$ or

(2) $x_1 \in N_G(g)$ and $y_1 = h$ or

(3) $x_1 \in N_G(g)$ and $y_1 \in N_H(h)$.

Similarly, $(x_2, y_2)(g, h) \in E(G \boxtimes H)$ and exactly one of the following three conditions holds.

(a) $x_2 = g$ and $y_2 \in N_H(h)$ or

(b) $x_2 \in N_G(g)$ and $y_2 = h$ or

(c) $x_2 \in N_G(g)$ and $y_2 \in N_H(h)$.

Now, there are nine cases.

Case 1. Both (a) and (1) hold. Then $y_1 \neq y_2$. Since $\langle N_H(h)\rangle$ is complete, we have $y_1y_2 \in E(H)$ so that $(x_1, y_1)(x_2, y_2) \in E(G \boxtimes H)$.

Case 2. Both (c) and (3) hold. Since $\langle N_G(g)\rangle$ is complete, either $x_1 = x_2$ or $x_1x_2 \in E(G)$. Similarly, we have either $y_1 = y_2$ or $y_1y_2 \in E(H)$. Since $(x_1, y_1) \neq (x_2, y_2)$, it follows that $(x_1, y_1)(x_2, y_2) \in E(G \boxtimes H)$. The other cases are similar. ■

Theorem 2.3. *Let G and H be connected graphs and S and T hull sets of G and H respectively. Then $S \times T$ is a hull set of $G \boxtimes H$.*

Proof. Let $W = S \times T$. We show that $[W]_{G\boxtimes H} = V(G \boxtimes H)$. Let $(x, y) \in V(G \boxtimes H)$. Now, since $[S]_G = V(G)$, it follows that there exists an integer

$m \geq 0$ such that $x \in I_G^m[S]$. We prove that $(x, y) \in [W]_{G\boxtimes H}$. The proof is by induction on m. Let $m = 0$. Then $x \in S$. Now, since $[T] = V(H)$, it follows that there exists an integer $n \geq 0$ such that $y \in I_G^n[T]$. We prove that $(x, y) \in [W]_{G\boxtimes H}$. The proof is by induction on n. If $n = 0$ then $y \in T$ and so $(x, y) \in S \times T \subseteq [W]_{G\boxtimes H}$. Assume that $(x, y) \in [W]_{G\boxtimes H}$ for all $y \in I_H^k[T]$. Let $y \in I_H^{k+1}[T]$ be such that $y \notin I_H^k[T]$. Then there exist y', y'' in $I_H^k[T]$ such that y lies on a $y' - y''$ geodesic $P : y' = y_0, y_1, \ldots, y_t = y''$ with $y \neq y', y''$. Now, by induction hypothesis, $(x, y'), (x, y'') \in [W]_{G\boxtimes H}$. Now, it follows from Theorem 1.1 that the walk $Q : (x, y') = (x, y_0), (x, y_1), \ldots, (x, y_t) = (x, y'')$ is a geodesic in $G \boxtimes H$ which contains the vertex (x, y). Hence $(x, y) \in [W]_{G\boxtimes H}$. Thus, by induction, $(x, y) \in [W]_{G\boxtimes H}$ for all $y \in V(H)$.

Assume that the result is true for $m = l$. Then $(x, y) \in [W]_{G\boxtimes H}$ for all $x \in I_G^l[S]$ and $y \in V(H)$. Let $x \in V(G)$ be such that $x \in I_G^{l+1}[S]$ and $x \notin I_G^l[S]$. Then there exist $x', x'' \in I_G^l[S]$ such that x lies on a $x' - x''$ geodesic $P' : x' = x_0, x_1, \ldots, x_j = x, \ldots, x_s = x'$ with $1 \leq j \leq s-1$. Now, by induction hypothesis, $(x', y), (x'', y) \in [W]_{G\boxtimes H}$. By Theorem 1.1, it follows that the walk $Q' : (x', y) = (x_0, y), (x_1, y) \ldots, (x_j, y) = (x, y), \ldots, (x_s, y) = (x'', y)$ is a geodesic. Hence $(x, y) \in [W]_{G\boxtimes H}$. Thus, by induction $(x, y) \in [W]_{G\boxtimes H}$ for all $x \in V(G)$ and $y \in V(H)$ so that $[W]_{G\boxtimes H} = V(G \boxtimes H)$. ■

Remark 2.4. The converse of Theorem 2.3 need not be true. Let G be the cycle $C_4 : u_1, u_2, u_3, u_4, u_1$ and let H be the complete graph K_2, with vertex set $\{v_1, v_2\}$. Let $S = \{u_1, u_3\}$ and $T = \{v_1\}$. Then, it is clear that $I^2_{G\boxtimes H}[S \times T] = V(G \boxtimes H)$ and so $S \times T$ is a hull set of $G \boxtimes H$. However, T is not a hull set of K_2.

Corollary 2.5. *Let G and H be connected graphs. Then* $\max\{2, e(G)e(H)\} \leq h(G \boxtimes H) \leq h(G)h(H)$.

Proof. Let S and T be minimum hull sets of G and H respectively. By Theorem 2.3, $W = S \times T$ is a hull set of G so that $h(G \boxtimes H) \leq h(G)h(H)$. The other inequality follows from Theorems 1.2 and 2.2. ■

Lemma 2.6. *Let G and H be connected graphs. Then, for any $x \in V(G)$ and $T \subseteq V(H)$, $x \times I_H^k[T] \subseteq I^k_{G\boxtimes H}[x \times T]$ for all $k \geq 0$.*

Proof. For $k = 0$, it is obvious. We first show that $x \times I_H[T] \subseteq I_{G\boxtimes H}[x \times T]$. Let $(x, y) \in x \times I_H[T]$. If $y \in T$, then $(x, y) \in I_{G\boxtimes H}[x \times T]$. If $y \notin T$, then y lies on a $y' - y''$ geodesic $P : y' = y_0, y_1, \ldots, y_i = y, \ldots, y_n = y''$ with $y', y'' \in T$. It follows from Theorem 1.1 that $Q : (x, y') = (x, y_0), (x, y_1), \ldots, (x, y_i) =$

$(x, y), \ldots, (x, y_n) = (x, y'')$ is a geodesic in $G \boxtimes H$ with $(x, y'), (x, y'') \in x \times T$ and so $(x, y) \in I_{G\boxtimes H}[x \times T]$. Hence $x \times I_H[T] \subseteq I_{G\boxtimes H}[x \times T]$.

Now, $x \times I_H^2[T] = x \times I_H[I_H[T]] \subseteq I_{G\boxtimes H}[x \times I_H[T]] \subseteq I_{G\boxtimes H}^2[x \times T]$. Proceeding like this, we see that $x \times I_H^k[T] \subseteq I_{G\boxtimes H}^k[x \times T]$ for all $k \geq 0$. ■

Theorem 2.7. *Let G and H be connected graphs. Then $h(G \boxtimes H) \leq \min\{h(H) + e(H)(h(G) - 1), h(G) + e(G)(h(H) - 1)\}$.*

Proof. Let S and T be minimum hull sets of G and H respectively. Let $W = (Ext(G) \times Ext(H)) \cup ((S - Ext(G)) \times Ext(H)) \cup (u \times (T - Ext(H)))$, where $u \in S$. Then $|W| \leq e(G)e(H) + (h(G) - e(G))e(H) + (h(H) - e(H)) = h(H) + e(H)(h(G) - 1)$. We prove that W is a hull set of $G \boxtimes H$.

Step 1. $u \times V(H) \subseteq [W]_{G\boxtimes H}$.

Let $y \in V(H)$. Since T is a hull set of H, it follows that $y \in I_H^k[T]$ for some $k \geq 0$ and so $(u, y) \in u \times I_H^k[T]$. Hence by Lemma 2.6, $(u, y) \in I_{G\boxtimes H}^k[u \times T]$. Since $u \in S$, it is clear from the definition of W that $u \times T \subseteq W$ and so $I_{G\boxtimes H}^k[u \times T] \subseteq I_{G\boxtimes H}^k[W] \subseteq [W]_{G\boxtimes H}$. Thus $(u, y) \in [W]_{G\boxtimes H}$ and so $u \times V(H) \subseteq [W]_{G\boxtimes H}$.

Step 2. If $x \in V(G)$ and $x \times V(H) \subseteq [W]_{G\boxtimes H}$, then $x' \times V(H) \subseteq [W]_{G\boxtimes H}$ for $x' \in N_G(x)$.

Let $y \in V(H)$. If $y \notin Ext(H)$, then there exist vertices $y', y'' \in N_H(y)$ such that y' and y'' are non-adjacent. It is clear that $Q : (x, y'), (x', y), (x, y'')$ is a geodesic in $G \boxtimes H$ with $(x, y'), (x, y'') \in [W]_{G\boxtimes H}$ and so $(x', y) \in [W]_{G\boxtimes H}$. Now, assume that $y \in Ext(H)$. Since S is a hull set of G, it follows that $x' \in I_G^l[S]$ for some $l \geq 0$. Thus $(x', y) \in I_G^l[S] \times y$. By Lemma 2.6, $(x', y) \in I_{G\boxtimes H}^l[S \times y]$. Since $y \in Ext(H)$, it is clear from the definition of W that $S \times y \subseteq [W]_{G\boxtimes H}$ and so $(x', y) \in [W]_{G\boxtimes H}$. Hence $x' \times V(H) \subseteq [W]_{G\boxtimes H}$.

Now, since G and H are connected, it follows from Step 1 and Step 2 that $V(G) \times V(H) \subseteq [W]_{G\boxtimes H}$ and so W is a hull set of $G \boxtimes H$. Hence $h(G \boxtimes H) \leq |W| \leq h(H) + e(H) + (h(G) - 1)$. Similarly, we can prove that $h(G \boxtimes H) \leq h(G) + e(G)(h(H) - 1)$. Thus the result follows. ■

Corollary 2.8. *Let G and H be connected graphs such that H has no extreme vertices. Then $h(G \boxtimes H) \leq h(H)$.*

Corollary 2.9. *Let G and H be connected graphs having no extreme vertices. Then $2 \leq h(G \boxtimes H) \leq \min\{h(G), h(H)\}$.*

Corollary 2.10. *For any connected graph G, $h(G \boxtimes K_{r_1,r_2,\ldots,r_n}) = 2$, where $n \geq 2$ and $r_i \geq 2$ for $i = 1, 2, \ldots n$.*

Proof. Let $n \geq 2$ and $r_i \geq 2$ for $i = 1, 2, \ldots, n$. Any two vertices of a partite set of $K_{r_1,r_2,\ldots,r_n}$ form a hull set and so $h(K_{r_1,r_2,\ldots,r_n}) = 2$. Also $K_{r_1,r_2,\ldots,r_n}$ has no extreme vertices. Hence the result follows from Corollary 2.8. ∎

Corollary 2.11. *For any connected graph G, $h(G \boxtimes C_{2n}) = 2$ for all $n \geq 2$.*

Proof. This follows from Corollary 2.8. ∎

In the following we introduce a class of graphs for which the upper bound of hull number is further improved.

Let $\Im$ denote the class of connected graphs G such that every non-extreme vertex of G has two non-adjacent neighbors which are not extreme. The graph G in Figure 2.1 belongs to the class $\Im$. Obviously, complete graphs and graphs having no extreme vertices belong to $\Im$.

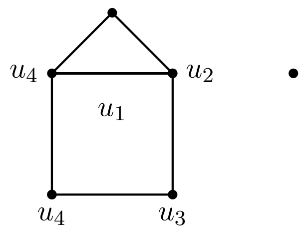

Figure 2.1.

Theorem 2.12. *If G and H are connected graphs having extreme vertices and belong to $\Im$, then $h(G \boxtimes H) \leq e(G)e(H) + h(G) + h(H) - e(G) - e(H)$.*

Proof. Let S and T be minimum hull sets of G and H respectively. Let $u \in Ext(G)$ and $v \in Ext(H)$. We show that $W = (Ext(G) \times Ext(H)) \cup ((S - Ext(G)) \times v) \cup (u \times (T - Ext(H)))$ is a hull set of $G \boxtimes H$. It is clear that $S \times v, u \times T \subseteq W \subseteq [W]_{G \boxtimes H}$.

Step 1. $V(G) \times v \subseteq [W]_{G \boxtimes H}$.

Let $x \in V(G)$. Since S is a hull set of G, we have $x \in I_G^k[S]$ for some $k \geq 0$ and so $(x, v) \in I_G^k[S] \times v$. By Lemma 2.6, $(x, v) \in I_{G \boxtimes H}^k[S \times v] \subseteq [W]_{G \boxtimes H}$. Hence $V(G) \times v \subseteq [W]_{G \boxtimes H}$.

Step 2. If $y \in V(H)$ and $(V(G) - Ext(G)) \times y \subseteq [W]_{G \boxtimes H}$, then $(V(G) - Ext(G)) \times y' \subseteq [W]_{G \boxtimes H}$ for $y' \in N_H(y)$.

Let $x \in V(G) - Ext(G)$. Since $G \in \Im$, there exist $x', x'' \in N_G(x)$ such that x', x'' are non-extreme and non-adjacent. Now, it is clear that $Q : (x', y), (x, y'), (x'', y)$ is a geodesic in $G \boxtimes H$ with $(x', y), (x'', y) \in (V(G) - Ext(G)) \times y \subseteq [W]_{G \boxtimes H}$ and so $(x, y') \in [W]_{G \boxtimes H}$. Thus $(V(G) - Ext(G)) \times y' \subseteq [W]_{G \boxtimes H}$.

Now, since G and H are connected, it follows from Step 1 and Step 2 that $(V(G) - Ext(G)) \times V(H) \subseteq [W]_{G \boxtimes H}$. Similarly, we can prove that $V(G) \times (V(H) - Ext(H)) \subseteq [W]_{G \boxtimes H}$. Also, by the definition of W, we have $Ext(G) \times Ext(H) \subseteq W \subseteq [W]_{G \boxtimes H}$. Hence $[W]_{G \boxtimes H} = V(G \boxtimes H)$ and so $h(G \boxtimes H) \leq |W| \leq e(G)e(H) + h(G) + h(H) - e(G) - e(H)$. ■

3. Exact Hull Numbers

In this section we determine the exact values of the hull numbers of the strong product for several classes of graphs. We also give several classes of graphs G and H with $h(G \boxtimes H) = 2$. It is to be noted that the graphs given in Corollaries 2.10 and 2.11 belong to this class. We also characterize graphs G and H for which $h(G \boxtimes H) = h(G)h(H)$.

Theorem 3.1. *Let G and H be connected graphs such that G has no extreme vertices. Then*

(i) *$h(G \boxtimes H) = 2$ if the girth of H is even,*

(ii) *$h(G \boxtimes H) \leq 3$ if the girth of H is odd and at least 5.*

Proof. (i) Let the girth of H be $2n (n \geq 2)$ and let $C : y_0, y_1, \ldots, y_{2n-1}, y_0$ be a cycle of length $2n$. For any $x \in V(G)$, we show that the set $W = \{(x, y_0), (x, y_n)\}$ is a hull set of $G \boxtimes H$. We first prove the following two steps.

Step 1. $V(G) \times \{y_0, y_n\} \subseteq [W]_{G \boxtimes H}$. Let $u \in V(G)$. We use induction on $d_G(x, u)$ to prove that $(u, y_0), (u, y_n) \in [W]_{G \boxtimes H}$. Let $d_G(x, u) = 0$ or 1. Since C is a shortest cycle in H, it follows that the path $P : y_0, y_1, \ldots, y_n$ and $P_1 : y_n, y_{n+1}, \ldots, y_{2n-1}, y_0$ are geodesics in H. Then it follows from Theorem 1.1 that $Q : (x, y_0), (x, y_1), \ldots, (x, y_{n-1}), (x, y_n)$ and $Q_1 : (x, y_n), (x, y_{n+1}), \ldots, (x, y_{2n-1}), (x, y_0)$ are geodesics in $G \boxtimes H$ and so $(x, y_1), (x, y_{2n-1}), (x, y_{n-1}), (x, y_{n+1}) \in [W]_{G \boxtimes H}$. It is clear that Q_2 :

$(x, y_1), (u, y_0), (x, y_{2n-1})$ and $Q_3 : (x, y_{n-1}), (u, y_n), (x, y_{n+1})$ are geodesics in $G \boxtimes H$. Hence $(u, y_0), (u, y_n) \in [W]_{G\boxtimes H}$.

Assume that the result is true for $d_G(x, u) = k$. Let u be a vertex such that $d_G(x, u) = k + 1$. Let $x = x_0, x_1, \ldots, x_k, x_{k+1} = u$ be a $x - u$ geodesic in G. By induction hypothesis, $(x_k, y_0), (x_k, y_n) \in [W]_{G\boxtimes H}$. As above, we see that $Q_3 : (x_k, y_0), (x_k, y_1), \ldots, (x_k, y_n)$ and $Q_4 : (x_k, y_n), (x_k, y_{n+1}), \ldots, (x_k, y_{2n-1}), (x_k, y_0)$ are geodesics in $G \boxtimes H$ so that $(x_k, y_1), (x_k, y_{2n-1}), (x_k, y_{n-1}), (x_k, y_{n+1}) \in [W]_{G\boxtimes H}$. It is clear that $Q_5 : (x_k, y_1), (u, y_0), (x_k, y_{2n-1})$ and $Q_6 : (x_k, y_{n-1}), (u, y_n), (x_k, y_{n+1})$ are geodesics in $G \boxtimes H$. Hence $(u, y_0), (u, y_n) \in [W]_{G\boxtimes H}$ and so $V(G) \times \{y_0, y_n\} \subseteq [W]_{G\boxtimes H}$.

Step 2. If $y \in V(H)$ and $V(G) \times y \in [W]_{G\boxtimes H}$, then $V(G) \times y' \subseteq [W]_{G\boxtimes H}$ for $y' \in N_H(y)$.

Let $u \in V(G)$. Since G has no extreme vertices, there exist vertices $u', u'' \in N_G(u)$ such that u' and u'' are non-adjacent. Now, it is clear that $Q_5 : (u', y), (u, y'), (u'', y)$ is a geodesic in $G \boxtimes H$ with $(u', y), (u'', y) \in [W]_{G\boxtimes H}$ and so $(u, y') \in [W]_{G\boxtimes H}$. Hence $V(G) \times y' \subseteq [W]_{G\boxtimes H}$. Now, since G and H are connected, it follows from Step 1 and Step 2 that $V(G) \times V(H) \subseteq [W]_{G\boxtimes H}$ and so W is a hull set of $G \boxtimes H$. Hence $h(G \boxtimes H) = 2$.

(ii) Let the girth of H be $2n+1 (n \geq 2)$ and let $C : y_0, y_1, \ldots, y_{2n}, y_0$ be a cycle of length $2n+1$. For any $x \in V(G)$, let $W = \{(x, y_0), (x, y_n), (x, y_{n+1}\}$. Then, as in (i), we can prove that W is a hull set of $G \boxtimes H$ and so $h(G \boxtimes H) \leq |W| = 3$. ∎

In the following we give a class of strong product graphs for which the bound in Theorem 2.7 is attained.

Theorem 3.2. *Let G and H be connected graphs and $S \subseteq V(G \boxtimes H)$. Then $I^k_G[\pi_G(S)] \subseteq \pi_G(I^k_{G\boxtimes H}[S])$.*

Proof. For $k = 0$, it is obvious. We first show that $I_G[\pi_G(S)] \subseteq \pi_G(I_{G\boxtimes H}[S])$. Let $x \in I_G[\pi_G(S)]$. If $x \in \pi_G(S)$, then there exists $y \in V(H)$ such that $(x, y) \in S \subseteq I_{G\boxtimes H}[S]$ and so $x \in \pi_G(I_{G\boxtimes H}[S])$. If $x \notin \pi_G(S)$, then there exist $g, g' \in \pi_G(S)$ such that x lies on a $g - g'$ geodesic $P : g = g_0, g_1, \ldots, g_i = x, g_{i+1}, \ldots, g_n = g'$ with $1 \leq i \leq n - 1$ so that $d_G(g, g') = n$. Since $g, g' \in \pi_G(S)$, there exist $h, h' \in V(H)$ such that $(g, h), (g', h') \in S$. Let $d_H(h, h') = m$. and let $Q : h = h_0, h_1, \ldots, h_m = h'$ be a $h - h'$ geodesic in H. We consider the following two cases.

Case 1. $m \geq n$. Then it follows from Theorem 1.1 that $Q' : (g, h) = (g_0, h_0), (g_1, h_1), \ldots, (g_i, h_i) = (x, h_i), (g_{i+1}, h_{i+1}), \ldots, (g_n, h_n), (g_n, h_{n+1}),$

$\ldots, (g_n, h_m) = (g', h')$ is a $(g, h) - (g', h')$ geodesic in $G \boxtimes H$ containing the vertex (x, h_i). Hence $(x, h_i) \in I_{G\boxtimes H}[S]$ and so $x \in \pi_G(I_{G\boxtimes H}[S])$.

Case 2. $m < n$. Then it follows from Theorem 1.1 that the walk Q'' : $(g, h) = (g_0, h_0), (g_1, h_1), \ldots, (g_m, h_m), (g_{m+1}, h_m), \ldots, (g_n, h_m) = (g', h')$ is a $(g, h) - (g', h')$ geodesic in $G \boxtimes H$ containing the vertex (x, h_i) for some i with $1 \leq i \leq m$. Hence $(x, h_i) \in I_{G\boxtimes H}[S]$ and so $x \in \pi_G(I_{G\boxtimes H}[S])$. Thus $I_G[\pi_G(S)] \subseteq \pi_G(I_{G\boxtimes H}[S])$.

Now, $I_G^2[\pi_G(S)] = I_G[I_G[\pi_G(S)]] \subseteq I_G[\pi_G(I_{G\boxtimes H}[S])] \subseteq \pi_G(I_{G\boxtimes H}^2[S])$. Proceeding like this, we get $I_G^k[\pi_G(S)] \subsetneq \pi_G(I_{G\boxtimes H}^k[S])$. ■

Remark 3.3. Strict inclusion can hold in Theorem 3.2. Let G and H be the paths $P_4 : u_1, u_2, u_3, u_4$ and $P_5 : v_1, v_2, v_3, v_4, v_5$ respectively. Let $S = \{(u_1, v_1), (u_2, v_2), (u_2, v_4)\}$. Then it is easily checked that $I_{G\boxtimes H}[S] = \{(u_1, v_1), (u_1, v_3), (u_1, v_2), (u_3, v_3), (u_2, v_3), (u_2, v_4), (u_2, v_2)\}$ and so $\pi_G(I_{G\boxtimes H}[S]) = \{u_1, u_2, u_3\}$. But $I_G[\pi_G(S)] = I_G[\{u_1, u_2\}] = \{u_1, u_2\} \subsetneq \pi_G(I_{G\boxtimes H}[S])$.

The following theorem shows that equality holds for the graph $H = K_m$ in the inclusion in Theorem 3.2.

Theorem 3.4. *Let G be a connected graph and $S \subseteq V(G \boxtimes K_m)$, where $m \geq 2$. Then $I_G^k[\pi_G(S)] = \pi_G(I_{G\boxtimes H}^k[S])$ for all $k \geq 0$.*

Proof. For $k = 0$, it is obvious. By Theorem 3.2, it is enough to show that $\pi_G(I_{G\boxtimes H}^k[S]) \subseteq I_G^k[\pi_G(S)]$. We first show that $\pi_G(I_{G\boxtimes H}[S]) \subseteq I_G[\pi_G(S)]$. Let $x \in \pi_G(I_{G\boxtimes H}[S])$. Then $(x, y) \in I_{G\boxtimes H}[S]$ for some $y \in V(K_m)$. If $(x, y) \in S$, then the result is trivial. If $(x, y) \notin S$, then there exist $(g, h), (g', h') \in S$ such that (x, y) lies on a $(g, h) - (g', h')$ geodesic $P : (g, h) = (g_0, h_0), (g_1, h_1), \ldots, (g_i, h_i) = (x, y), \ldots, (g_n, h_n) = (g', h')$ of length $n \geq 2$ with $1 \leq i \leq n - 1$. Since $d_H(h, h') = 1$, it follows that $d_G(g, g') > d_H(h, h')$. By Proposition 2.1, $\pi_G(P)$ is a $g - g'$ geodesic in G containing the vertex x, where $g, g' \in \pi_G(S)$. Hence $x \in I_G[\pi_G(S)]$. Thus $\pi_G(I_{G\boxtimes H}[S]) \subseteq I_G[\pi_G(S)]$. Now, $\pi_G(I_{G\boxtimes H}^2[S]) \subseteq I_G[\pi_G(I_{G\boxtimes H}[S])] \subseteq I_G^2[\pi_G(S)]$.
Proceeding like this, we get $\pi_G(I_{G\boxtimes H}^k[S]) = I_G^k[\pi_G(S)]$ for all $k \geq 0$. ■

Theorem 3.5. *For a connected graph G, $h(G \boxtimes K_m) = h(G) + e(G)(m - 1)$.*

Proof. Let S be a minimum hull set of G and let $W = Ext(G) \times V(K_m) \cup ((S - Ext(G)) \times v)$, where $v \in V(K_m)$. Then, as in the proof of Theorem 2.7, we can prove that W is a hull set of $G \boxtimes K_m$. Hence $h(G \boxtimes K_m) \leq |W| = h(G) + e(G)(m-1)$. On the other hand, if there exists a hull set W' of $G \boxtimes K_m$ such that $|W'| < |W|$, then it follows from Theorems 1.2 and 2.2 that $W' = Ext(G) \times V(K_m) \cup T$, where $T \cap (Ext(G) \times V(K_m)) = \phi$. This implies that $|T| < |(S - Ext(G)) \times v| = |S - Ext(G)| = h(G) - e(G)$. Now, since W' is a hull set of $G \boxtimes K_m$, there exists an integer $k \geq 0$ such that $[W']_{G \boxtimes K_m} = I^k_{G \boxtimes K_m}[W'] = V(G \boxtimes K_m)$. By Theorem 3.4, $I^k_G[\pi_G(W')] = \pi_G(I^k_{G \boxtimes K_m}[W']) = V(G)$ and so $\pi_G(W')$ is a hull set of G. It is easily seen that $\pi_G(W') = Ext(G) \cup \pi_G(T)$ and so $|\pi_G(W')| \leq |Ext(G)| + |\pi_G(T)| \leq |Ext(G)| + |T| < e(G) + h(G) - e(G) = h(G)$. Thus $\pi_G(W')$ is a hull set of G such that $|\pi_G(W')| < h(G)$, which is a contradiction. Hence W is a minimum hull set of $G \boxtimes H$ so that $h(G \boxtimes K_m) = |W| = h(G) + e(G)(m-1)$. ■

Let $G \circ H$ denote the composition of two graphs G and H. By proving complicated lemmas and theorems, it is proved in [2] that $h(G \circ K_m) = h(G) + e(G)(m-1)$. We observe that $G \circ K_m = G \boxtimes K_m$ and so the following corollary gives a very simple and alternate proof of the above result proved in [2].

Corollary 3.6. *For a connected graph G, $h(G \circ K_m) = h(G) + e(G)(m-1)$.*

4. EXTREME HULL GRAPHS

In this section we characterize the class of graphs for which the upper bound in Corollary 2.5 is attained.

Definition. A graph G is an *extreme hull graph* if the set of extreme vertices of G is a hull set of G.

Example 4.1. For the graph G in Figure 4.1, the set $S = \{u_1, u_5\}$ of extreme vertices is a hull set of G so that G is an extreme hull graph.

Remark 4.2. Every extreme geodesic graph is an extreme hull graph. The graph G given in Figure 4.1 is an extreme hull graph, which it is not an extreme geodesic graph.

By Theorem 1.2, $0 \leq e(G) \leq h(G)$ for every graph G. The following theorem is a realization of this result.

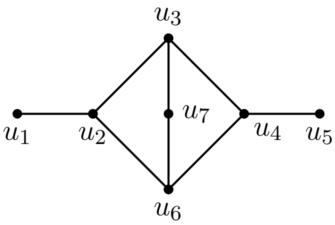

Figure 4.1.

Theorem 4.3. *For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $e(G) = a$ and $h(G) = b$.*

Proof. If $a = b$, then $a \geq 2$ and $G = K_a$ has the desired properties. Thus we assume that $a < b$. Let G_i $(1 \leq i \leq b - a)$ be the graphs given in Figure 4.2. Let H be the graph obtained from $\bigcup_{i=1}^{b-a} G_i$ by adding a new vertex w and joining w to x_i and z_i $(1 \leq i \leq b-a)$. Now, let G be the graph obtained from H by adding the new vertices $s_1, s_2, \ldots, s_a$ and joining these to w. The graph G is shown in Figure 4.3.

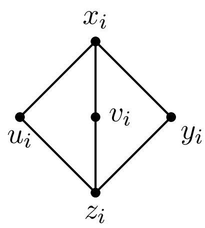

Figure 4.2.

Then $S = \{s_1, s_2, \ldots, s_a\}$ is the set of extreme vertices of G and so $e(G) = a$. We prove that $h(G) = b$. By Theorem 1.2, the vertices $s_1, s_2, \ldots, s_a$ belong to every hull set of G. Since $V(G) - V(G_i)$ is a convex set for each $i = 1, 2, \ldots, b - a$, it follows that every hull set of G contains at least one vertex from each G_i. Hence $h(G) \geq a + b - a = b$. Now, since the set $S' = S \cup \{v_1, v_2, \ldots, v_{b-a}\}$ is a hull set of G, we have $h(G) = b$. ∎

Theorem 4.4. *For every pair a, b of integers with $2 \leq a \leq b$, there exists an extreme hull graph G with $h(G) = a$ and $g(G) = b$.*

Proof. If $a = b$, then $G = K_a$ has the desired properties. Thus we assume that $a < b$. We construct a graph G with the required geodetic number a

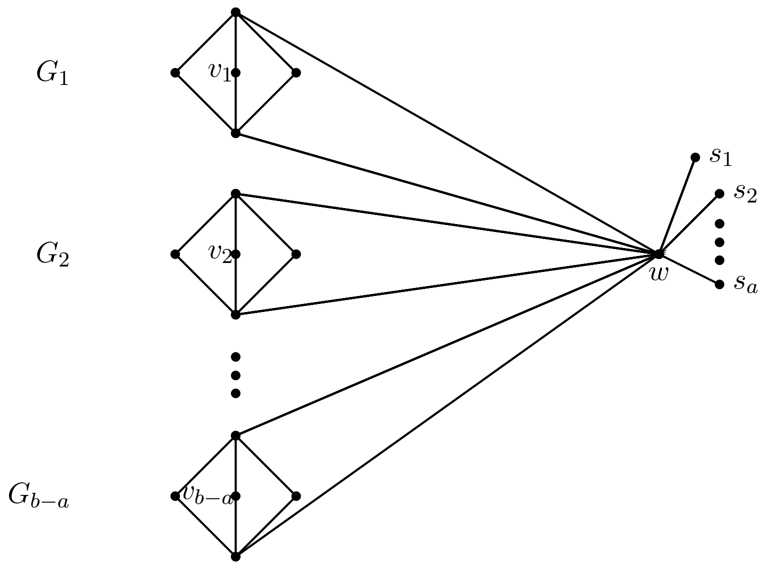

Figure 4.3.

and hull number b. Let G_i $(1 \leq i \leq b-a)$ be the graphs given in Figure 4.2. Let G be the graph obtained from $\bigcup_{i=1}^{b-a} G_i$ by adding the new vertices w_i $(1 \leq i \leq a)$ and the edges (1) $w_i u_1$ $(1 \leq i \leq a-1)$, $w_a y_{b-a}$ and (2) $y_i u_{i+1}$ $(1 \leq i \leq b-a-1)$. The graph G is shown in Figure 4.4.

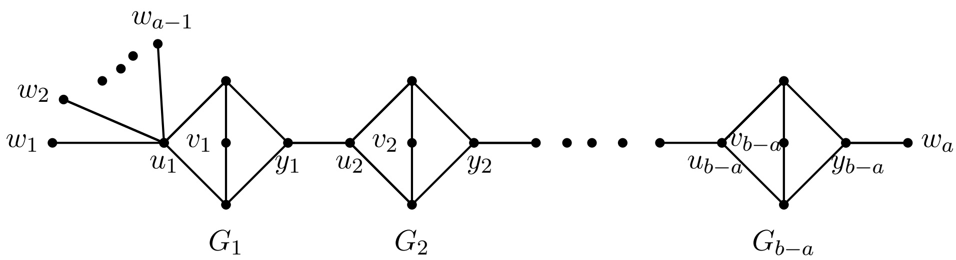

Figure 4.4.

Let $S = \{w_1, w_2, \ldots, w_a\}$ be the set of extreme vertices of G. Then it is clear that $I[S] = V(G) - \{v_1, v_2, \ldots, v_{b-a}\}$ and $I^2[S] = V(G)$. Hence by Theorem 1.2, S is the unique minimum hull set of G and so $h(G) = a$.

Next, we show that $g(G) = b$. It is clear that $I[S] = V(G) - \{v_1, v_2, \dots, v_{b-a}\}$ and each v_i must belong to every minimum geodetic set of G. Since $W = S \cup \{v_1, v_2, \dots, v_{b-a}\}$ is a geodetic set of G, it follows from Theorem 1.3 that $g(G) = b$. ■

Theorem 4.5. *Let G and H be connected graphs. Then $h(G \boxtimes H) = h(G)h(H)$ if and only if both G and H are extreme hull graphs.*

Proof. Let G and H be extreme hull graphs. Then $Ext(G)$ and $Ext(H)$ are minimum hull sets of G and H respectively. Therefore, $h(G) = e(G)$ and $h(H) = e(H)$. Now, it follows from Theorems 2.2 and 2.3 that $Ext(G \boxtimes H) = Ext(G) \times Ext(H)$ is a hull set of $G \boxtimes H$. Hence by Theorem 1.2, we have $h(G \boxtimes H) = e(G)e(H) = h(G)h(H)$.

Conversely, assume that $h(G \boxtimes H) = h(G)h(H)$. Let S and T be minimum hull sets of G and H respectively. If $Ext(G) = \emptyset$, then, by Corollary 2.8, $h(G \boxtimes H) \leq h(G) < h(G)h(H)$, which is a contradiction. Hence $Ext(G) \neq \emptyset$. Similarly, we can prove that $Ext(H) \neq \emptyset$. Now, by Theorem 2.7, $h(G \boxtimes H) \leq h(H) + e(H)(h(G) - 1)$. Hence $h(G)h(H) \leq h(H) + e(H)(h(G) - 1)$. This implies that $h(H)(h(G) - 1) \leq e(H)(h(G) - 1)$. Since $h(G) \geq 2$, we have $h(H) \leq e(H)$. Hence it follows from Theorem 1.2 that $h(H) = e(H)$ and so H is an extreme hull graph. Similarly, G is also an extreme hull graph. ■

Corollary 4.6. *Let G and H be connected graphs. If G and H are extreme hull graphs, then $G \boxtimes H$ is an extreme hull graph.*

Proof. This follows from Theorems 2.2 and 4.5. ■

The converse of the above corollary seems to be a diffcult problem and we leave it open.

Problem 4.7. Let G and H be graphs such that $G \boxtimes H$ is an extreme hull graph. Is it true that G and H are extreme hull graphs ?

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