

## BOUNDING NEIGHBOR-CONNECTIVITY OF ABELIAN CAYLEY GRAPHS

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### Abstract

For the notion of neighbor-connectivity in graphs whenever a vertex is subverted the entire closed neighborhood of the vertex is deleted from the graph. The minimum number of vertices whose subversion results in an empty, complete, or disconnected subgraph is called the neighbor-connectivity of the graph. Gunther, Hartnell, and Nowakowski have shown that for any graph, neighbor-connectivity is bounded above by  $\kappa$ . Doty has sharpened that bound in abelian Cayley graphs to approximately  $\frac{1}{2}\kappa$ . The main result of this paper is the constructive development of an alternative, and often tighter, bound for abelian Cayley graphs through the use of an auxiliary graph determined by the generating set of the abelian Cayley graph.

**Keywords:** Cayley graphs, neighbor-connectivity bound.

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### 1. INTRODUCTION

In a series of papers, ([5, 6, 7]), Gunther and Hartnell introduced the notion of neighbor-connectivity in graphs. If a graph is being used to model a communication network, the failure (“subversion”, in the terminology originated by Gunther and Hartnell) of a vertex causes the failure (or purposeful shut-down) of all its immediate neighbors as well. Whenever a vertex is subverted the entire closed neighborhood of the vertex is deleted from the graph. The

minimum number of vertices whose subversion results in an empty, complete, or disconnected subgraph is called the neighbor-connectivity of the graph. Gunther [5] particularly noted that complete graphs are included with empty and disconnected graphs in this definition because they are “very vulnerable to attack”—subverting a single vertex will destroy the entire graph. Although neighbor-connectivity uses the language of spy networks it can be applied as well to electronic or physical networks in which failure of one node causes neighboring nodes to be shut down. Neighbor-connectivity and an idea closely related to neighbor-connectivity, namely that of efficient domination, are usefully studied in Cayley graphs because the vertex connectivity of Cayley graphs enables the design of efficient routing algorithms for computer networks [10]. Vertex dominating sets and efficient dominating sets in Cayley graphs have been described by Dejter and Serra [1], by Obradović, Peters and Ružić [10] and by Huang and Xu [9]. Doty, Goldstone, and Suffel [3] have used the algebraic structure of Cayley graphs to characterize abelian Cayley graphs with neighbor-connectivity equal to one. Doty [2] has shown that neighbor-connectivity of abelian Cayley graphs is bounded by approximately half the size of the graph’s generating set. This bound can be a significant overestimate of the actual neighbor-connectivity of specific graphs as shown in the following example.

**Example 1.** In the Cayley graph with vertices in  $Z_{60}$  and with generating set  $S = \{\pm 1, \pm 6, \pm 7, \pm 13, \pm 15, \pm 19, \pm 21, \pm 27\}$ , the upper bound on neighbor-connectivity given in [2] is  $\frac{1}{2}|S| + 1 = 9$ . It can be shown that the actual neighbor-connectivity is no more than four by noting that the closed neighborhood of  $\{\pm 12, \pm 24\}$  contains all elements of  $S$ . Thus the removal of  $\{\pm 12, \pm 24\}$  and their neighbors results in a graph with one component consisting of the isolated vertex 0.

In this paper we determine an alternative bound—one that is often substantially less than the known bounds. This new bound is computationally simple. It uses an auxiliary graph that is easy to construct and bounds neighbor-connectivity by the minimum vertex degree in this graph. In the previous example this new bound’s value is six, a significant improvement. Furthermore, the construction used in the proof gives a simple method of finding the effective subversion strategy  $\{\pm 12, \pm 24\}$  of the previous example, an effective subversion strategy with an even smaller number of elements than the numerical bound itself.

## 2. DEFINITIONS AND PRELIMINARY RESULTS

In order to explain these ideas more precisely, the following definitions are used: Suppose  $\Gamma$  is a graph with vertex set  $V$ . For any subset  $A$  of  $V$ ,  $N[A] = A \cup \{v \in V \mid v \text{ is adjacent to } a \text{ for some } a \in A\}$  is called the *closed neighborhood of  $A$* . If  $A = \{a\}$ , then we write  $N[a]$ . The remaining definitions essentially follow Gunther and Hartnell [7] and Gunther, Hartnell, Nowakowski [8]. To *subvert a vertex*  $v \in \Gamma$  means to remove all elements of  $N[v]$  from  $\Gamma$ . The resulting induced subgraph, called the *survival subgraph*, is exactly the subgraph of  $\Gamma$  induced by  $V \setminus N[v]$ . For a set of vertices  $B$ , the *survival subgraph for  $B$*  is the subgraph of  $\Gamma$  induced by  $V \setminus N[B]$ . We denote this survival subgraph by  $\Gamma \setminus N[B]$  and refer to the set  $B$  as a *subversion strategy*. If the survival subgraph for  $B$  is empty, complete, or disconnected, then  $B$  is called an *effective subversion strategy*. We say that a graph  $\Gamma$  has *neighbor-connectivity  $k$* , and we write  $\text{NC}(\Gamma) = k$ , if  $k$  is the minimum size of an effective subversion strategy.

For abelian group  $G$  and subset  $S$  of  $G \setminus \{0\}$  for which  $-S = S$ , the abelian Cayley graph, denoted  $\text{Cay}(G, S)$ , is the graph with vertices in  $G$ ; two vertices  $v, w \in G$  are adjacent in  $\text{Cay}(G, S)$  whenever  $v + s = w$ , for some  $s \in S$ . Elements of  $S$  should be thought of as generators of edges of the graph, rather than in the traditional group theoretic sense. Thus for our purposes, a *non-generator* is simply an element of  $G \setminus S$ . The generating set  $S$  lists all generators including inverses; by following this convention we have  $|S|$  equal to the degree of each vertex. We use  $S_0$  to abbreviate  $S \cup \{0\}$ .

To find an effective subversion strategy for  $\Gamma = \text{Cay}(G, S)$ , we construct an *auxiliary graph*, denoted  $\text{AUX}_\Gamma$  (or simply  $\text{AUX}$  if  $\Gamma$  is clear from context), with  $|S|$  vertices, one for each element of  $S$ . In other words, the elements of  $S$  are the vertices of  $\text{AUX}_\Gamma$ . Two vertices  $s$  and  $t$  are adjacent if and only if  $s + t \notin S_0$ . Note that  $s + s \notin S_0$  if and only if there is a loop at vertex  $s$  in  $\text{AUX}_\Gamma$ . This definition is a slight modification of the one first given in [2] where loops were specifically forbidden. In the auxiliary graph each edge represents a sum of generators in  $G$  and thus an element of  $G$ . If this sum, say  $s + t$ , is subverted, then the elements  $s$  and  $t$  will not be in the survival subgraph since  $s, t \in N[s + t]$ . If, in addition,  $s + t$  is not in  $S_0$ , then the vertex  $0$  will remain in the survival subgraph. Thus our basic method for creating an effective subversion strategy for  $\Gamma$  is to try to construct a set of sums like  $s + t$  that are not generators and to include each element of  $S$  in at least one such sum. Upon subversion of this set of sums, in some

cases, we will have isolated 0 in the survival subgraph, while in others we will only be able to guarantee that 0 is in a clique. By definition, however, this is sufficient to conclude that the subversion strategy is effective. The size of the effective subversion strategy will give an upper bound on the neighbor-connectivity of  $\Gamma$ .

To analyze  $\text{Cay}(G, S)$  and its auxiliary graph we select a specific element  $s_*$  of  $S$  and construct the quotient group  $G/\langle s_* \rangle$ , where  $\langle s_* \rangle$  represents the subgroup generated by  $s_*$ . Before beginning our analysis of how the structure of  $\text{AUX}_\Gamma$  can be used to bound  $\text{NC}(\Gamma)$ , we need some technical apparatus to describe the quotient group in terms of elements of  $S$ . In quotient group  $G/\langle s_* \rangle$ , if a coset  $t + \langle s_* \rangle \subseteq S$ , then  $t + \langle s_* \rangle$  is called an *all-generator coset*. If  $t + \langle s_* \rangle \subseteq G \setminus S$ , then  $t + \langle s_* \rangle$  is called an *all-vertex coset*. Note that with these definitions  $\langle s_* \rangle$  is neither an all-vertex nor all-generator coset.

**Lemma 1.** *Let  $\Gamma = \text{Cay}(G, S)$  and let  $s_*$  be a vertex in  $\text{AUX}$ . For any  $v \in \langle s_* \rangle$ .*

- (a)  *$N(v)$  contains all elements of every all-generator coset, and*
- (b)  *$N(v)$  contains no element of any all-vertex coset.*

**Proof.** Let  $v \in \langle s_* \rangle$ , and let  $x$  be an element in an all-generator coset. Then  $x = v + x - v$ . Since  $x$  is in an all-generator coset,  $x - v \in S$ . Thus  $x \in N(v)$ . Let  $y$  be in an all-vertex coset. If  $y \in N(v)$ , then  $y = v + s$ , for some  $s \in S$ . Since  $v \in \langle s_* \rangle$ ,  $s = y - v \in y + \langle s_* \rangle$ , contradicting the hypothesis that  $y + \langle s_* \rangle$  is an all-vertex coset. ■

We can use this lemma immediately to determine neighbor-connectivity of  $\Gamma = \text{Cay}(G, S)$  when  $\text{AUX}_\Gamma$  has an isolated vertex with a loop.

**Lemma 2.** *Let  $\Gamma = \text{Cay}(G, S)$ . If  $\Gamma$  has an isolated vertex,  $s_*$ , with a loop in  $\text{AUX}$ , then*

- (a)  *$\text{NC}(\Gamma) = 1$ , or*
- (b) *every coset of  $G \setminus S$  except  $\langle s_* \rangle$  is an all-generator coset,  $\text{Cay}(\langle s_* \rangle, \langle s_* \rangle \cap S)$  is a cycle with at least six vertices, and  $\text{NC}(\Gamma) = 2$ .*

**Proof.** Let  $\Gamma = \text{Cay}(G, S)$  and let  $s_*$  be an isolated vertex with a loop in  $\text{AUX}$ . Since  $s_*$  is isolated in  $\text{AUX}$ , for each  $t \in S$ ,  $t + s_* \in S_0$ . In other words, for each  $t \in S$ ,  $t + \langle s_* \rangle \subseteq S_0$ . Thus every coset of  $G \setminus S$  except  $\langle s_* \rangle$  is either all-generator or all-vertex. Since  $2s_* \notin S_0$  and  $s_*$  has degree two in  $\text{AUX}$ ,

no element of  $\langle s_* \rangle \setminus \{\pm s_*\}$  is in  $S$ . Thus  $\text{Cay}(\langle s_* \rangle, \langle s_* \rangle \cap S)$  is a cycle. There are two cases to consider. If there is at least one all-vertex coset in  $G/S$ , then let  $X = \{2s_*\}$ . By Lemma 1(a),  $N[X]$  contains  $S \setminus \langle s_* \rangle = S \setminus \{\pm s_*\}$ . Since  $0 \notin N[X]$ ,  $\langle s_* \rangle$  contains a non-empty component of  $\Gamma \setminus N[X]$ . Thus  $X$  is an effective subversion strategy with 0 in one component of  $\Gamma \setminus N[X]$  and, by Lemma 1(b), vertices of an all-vertex coset in another. If there are no all-vertex cosets in  $G/S$ , then by Lemma 1(a),  $\Gamma \setminus N[X]$  is a subgraph of the cycle  $\text{Cay}(\langle s_* \rangle, \langle s_* \rangle \cap S)$ . If the cycle has no more than five vertices, then  $\Gamma \setminus N[X]$  is a complete graph, and so  $\text{NC}(\Gamma) = 1$ . If the cycle has at least six vertices, then  $Y = \{\pm 2s_*\}$  is an effective subversion strategy for the cycle. Since  $\Gamma \setminus N[Y]$  is contained in the cycle,  $Y$  is also an effective subversion strategy for  $\Gamma$ . Thus  $\text{NC}(\Gamma) = 2$ . ■

When  $\text{AUX}_\Gamma$  has an isolated vertex that does not have a loop, i.e., when  $\text{AUX}_\Gamma$  has a vertex of degree zero,  $\Gamma = \text{Cay}(G, S)$  has a well-defined structure.

**Example 2.** In the Cayley graph with vertices in  $Z_{45}$  and with generating set  $S = \{\pm 1, \pm 7, \pm 8, \pm 14, \pm 15, \pm 16, \pm 22\}$ ,  $\deg_{\text{AUX}} 15 = 0$ . In other words,  $15 + S_0 \subseteq S_0$ . In the quotient group  $Z_{45}/\langle 15 \rangle$ , the cosets  $1 + \langle 15 \rangle, 7 + \langle 15 \rangle, 8 + \langle 15 \rangle, 14 + \langle 15 \rangle$  are all-generator cosets and the other non-identity cosets are all-vertex cosets. The coset  $\langle 15 \rangle \subseteq S_0$ . This structure invites one to collapse the cosets and investigate  $\text{Cay}(Z_{15}, \{\pm 1, \pm 7\})$  instead of the original graph. The discussion following this example gives the details necessary to establish that neighbor-connectivity is unchanged by such a collapse.

Note that the condition  $\deg_{\text{AUX}} s = 0$  is equivalent to  $s + S_0 \subseteq S_0$ . If  $H$  is a subgroup of abelian group  $G$ , then subset  $Y \subseteq G$  is *H-periodic* (or simply periodic) whenever  $H + Y \subseteq Y$ . This condition is equivalent to  $Y$  being a union of cosets of  $H$ . Whenever we have a subgroup  $H$  of  $G$  in  $\text{Cay}(G, S)$  for which  $H + S_0 \subseteq S_0$  and  $H \subseteq S_0$ , the periodicity of  $S_0$  is well-suited to factoring. Since  $H \subseteq S_0$ , the vertices of each coset induce a clique. Whenever a vertex in  $v + H$  is adjacent to a vertex of  $u + H$ , we have  $(v - u) + H \subseteq S_0$  because of the  $H$ -periodicity of  $S_0$ . Consequently each vertex of  $v + H$  is adjacent to each vertex of  $u + H$ . So when  $S_0$  is  $H$ -periodic and  $H \subseteq S_0$ ,  $\text{Cay}(G, S)$  is a graph whose vertex set is partitioned (by  $H$ -cosets) into cliques that are either adjacent to each other or not adjacent to each other unambiguously. For  $\text{Cay}(G, S)$  with subgroup  $H$  for which  $S_0$  is  $H$ -periodic and  $H \subseteq S_0$ , we define an associated

Cayley graph. Let  $\phi : G \rightarrow G/H$  be the quotient homomorphism and let  $R = \phi S \setminus \{0\}$ . Then  $R = R^{-1}$  and  $\phi S_0 = R_0$ . The vertices of  $\text{Cay}(\phi G, R)$  correspond to  $H$ -cosets; indeed, each  $H$ -coset of  $\text{Cay}(G, S)$  is a clique that collapses to a vertex in  $\text{Cay}(\phi G, R)$ . Moreover, two  $H$ -cliques are adjacent (by complete joins) in  $\text{Cay}(G, S)$  precisely when the vertices they collapse to in  $\text{Cay}(\phi G, R)$  are adjacent. Note that the only time this factoring process fails to produce a non-periodic generating set occurs when  $\text{Cay}(G, S)$  is a clique, i.e.,  $S_0 = G$ . The neighbor-connectivity of any clique is 0, and so this case is of no interest. Thus we adopt the convention that  $S_0 \neq G$  for the rest of the paper. Now we establish useful connections between effective subversion strategies of  $\text{Cay}(G, S)$  and  $\text{Cay}(\phi G, R)$  for a specific subgroup of  $G$  containing all elements of  $S$  that have degree zero in the auxiliary graph. A complete description of the structure of  $\text{Cay}(G, S)$  can be found in [3]. The specific result we use is given in this theorem.

**Theorem [2].** *Let  $\Gamma$  be the abelian Cayley graph  $\text{Cay}(G, S)$ , and let  $H = \{s \in S_0 \mid s + S_0 \subseteq S_0\}$ . Let  $\phi : G \rightarrow G/H$  be the quotient homomorphism and let  $R = \phi S \setminus \{0\}$  with  $R_0 = \phi S_0$ .*

- (a)  *$H$  is a subgroup of  $G$ .*
- (b) *Let  $\hat{\Gamma} = \text{Cay}(\phi G, R)$  be the quotient Cayley graph defined above. Then  $\hat{\Gamma}$  has no element  $r \in R$  such that  $r + R_0 \subseteq R_0$ .*
- (c) *If  $\hat{X}$  is an effective subversion strategy for  $\hat{\Gamma} = \text{Cay}(\phi G, R)$ , then there is an effective subversion strategy  $X$  for  $\Gamma$  with  $|X| = |\hat{X}|$ . Any clique component of  $\hat{\Gamma} \setminus N[\hat{X}]$  corresponds to a clique component of  $\Gamma \setminus N[X]$ .*
- (d) *If  $\hat{\Gamma} = \text{Cay}(\phi G, R)$ , then  $\text{NC}(\Gamma) \leq \text{NC}(\hat{\Gamma})$*

**Remark 3.** In light of this result we assume for the rest of the analysis that the Cayley graph under consideration has undergone the preliminary quotient reduction of the previous theorem. By part (b) of the theorem, in the auxiliary graph of the Cayley graph being analyzed the minimum degree is at least one.

### 3. MAIN RESULT

We now consider the case when the minimum degree in the auxiliary graph of  $\Gamma = \text{Cay}(G, S)$  is at least one. Let  $s_*$  be a fixed vertex in  $\text{AUX}_\Gamma$ . For this analysis, we exploit the fact that the cyclic nature of  $\langle s_* \rangle$  induces an order within each coset. For ease in referring to sequences of elements within cosets

we use the symbol  $\langle s_* \rangle_a^b$ ,  $a \leq b$ , to represent the sequence  $as_*, (a+1)s_*, (a+2)s_*, \dots, bs_*$ . Note that  $\{ks_*\} + \langle s_* \rangle_a^b = \langle s_* \rangle_{k+a}^{k+b}$ , and  $\{ks_*\} \langle s_* \rangle_a^b = \langle s_* \rangle_{k-b}^{k-a}$ . For each generator  $t \in S$  such that  $t + s_* \notin S$  let  $n_t$  be the largest positive integer such that  $t - \langle s_* \rangle_0^{n_t-1}$  are all in  $S$ . This ordered list of generators is called a  $t$ -string, or simply a *string* if the specific element  $t$  is not important. The *boundary* of the  $t$ -string,  $t - \langle s_* \rangle_0^{n_t-1}$ , is  $\{t + s_*, t - n_t s_*\} \subseteq G \setminus S$ . The number of generators,  $n_t$ , is called the *length of the  $t$ -string*. Note that by definition,  $1 \leq n_t \leq (|\langle s_* \rangle| - 1)$  since  $t$ -strings are only defined when coset  $t + \langle s_* \rangle$  is not an all-generator coset. The important fact about strings is that each one (except the string containing  $s_*$  and the string containing  $-s_*$ ) corresponds to exactly one vertex of  $\text{AUX}_\Gamma$ ,  $t + s_*$ , that is adjacent to vertex  $s_*$  in  $\text{AUX}_\Gamma$ . Hence the total number of  $t$ -strings in  $G/\langle s_* \rangle$  is closely related to  $\deg_{\text{AUX}} s_*$ . In fact, as we show as part of the next lemma, if  $s_*$  has a loop in  $\text{AUX}$ , the number of  $t$ -strings is equal to  $\deg_{\text{AUX}} s_*$ . Several elementary and useful facts about  $t$ -strings are collected in the next lemma.

**Lemma 4.** *Let  $\Gamma$  be the abelian Cayley graph  $\text{Cay}(G, S)$ . Let  $t, s_* \in S$  and form  $G/\langle s_* \rangle$ .*

- (a) *If  $t - \langle s_* \rangle_0^{n_t-1}$  is a  $t$ -string, then  $-(t - \langle s_* \rangle_0^{n_t-1}) = -t - \langle s_* \rangle_{-n_t+1}^0$  is a string of length  $n_t$  and is the string containing  $-t$ .*
- (b) *If  $t = -t$ , then  $n_t = 1$ .*
- (c) *If  $2s_* \notin S_0$ , the total number of  $t$ -strings in  $G$  equals  $\deg_{\text{AUX}} s_*$ .*
- (d) *If the length of a  $t$ -string is at least two, then  $N[\pm 2s_*]$  contains the boundary of the string.*
- (e) *If the length of a  $t$ -string is at least four, then  $N[\pm 2s_*]$  contains all the generators in the string.*
- (f) *If the length of a  $t$ -string is three, then there is an element  $y_t \in S$  such that  $t - s_* + y_t \notin S_0$ , and  $N[\{\pm 2s_*, t - s_* + y_t\}]$  contains all the generators in the string.*
- (g) *If the length of a  $t$ -string is one and  $t \neq -s_*$ , then the element  $t + s_* \notin S_0$ . Furthermore,  $N[t + s_*]$  contains the only generator in the string. For the only other  $t$ -string of length one, namely the  $-s_*$ -string, the element  $t - s_* = -2s_* \notin S_0$  and  $-2s_* \in N[t - s_*] = N[-2s_*]$ .*

**Proof.** The proofs of parts (a) and (b) follow directly from the fact that  $-S = S$ . For part (c) observe that for each  $t$ -string except the  $-s_*$ -string and the  $s_*$ -string,  $t$  contributes exactly one to the degree of  $s_*$  in  $\text{AUX}_\Gamma$ .

Since  $2s_* \notin S$ , the loop at  $s_*$  contributes two to the degree, one for the  $-s_*$ -string and one for the  $s_*$ -string. For part (d), let  $t\langle s_* \rangle_0^{n_t-1}$  be a string with  $n_t \geq 2$ . Note that  $N[2s_*]$  contains  $t - \langle s_* \rangle_{-2}^{n_t-3}$  and so  $t + s_* \in N[2s_*]$ . Similarly  $N[-2s_*]$  contains  $t - \langle s_* \rangle_2^{n_t+1}$  and so  $t - n_t s_* \in N[-2s_*]$ . For part (e), let  $t\langle s_* \rangle_0^{n_t-1}$  be a string with  $n_t \geq 4$ . Again  $N[2s_*]$  contains  $t - \langle s_* \rangle_{-2}^{n_t-3}$  and  $N[-2s_*]$  contains  $t - \langle s_* \rangle_2^{n_t+1}$ . Since  $n_t \geq 4$  the union of these two sets contains all elements in the  $t$ -string. For part (f), a similar argument shows that  $N[\pm 2s_*]$  contains the first and last elements of the string. By Theorem (2)(b), there is an element  $y_t \in S$  such that  $t - s_* + y_t \notin S$ , and  $N[t - s_* + y_t]$  contains  $t - s_*$ , the middle element in the string. Hence  $N[\{\pm 2s_*, t - s_* + y_t\}]$  contains all three generators in the string. Part (g) follows immediately from definition of  $t$ -string. ■

**Example 3.** To illustrate some of the ideas of the previous lemma we consider the Cayley graph with vertices in  $Z_4 \times Z_6$  and with generating set  $S = \{\pm(1, 0), \pm(0, 1), \pm(1, 1), \pm(1, 2), \pm(2, 1), \pm(2, 4)\}$ . Using  $s_* = (2, 4)$  and factoring by  $\langle(2, 4)\rangle$  we have the cosets  $H = \langle(2, 4)\rangle$ ,  $(0, 1) + H$ ,  $(1, 0) + H$ ,  $(1, 1) + H$  shown in Figure 1. For clarity only the string names are labelled in the diagram. Observe that in any coset,  $N[2s_*] = N[(0, 2)]$  can be visualized by shifting the generators in the coset down by two positions. Similarly within a coset,  $N[-(0, 2)]$  can be visualized by shifting the generators up by two positions. Thus we see that every element in the  $(3, 0)$ -string is in  $N[\pm(0, 2)]$ . By contrast, no element in the shorter  $(3, 5)$ -string is in  $N[\pm(0, 2)]$ , although the string's boundary is in  $N[\pm(0, 2)]$ .

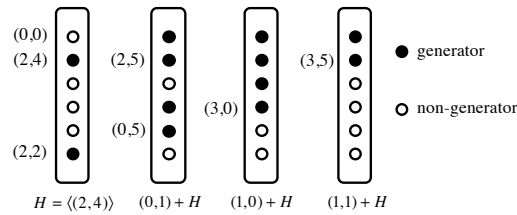


Figure 1. Visualizing Strings and Cosets.

The last three parts of the previous lemma together with Lemma 1(a) show that  $X = \{\pm 2s_*\}$  together with one non-generator for each string of length one or three effectively subverts all generators of  $\text{Cay}(G, S)$  except those



in strings of length two. To be specific, let  $\hat{X}$  be the subversion strategy constructed using the last three parts of the previous lemma. That is,  $\hat{X}$  contains  $\pm 2s_*$  and for each  $t$ -string of length one or three a single element  $y_t$  whose existence is guaranteed by parts (g) and (f) of the previous lemma. Then  $N[\hat{X}]$  contains all the generators in all strings of length different from two. By Lemma 1(a),  $N[\hat{X}]$  contains all generators in all of the all-generator cosets. Moreover,  $|\hat{X}| \leq M$  where  $M$  is the total number of strings of length different from two. So for the remainder of the analysis we describe how to add elements to  $\hat{X}$  that will effectively subvert all elements in strings of length two. We will add elements so that no more than one element is added for each such string.

First we divide the elements of strings of length two into two disjoint sets. Define  $W = \{w \mid w, w - s \text{ form a string of length two and } -w \neq w - s_*\}$ , and define  $T = \{t \mid t, t - s \text{ is a string of length two and } -t = t - s_*\}$ . By Lemma 4(a),  $z$  is in a string of length two if and only if  $-z$  is in a string of length two. Hence by Lemma 4(b),  $-z \neq z$  whenever  $z$  is in a string of length two. Thus for  $z$  in a string of length two,  $-z \neq z + s_*$  is equivalent to  $z$  and  $-z$  being in distinct strings. So  $W$  consists of all generators  $z$  in strings of length two for which  $z$  and  $-z$  are in distinct strings, while  $T$  consists of all generators  $z$  in strings of length two for which  $z$  and  $-z$  are in the same string. Finally, note that both  $W$  and  $T$  are closed under inverses. In Example 3,  $W = \{(2, 5), (0, 5)\}$  since  $(2, 1) = -(2, 5)$  is not in the  $(2, 5)$ -string and  $(0, 1) = -(0, 5)$  is not in the  $(0, 5)$ -string. In the same example we have  $T = \{(3, 5)\}$  since  $(1, 1) = -(3, 5)$  is in  $(3, 5)$ -string.

The next lemma shows how to construct an extension of  $\hat{X}$  that effectively subverts all generators in  $W$ . Note that for the rest of the analysis we have the hypotheses  $\Gamma = \text{Cay}(G, S)$ ,  $\text{AUX}_\Gamma$  has no vertices of degree zero, and  $s_*$  is a fixed vertex of  $\text{AUX}_\Gamma$  for which  $2s_* \notin S_0$ .

The next proofs repeatedly use the fact that

$$(1) \quad a, b \in N[a + b] \text{ when } a, b \in S, \text{ and } 0 \notin N[a + b] \text{ when } a + b \notin S_0.$$

**Lemma 5.** *For all  $v \in W$ , there exist  $x_1, x_2 \notin S_0$  such that  $N[\hat{X} \cup \{x_1, x_2\}]$  contains  $\pm v$  and  $\pm(v - s_*)$  and  $0 \notin N[\hat{X} \cup \{x_1, x_2\}]$ . Thus  $N[\hat{X} \cup \{x_1, x_2\}]$  contains all elements in the two strings containing  $v$  and  $-v$ .*

**Proof.** Let  $v \in W$ . Then  $\pm v, \pm(v - s_*) \in S$  and  $\pm(v - 2s_*), \pm(v + s_*) \notin S_0$ . We consider cases depending on the nature of the elements  $2v - 2s_*, 2v - s_*, 2v$ .

*Case 1.* If  $v + (v - s_*) = 2v - s_* \notin S_0$ , then by (1)  $v, v - s_* \in N[2v - s_*]$  since  $2v - s_* = v + (v - s_*)$ . Again by (1),  $0 \notin N[2v - s_*]$ . Define  $x_1 = 2v - s_*$  and  $x_2 = -(2v - s_*)$ .

*Case 2.* If  $2v - 2s_* \in S$ , then by (1),  $-v, v - s_* \in N[v - 2s_*]$  since  $(2v - 2s_*) - v = (v - s_*) - s_* = (v - 2s_*) \notin S_0$ . Again by (1),  $0 \notin N[v - 2s_*]$ . Define  $x_1 = v - 2s_*$  and  $x_2 = -(v - 2s_*)$ .

*Case 3.* If  $2v \in S$ , then by (1)  $v, -v + s_* \in N[v + s_*]$  since  $(-v + s_*) + 2v = v + s_* \notin S_0$ . Again by (1),  $0 \notin N[v + s_*]$ . Define  $x_1 = v + s_*$  and  $x_2 = -(v + s_*)$ .

*Case 4.* If none of the first three cases holds, then  $2v - 2s_* \notin S$  and  $2v - s_* \in S_0$  and  $2v \notin S$ . Note first that  $v \in W \Rightarrow -v \neq v - s_* \Rightarrow 2v - s_* \neq 0$ . Next  $v \in W \Rightarrow v \neq -v \Rightarrow 2v \neq 0$ . So  $2v - s_* \neq -s_*$ . Hence  $2v - s_*$  is a string of length one different from  $-s_*$ . Thus by construction  $2v \in \hat{X}$  and  $v \in N[\hat{X}]$ . Furthermore,  $2v - s_* \neq s_*$ , for otherwise  $v - s_* = -v + s_*$  which means  $-v, -v + s_* = v - s_*, v \in S$  contradicting the fact that  $v$  is in a string of length two. Since  $2v - s_*$  is a string of length one different from both  $\pm s_*$ ,  $-2v + s_*$  is also a string of length one (Lemma 4 (a)) different from  $\pm s_*$ . So by construction,  $-2v + 2s_* \in \hat{X}$ . Thus  $-v + s_* \in N[\hat{X}]$ . To get  $v - s_* \in N[\{x_1\}]$ , we define  $x_1 = v - 2s_*$ . To get  $-v \in N[\{x_2\}]$ , we define  $x_2 = -v - s_*$ . Since  $v - 2s_*, -v - s_* \notin S_0$ ,  $0 \notin N[\hat{X} \cup \{x_1, x_2\}]$ . ■

**Remark 6.** Note that  $|W|$  is even because for each  $v \in W$ ,  $v$  is in a string of length two, so  $v \neq -v$  and  $-v$  is in a different string of length two. In Example 3,  $W = \{(0, 5), (2, 5)\}$ . As is typical of generators in  $W$ ,  $-(0, 5) = (0, 1)$  is in the  $(2, 5)$ -string and  $-(2, 5) = (2, 1)$  is in the  $(0, 5)$ -string. Since  $(0, 1) + (2, 5) = (2, 0) \notin S_0$ , Case 1 of the previous lemma applies. Thus  $(2, 0)$  is added to the subversion strategy we are building. In this same example, we happen to have  $-(2, 0) = (2, 0)$  and so the one element added insures that all four generators in these two strings are in the closed neighborhood of the subversion strategy. In the following example, two elements must be added.

**Example 4.** For the Cayley graph  $\text{Cay}(Z_{20}, \{\pm 1, \pm 5, \pm 6, \pm 7\})$  if we factor by  $\langle 5 \rangle$ ,  $W = \{6, 19\}$  (see Figure 2). Since  $6 + 1 = 7 \in S$ ,  $6 + 1 - 5 = 2 \notin S$ , and  $2 \cdot 6 = 12 \notin S$ , we see that case 4 of the previous lemma applies and so 7 is a string of length one. Thus  $13 = -7$  is also a string of length one. Since 7 and 13 are strings of length one, we have already used Lemma 4(g)

and put 12 and 18 in the subversion strategy we are building to insure the removal of 7 and 13 from the graph. According to Case 4, 6 and 19 are removed from the graph when 12 and 18 are subverted. The remaining two generators in the 6-string and the 19-string,  $-19 = 1$  and  $-6 = 14$ , will be removed from the graph when  $1 - 5 = 16$  ( $x_1$  in the proof) and  $14 - 5 = 9$  ( $x_2$  in the proof) are subverted.

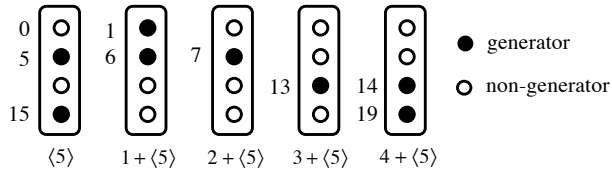


Figure 2. Visualizing Example 4.

**Lemma 7.** *Let  $t_1 \neq t_2$  be elements of  $T$ . There exist  $x_1, x_2 \notin S_0$  such that  $N[\{x_1, x_2\}]$  contains  $\pm t_1$  and  $\pm t_2$ . In other words,  $N[\{x_1, x_2\}]$  contains all elements in the two strings containing  $t_1$  and  $t_2$  and  $0 \notin N[\hat{X} \cup \{x_1, x_2\}]$ .*

**Proof.** Since  $t_1 \neq t_2$  are elements of  $T$ ,  $t_1$  and  $t_2$  are in distinct strings. Recall from definition of  $T$ ,  $-t_1 = t_1 - s_*$  and  $-t_2 = t_2 - s_*$ . Thus  $t_1 \neq \pm t_2$  and  $\pm(t_1 + s_*) \notin S_0$ . If  $t_1 + t_2 \notin S$ , then let  $x_1 = t_1 + t_2, x_2 = -(t_1 + t_2)$ . By equation (1),  $\pm t_1, \pm t_2 \in N[\{x_1, x_2\}]$  and  $0 \notin N[\{x_1, x_2\}]$ . If  $t_1 + t_2 \in S$ , then  $t_2$  is adjacent to  $t_1 + s_*$  in  $\text{Cay}(G, S)$  because  $(t_1 + s_*) - (t_1 + t_2) = -t_2 + s_* = t_2$ . Hence  $t_1, t_2 \in N[t_1 + s_*]$ . So we have  $\pm t_1, \pm t_2 \in N[\{\pm(t_1 + s_*)\}]$ , as required. ■

The previous lemma applies only when  $|T| \geq 2$  and will provide an effective way to subvert all elements of  $T$  only when  $|T|$  is even. The proof of the main theorem deals with the case in which  $|T|$  is odd. For this case we need two more technical lemmas.

**Lemma 8.** *Let  $\Gamma$  be the abelian Cayley graph  $\text{Cay}(G, S)$ ,  $s_* \in S$  be a vertex in  $\text{AUX}$  for which  $2s_* \notin S_0$ , and  $Y = \{y + s_* \mid y \neq \pm s_*\}$  is a string of length one in  $G/\langle s_* \rangle$ . Let  $t \in T$ .*

- (a) *If there is a string of length one,  $v \neq \pm s_*$ , such that  $t + v \in S$ , then  $\pm t \in N[Y \cup \{-t - s_*\}]$  and  $0 \notin N[Y \cup \{-t - s_*\}]$ .*

- (b) If there is a string of length one,  $v \neq \pm s_*$ , such that  $t + v \notin S$ , then  $t + v \notin S_0$  and  $v, \pm t \in N[(Y \setminus \{v + s_*\}) \cup \{t + v, -t - s_*\}]$  and  $0 \notin N[(Y \setminus \{v + s_*\}) \cup \{t + v, -t - s_*\}]$ .

**Proof.** Recall  $t \in T$  means  $t, -t = t - s_*$  is a string of length two in  $G/\langle s_* \rangle$ . For part (a), note that by equation (1)  $N[v + s_*]$  contains  $t = -t + s_*$  because  $v + s_* = (v + t) + (-t + s_*)$ . Since  $v + s_* \in Y$  and  $-t \in N[-t - s_*]$  and  $v + s_*, -t - s_* \notin S_0$ , the result follows. For part (b), first note that since  $v$  is a string of length one and  $t$  is in a string of length two,  $t + v \neq 0$  by Lemma 4(a). By equation 1,  $t, v \in N[t + v]$  and  $-t \in N[-t - s_*]$ . Since  $t + v, -t - s_* \notin S_0$ ,  $0 \notin N[t + v] \cup N[-t - s_*]$ , and so  $N[(Y \setminus \{v + s_*\}) \cup \{t + v, -t - s_*\}]$  has the required properties. ■

Recall in the earlier discussion we had described how to construct a subversion strategy whose closed neighborhood contains all generators in all strings of length different from two and all generators in every all-generator coset. Lemmas 5, 7, and 8 allow us to complete an extension of this subversion strategy so that the subversion strategy removes all strings of length two provided there is at least one string of length one different from  $\pm s_*$  in  $G/s_*$ . The final lemma deals with the case in which  $G/\langle s_* \rangle$  has no strings of length one except  $\pm s_*$  as happens in Example 3 with  $T = \{(3, 5)\}$ .

**Lemma 9.** Let  $\Gamma$  be the abelian Cayley graph  $\text{Cay}(G, S)$ ,  $s_* \in S$  be a vertex in  $\text{AUX}_\Gamma$  for which  $2s_* \notin S_0$ , and suppose  $G/\langle s_* \rangle$  has no strings of length one except  $\pm s_*$ -strings. Let  $t \in T$ .

- (a) If  $v, v - s_* \in S$  such that  $v + t - s_* \notin S_0$ , then  $\pm t \in N[v + t - s_*]$  and  $0 \notin N[v + t - s_*]$ .
- (b) If  $v, v - s_* \in S$  such that  $v + t - s_* = 0$ , then  $v = t$  and  $\{\pm t\} + \{v, v - s_*\} = \{\pm s_*, 0\}$ .
- (c) If  $v, v - s_*$  is a string of length two in  $G/\langle s_* \rangle$  and  $v + t - s_* \in S$ , then  $\{\pm t\} + \{v, v - s_*\}$  is a subset of (i) a string of length at least two together with the string's boundary or (ii) an all-generator coset.
- (d) If  $v, v - s_*, v - 2s_*$  is a string of length three in  $G/\langle s_* \rangle$  such that  $v + t - s_* \notin S_0$  or  $v + t - 2s_* \notin S_0$ , then  $\pm t \in N[v + t - s_*]$  or  $\pm t \in N[v + t - 2s_*]$ . Furthermore,  $0 \notin N[v + t - s_*]$  when  $v + t - s_* \notin S_0$ , and  $0 \notin N[v + t - 2s_*]$  when  $v + t - 2s_* \notin S_0$ .
- (e) If  $v, v - s_*, v - 2s_*$  is a string of length three in  $G/\langle s_* \rangle$  such that  $v + t - s_* \in S_0$  and  $v + t - 2s_* \in S_0$ , then  $\{\pm t\} + \{v, v - s_*, v - 2s_*\}$  is a subset

*of a string of length at least two together with the string's boundary or is a subset of an all-generator coset.*

**Proof.** Recall  $t \in T$  means  $t, -t = t - s_*$  is a string of length two in  $G/\langle s_* \rangle$ . For (a),  $v + (t - s_*) = (v - s_*) + t$  and since  $v, v - s_* \in S$  we have  $t, -t = t - s_* \in N[v + t - s_*]$ . By (1),  $0 \notin N[v + t - s_*]$  since  $v + t - s_* \notin S_0$ . For (b),  $v + t - s_* = 0$  implies  $v = -t + s_* = t$ . So  $v + t - s_* = 2t - s_* = 0$ . This means  $\{\pm t\} + \{v, v - s_*\} = \{2t - 2s_*, 2t - s_*, 2t\} = \{-s_*, 0, s_*\}$ . For (c) note first that  $v + t \neq 0$ , for otherwise  $v = -t = t - s_*$  and  $v - s_*, v = t - s_*, v + s_* = t$  is a string of length three contrary to hypothesis. Now  $v + t \neq 0$  implies  $v + t - s_* \neq -s_*$ . Next note that  $v + t - s_* \neq s_*$ , for otherwise  $v = -t + 2s_* = t + s_*$  contradicting the hypotheses that  $v \in S$  and  $t, t - s_*$  is a string of length two. By hypothesis there are no strings of length one except  $\pm s_*$ -strings. Thus  $v + t - s_* \in S$  and  $v + t - s_* \neq \pm s_*$  implies that  $v + t - 2s_* \in S$  or  $v + t \in S$ . Now  $\{\pm t\} + \{v, v - s_*\} = \{v + t - 2s_*, v + t - s_*, v + t\}$  and at least two consecutive elements of this set are generators. Hence  $\{\pm t\} + \{v, v - s_*\}$  is a subset of (i) a string of length at least two and its boundary or (ii) an all-generator coset of  $G/\langle s_* \rangle$ . The proof of part (d) when  $v + t - s_* \notin S_0$  is identical to the proof of part (a). When  $v + t - 2s_* \notin S_0$ , the proof is analogous. For (e), note that  $v + t - s_* \neq 0$ , for otherwise  $v = -t + s_* = t$  which is a contradiction because  $v$  and  $t$  are in strings of different lengths. Further,  $v + t - 2s_* \neq 0$ , for otherwise  $v = -t + 2s_* \notin S_0$  contradicting the fact that  $v \in S$ . Hence  $v + t - 2s_*, v + t - s_* \in S$ . So  $\{\pm t\} + \{v, v - s_*\} = \{v + t - 3s_*, v + t - 2s_*, v + t - s_*, v + t\}$  is a subset of (i) a string of length at least two and its boundary or (ii) an all-generator coset of  $G/\langle s_* \rangle$ . ■

**Main Theorem.** *Let  $\Gamma$  be the abelian Cayley graph  $\text{Cay}(G, S)$  and let  $\text{AUX}$  be the auxiliary graph of  $\Gamma$ . Let  $s_* \in S$  be a vertex of minimum degree among all vertices that have loops in  $\text{AUX}$ . There exists an effective subversion strategy  $X$  for which  $|X| \leq \deg_{\text{AUX}} s_*$ . Hence  $\text{NC}(\Gamma) \leq \deg_{\text{AUX}} s_*$ .*

**Proof.** If  $\text{AUX}$  has an isolated vertex with a loop, then by Lemma 2,  $\text{NC}(\Gamma) \leq 2$  and since the vertex has a loop we are done. So now suppose  $\text{AUX}_\Gamma$  has no isolated vertices with loops, and let  $s_*$  be a vertex of minimum degree among all vertices that have loops in  $\text{AUX}_\Gamma$ . We construct an effective subversion strategy in stages. First for each  $t$ -string of length three, by Lemma 4(f) there exists  $y_t \in S$  such that  $t - s_* + y_t \notin S$  and  $N[\{\pm s_*, t - s_* + y_t\}]$  contains all three generators in the  $t$ -string. Define  $X_1 = \{y_t \mid t$

is a string of length three}  $\cup \{\pm 2s_*\} \cup \{v + s_* \mid v \neq \pm s_* \text{ is a string of length one}\}$ . By Lemma 4(e),(f),(g) (strings of length at least 4, length 3, length 1, respectively) and Lemma 1(a) (all-generator cosets),  $N[X_1]$  contains all generators of  $\text{Cay}(G, S)$  except those in strings of length two. Now for each  $w \in W$ , by Lemma 5, there exist  $x_{1,w}, x_{2,w}$  such that  $W \subseteq N[\{\pm 2s_*\} \cup \{x_{1,w}, x_{2,w} \mid w \in W\}]$ . Define  $X_2 = X_1 \cup \{x_{1,w}, x_{2,w} \mid w \in W\}$ . Then  $N[X_2]$  contains all generators of  $\text{Cay}(G, S)$  except possibly those in  $t$ -strings where  $t \in T$ .

Now consider  $t$ -strings with  $t \in T$ . Note that  $T$  has one element for each  $t$ -string of length two. Let  $T_e$  be a maximum cardinality subset of  $T$  such that  $|T_e|$  is even and write the elements of  $T_e$  as pairs:  $(t_1, t_2), (t_3, t_4), \dots, (t_{|T_e|-1}, t_{|T_e|})$ . By Lemma 7, for each such pair  $(t_i, t_{i+1})$ , there exist  $x_{1,t_i}, x_{2,t_i} \notin S_0$  such that  $N[\{x_{1,t_i}, x_{2,t_i}\}]$  contains all four elements in the two strings containing  $t_i$ , and  $t_{i+1}$ . Define  $X_3 = X_2 \cup \bigcup_I \{x_{1,t_i}, x_{2,t_i}\}$ , where  $I = \{i \mid i \text{ is odd and } 1 \leq i \leq |T_e|\}$ . If  $T \setminus T_e = \emptyset$ , then  $N[X_3]$  contains all generators of  $\text{Cay}(G, S)$ . If  $T \setminus T_e \neq \emptyset$ , then  $|T \setminus T_e| = 1$  and  $N[X_3]$  contains all generators of  $\text{Cay}(G, S)$  except the ones in the  $t$ -string for which  $t \in T \setminus T_e$ . Moreover,  $X_3$  contains no more than one element for each string in  $G/S$  except the one for which  $t \in T \setminus T_e$ . Let  $M$  be the total number of strings in  $G/S$ . Note that  $M = \deg_{\text{AUX}} s_*$ , by Lemma 4(c). When  $T \setminus T_e = \emptyset$ , then  $X_3$  is an effective subversion strategy that isolates 0 since  $S \subseteq N[X_3]$ . Since  $|X_3| \leq M = \deg_{\text{AUX}} s_*$ ,  $X_3$  is the required strategy. When  $T \setminus T_e \neq \emptyset$ , then  $|X_3| \leq M - 1$ . Let  $t, t - s_*$  be the single  $t$ -string for which  $t \in T \setminus T_e$ . Here we will modify  $X_3$  so that the new strategy (i) has at most  $M$  elements and (ii) results in a survival subgraph that has a clique component whose vertices are in  $\{0, \pm t\}$ . We use three cases.

*Case 1.* There is a string of length at least four in  $G/S$ . Then  $|X_3| \leq M - 2$  since  $2s_*$  subverts  $s_*$  and all strings of length at least four. Define  $X = X_3 \cup \{t + s_*, t - 2s_*\}$ . Since  $t + s_*, t - 2s_* \notin S_0$  and  $S \subseteq N[X]$ ,  $X$  is an effective subversion strategy that isolates 0 and  $|X_3| \leq M$ .

*Case 2.* There is a string,  $z \neq \pm s_*$ , of length one in  $G/S$ . If  $t + z \in S$ , then by Lemma 8(a)  $X = X_3 \cup \{-t - s_*\}$  is an effective subversion strategy of the required cardinality. If  $t + z \notin S$ , then by Lemma 8(b),  $X = (X_3 \setminus \{z + s_*\}) \cup \{t + z, -t - s_*\}$  is an effective subversion strategy of the required cardinality.

*Case 3.* All strings of  $G/S$  except  $s_*$ -string and  $(-s_*)$ -string have length two or three. Recall  $N[X_3]$  contains  $S \setminus \{\pm t\}$ . Combining this with Lemma

4(d), we know  $N[X_3]$  contains all elements of  $S \setminus \{\pm t\}$  together with all boundary elements except 0 of all strings in  $G/\langle s_* \rangle$ .

- If there is an all-generator coset  $u + \langle s_* \rangle$  such that  $t + u + \langle s_* \rangle \not\subseteq S_0$ , then let  $t + u_t \in (t + u + \langle s_* \rangle) \cap (G \setminus S_0)$ . Since  $u + \langle s_* \rangle \subseteq$ , we have  $t + \langle s_* \rangle \subseteq N[t + u_t]$ . Thus  $\pm t \in N[t + u_t]$  and so  $S \subseteq N[X_3 \cup \{t + u_t\}]$ . Since  $0 \notin N[X_3 \cup \{t + u_t\}]$  we have an effective subversion strategy that isolates 0 and has the required cardinality.
- If there is a  $v$ -string of length two or three (with elements  $v, v - s_*$ , and  $v - 2s_*$ , if it has length three) such that  $v + t - s_* \notin S_0$ , then by Lemma 9(a), (d),  $\pm t \in N[v + t - s_*]$ . So  $S \subseteq N[X_3 \cup \{v + t - s_*\}]$ . Since  $0 \notin N[X_3 \cup \{v + t - s_*\}]$  we have an effective subversion strategy that isolates 0 and has the required cardinality.
- If there is a  $v$ -string of length three such that  $v + t - 2s_* \notin S_0$ , then by Lemma 9(d),  $\pm t \in N[v + t - 2s_*]$ . So  $S \subseteq N[X_3 \cup \{v + t - 2s_*\}]$ . Since  $0 \notin N[X_3 \cup \{v + t - 2s_*\}]$  we have an effective subversion strategy that isolates 0 and has the required cardinality.

Now the only remaining possibility is that  $G/\langle s_* \rangle$  has all these properties:

- (1) For every all-generator coset  $u + \langle s_* \rangle$ ,  $t + u + \langle s_* \rangle$  is an all-generator coset and so by Lemma 1(a),  $\pm t + u + \langle s_* \rangle \subseteq N[\pm 2s_*] \subseteq N[X_3]$ .
- (2) For every string of length two,  $v, v - s_*$ , with  $v \neq t$ ,  $t + v - s_* \in S_0$ . By Lemma 9(b),  $t + v - s_* \in S$ . Then by Lemma 9(c),  $\pm t + \{v, v - s_*\}$  is contained in (i) an all-generator coset or (ii) a string of length at least two and its boundary. By Lemmas 1(a) and 4(d),  $\pm t + \{v, v - s_*\} \subseteq N[X_3]$ .
- (3) For every string of length three,  $v, v - s_*, v - 2s_*$ , the element  $t + v - s_* \in S_0$  and  $t + v - 2s_* \in S_0$ . So by Lemma 9(d),  $\pm t + \{v, v - s_*, v - 2s_*\}$  is contained in (i) an all-generator coset or (ii) a string of length at least two and its boundary. Again by Lemmas 1(a) and 4(d),  $\pm t + \{v, v - s_*, v - 2s_*\} \subseteq N[X_3]$ .

Since all three properties are true, we have  $\{\pm t, 0\} + S_0 \subseteq \{\pm t, 0\} \cup \{\pm s_*, 0\} \cup N[X_3] \subseteq \{\pm t, 0\} \cup N[X_3]$ . This means  $\{\pm t, 0\}$  contains the vertices of a component of  $\Gamma \setminus N[X_3]$ . Since  $0 \notin N[X_3]$  and  $\pm t \in S$  we know this component is non-empty and is a clique. Hence  $N[X_3]$  is an effective subversion strategy with  $|X_3| \leq M$ , as required. ■

We conclude by illustrating the complete construction of an effective subversion strategy for some of the graphs described earlier.

**Example.** For  $\Gamma = \text{Cay}(Z_{45}, S)$  with  $S = \{\pm 1, \pm 7, \pm 8, \pm 14, \pm 15, \pm 16, \pm 22\}$ , we have already noted that  $\text{NC}(\Gamma) = \text{NC}(\hat{\Gamma})$  where  $\hat{\Gamma} = \text{Cay}(Z_{15}, \{\pm 1, \pm 7\})$ . In  $\text{AUX}_{\hat{\Gamma}}$ , vertex 1 has a loop and degree three. In  $Z_{15}$  there are three strings: 1;  $-1$ ; 7, 8. Since  $-8 = 7$ ,  $8 \in T$ . As in the last part of the proof of the Main Result we note that  $\pm 8 + \{\pm 8\} \subseteq \{\pm 1, 0\}$ . Since  $\{\pm 1\} \subseteq N[\pm 2]$ , we have  $\{\pm 8\} + S_0 \subseteq \{\pm 8, 0\} \cup N[\pm 2]$ . Thus  $\{\pm 2\}$  is an effective subversion strategy of  $\hat{\Gamma}$ , and so  $\text{NC}(\Gamma) \leq 2$ .

**Example.** For  $\text{Cay}(Z_{60}, S)$  with  $S = \{\pm 1, \pm 6, \pm 7, \pm 13, \pm 15, \pm 19, \pm 21, \pm 25\}$ , vertex 6 is a vertex of  $\text{AUX}$  with minimum degree among the vertices with loops. So we take  $s_* = 6$ . The quotient group  $Z_{60}/\langle 6 \rangle$  has six strings: 6;  $-6$ ; 21, 15; 45, 39; 25, 19, 13, 7, 1; and 59, 53, 47, 41, 35.  $N[\{\pm 12\}]$  contains all generators in the two strings of length one and in the two strings of length five. Since each of the 21-string and 45-string have length two and does not contain its own inverse,  $21, 45 \in W$ . We use Lemma 5 Case 1 to determine that  $v + (v - s_*) = 21 + 15 = 36$  and  $45 + 39 = 24$  should be added to the subversion strategy. Then  $\{\pm 12, 24, 36\}$  is an effective subversion strategy that isolates 0.

**Example.** For  $\text{Cay}(Z_4 \times Z_6, S)$  with  $S = \{\pm(1, 0), \pm(0, 1), \pm(1, 1), \pm(1, 2), \pm(2, 1), \pm(2, 4)\}$ , vertex  $(2, 4)$  is a vertex of  $\text{AUX}$  with minimum degree among the vertices with loops. So we take  $s_* = (2, 4)$ . Recall from Example 3 and Remark 6 we already know that  $X_3 = \{\pm(0, 2), (2, 0)\}$  is a subversion strategy whose closed neighborhood includes all the generators in the graphs except possibly those in the  $(3, 5)$ -string. To complete the construction of an effective subversion strategy, we note  $T = \{(3, 5)\}$ . In the language of the proof of the Main Theorem,  $(3, 5), (1, 1)$  is the single  $t$ -string that remains to be dealt with. Case 1 of the theorem applies since there is a string of length four in the quotient. Thus we add the boundary of the  $(3, 5)$ -string,  $(1, 3)$ ,  $(3, 3)$ , to  $X_3$  to create an effective subversion strategy  $\{\pm(0, 2), (2, 0), (1, 3), (3, 3)\}$  that isolates 0.

The usefulness of the auxiliary graph is its ability to identify easily a candidate for factoring  $G$  that will result in a quotient group with a small number of strings. This eliminates a computationally more difficult search through all subgroups of  $G$ . It is also worth noting that while the Main Result gives



an upper bound of six for neighbor-connectivity of the graph in Example 1, the actual construction described in the proof yields a lower value for the upper bound because there are long strings in the quotient group. This raises the question of whether there are ways to analyze the auxiliary graph that will allow one to detect the existence of long strings without actually factoring  $G$ .

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