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DEFECTIVE CHOOSABILITY OF GRAPHS IN SURFACES

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Abstract

It is known that if G is a graph that can be drawn without edges crossing in a surface with Euler characteristic ϵ, and k and d are positive integers such that $k \geqslant 3$ and d is sufficiently large in terms of k and ϵ, then G is $(k,d)^*$-colorable; that is, the vertices of G can be colored with k colors so that each vertex has at most d neighbors with the same color as itself. In this paper, the known lower bound on d that suffices for this is reduced, and an analogous result is proved for list colorings (choosability). Also, the recent result of Cushing and Kierstead, that every planar graph is $(4,1)^*$-choosable, is extended to $K_{3,3}$-minor-free and K_5-minor-free graphs.

Keywords: list coloring, defective coloring, minor-free graph.

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1. Introduction

Let k and d be integers with $k \geqslant 1$ and $d \geqslant 0$. The *defect* of a vertex v in a (vertex-)colored graph is the number of neighbors of v that have the same color as v, and the *defect* of a coloring is the maximum defect of the colored vertices. A graph G is $(k,d)^*$*-colorable* if it has a coloring with k colors and defect at most d. If each vertex $v \in V(G)$ is assigned a list $L(v)$ of colors, then G is $(L,d)^*$*-colorable* if it has a coloring with defect at most d in which each vertex is colored with a color from its own list. Finally, G is $(k,d)^*$*-choosable* if it is $(L,d)^*$-colorable whenever $|L(v)| \geqslant k$ for each vertex v. Clearly $(k,0)^*$-*colorable* and $(k,0)^*$-*choosable* mean the same as (properly) k-*colorable* and k-*choosable*, respectively.

Table 1. Values (k,d) for which every graph in the named class is $(k,d)^*$-colorable.

$K_{2,3}$-minor-free	Outerplanar	K_4-minor-free
$(4,0)$		
$(3,1)$	$(3,0)$	$(3,0)$
$(2,2)$	$(2,2)$	
Not $(3,0)$: K_4		
Not $(2,1)$: see right	Not $(2,1)$: $K_1+2K_{1,2}$	Not $(2,d)$:
Not $(1,d)$: see right	Not $(1,d)$: $K_{1,d+1}$	$K_1+(d+1)K_{1,d+1}$
$K_{3,3}$-minor-free	Planar	K_5-minor-free
$(5,0)$ ([10])		
$(4,1)$ ([17])	$(4,0)$ (4CT)	$(4,0)$ (4CT + WET)
$(3,2)$ ([16])	$(3,2)$ ([2], [16])	
Not $(4,0)$: K_5	Not $(3,1)$:	Not $(3,d)$: $K_1+(d+1)\cdot$
Not $(3,1)$: see right	$K_1+2[K_1+2K_{1,2}]$	$[K_1+(d+1)K_{1,d+1}]$
Not $(2,d)$: see right	Not $(2,d)$:	
	$K_1+(d+1)K_{1,d+1}$	

If a graph G is $(k,d)^*$-colorable or $(k,d)^*$-choosable, then it is clearly $(k',d')^*$-colorable or $(k',d')^*$-choosable, respectively, whenever $k' \geqslant k$ and $d' \geqslant d$. Hence, to specify the pairs (k,d) for which G has one of these properties, it suffices to list the pairs that are minimal in each coordinate. With this convention, the pairs (k,d) such that all graphs in various classes are $(k,d)^*$-colorable are listed in Table 1, which is copied from [17]; here 4CT refers to the Four-Color Theorem, and WET is Wagner's Equivalence Theorem [15]. Examples are included to show that the results cannot be improved; here $+$ denotes 'join'.

The analogous results for $(k,d)^*$-choosability are given in Table 2. This must differ from Table 1, since it is known ([14], see also [7, 9]) that not every planar graph, or, therefore, every K_5-minor-free graph, is 4-choosable; in fact, this is the only difference. The results in Table 2 have all been known for at least ten years, except for the three results involving $(4,1)^*$-choosability. Recently, in a deep and impressive paper, Cushing and Kierstead [4] have proved that all planar graphs are $(4,1)^*$-choosable. The first thing we do in this paper, in Section 2, is to give the following easy extension of this result:

Theorem 1. *All $K_{3,3}$-minor-free and all K_5-minor-free graphs are $(4,1)^*$-choosable.*

Table 2. Values (k,d) for which every graph in the named class is $(k,d)^*$-choosable.

$K_{2,3}$-minor-free	Outerplanar	K_4-minor-free
$(4,0)$		
$(3,1)$	$(3,0)$	$(3,0)$ (2-degenerate [5])
$(2,2)$	$(2,2)$ ([12], [6])	
$K_{3,3}$-minor-free	**Planar**	**K_5-minor-free**
$(5,0)$ ([17])	$(5,0)$ ([13])	$(5,0)$ ([11], [8])
$(4,1)$	$(4,1)$ ([4])	$(4,1)$
$(3,2)$ ([17])	$(3,2)$ ([12], [6])	

We then consider graphs in other surfaces. Cowen, Cowen and Woodall [2] proved that every graph that can be drawn without edges crossing in a surface S with Euler characteristic $\epsilon \leqslant 2$ is $(4,d)^*$-colorable, where $d = \max\{14, \lceil -4\epsilon/3 \rceil - 1\}$. They conjectured that there exists a d such that every such graph is $(3,d)^*$-colorable. This was proved by Archdeacon [1], with $d = \max\{15, \lceil -3\epsilon/2 \rceil - 1\}$. His argument can easily be extended to prove that (with a larger value of d) every graph in S is $(3,d)^*$-choosable, although he did not do this in [1]. (The concept of defective choosability was not invented until later.) Archdeacon's value of d was improved to $\max\{12, \lceil 6 + \sqrt{12 - 6\epsilon} \rceil\}$ by Cowen, Goddard and Jesurum [3], who also proved that every toroidal graph is $(3,2)^*$-colorable and $(5,1)^*$-colorable.

Here we will use some technical lemmas to improve and generalize the results for an arbitrary surface S. We will prove the following theorem.

Theorem 2. *Let S be a compact connected surface with Euler characteristic $\epsilon \leqslant 2$, and let G be a graph drawn in S without edges crossing.*

(a) *G is $(k,d)^*$-colorable in the following cases*:

(i) *$k=3$, $d \geqslant 9$ and $d^2-4d-(9-4\epsilon)>0$.*

(ii) *$k=4$, $d \geqslant 4$ and $3d^2-d-(8-4\epsilon)>0$.*

(iii) *$k=5$, $d \geqslant 2$ and $6d^2+5d-(5-4\epsilon)>0$.*

(iv) *$k \geqslant 6$ and $\frac{1}{2}k(k-1)d^2+(\frac{3}{2}k(k-1)-4k)d+(k^2-5k-6+6\epsilon)>0$.*

(b) *G is $(k,d)^*$-choosable in the following cases:*

(i) *$k=3$ and $d>\max\{9, 3-\frac{4}{3}\epsilon\}$ or $d=9$ and $\epsilon \geqslant -2$.*

(ii) *$k=4$ and $d>\max\{4, 1-\frac{1}{2}\epsilon\}$ or $d=4$ and $\epsilon \geqslant -4$.*

(iii) *$k=5$ and $d>\max\{2, \frac{1}{3}-\frac{4}{15}\epsilon\}$ or $d=2$ and $\epsilon \geqslant -5$.*

(iv) *$k \geqslant 6$ and $2k(k-3)d+(k^2-5k-6+6\epsilon)>0$.*

Table 3. Values of d for which every graph in a surface of characteristic ϵ is shown to be $(k,d)^*$-colorable or $(k,d)^*$-choosable (excluding special results for the plane and torus)

$\epsilon =$	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18	−19	−20
$(k,d)^*$-colorable ([1]–[3]):																							
$k=3$ ([1])	15	15	15	15	15	15	15	15	15	15	15	15	15	16	17	19	20	22	23	25	26	28	29
$k=3$ ([3])	12	12	12	12	12	12	12	13	13	14	14	15	15	15	16	16	16	17	17	17	17	18	18
$k=4$ ([2])	14	14	14	14	14	14	14	14	14	14	14	14	14	14	15	17	18	19	21	22	23	25	26
$(k,d)^*$-colorable (this paper):																							
$k=3$	9	9	9	9	9	9	9	9	9	9	9	10	10	10	10	11	11	11	11	12	12	12	12
$k=4$	4	4	4	4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	6	6	6	6	6
$k=5$	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	4	4	4	4	4
$k=6$	0	0	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3
$k=7$	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2
$(k,d)^*$-choosable (this paper):																							
$k=3$	9	9	9	9	9	10	10	10	12	13	14	16	17	18	20	21	22	24	25	26	28	29	30
$k=4$	4	4	4	4	4	4	4	5	5	5	6	6	7	7	8	8	9	9	10	10	11	11	12
$k=5$	2	2	2	2	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5	6	6	6
$k=6$	0	0	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4
$k=7$	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	3

The inequalities in Theorem 2(a) can be rephrased more explicitly by using the quadratic formula; for example, the condition in Theorem 2(a)(i) can be rewritten as $d>\max\{8, 2+\sqrt{13-4\epsilon}\}$ or, equivalently, $d \geqslant \max\{9, 2+\sqrt{14-4\epsilon}\}$. For $k \leqslant 7$ and $\epsilon \geqslant -20$, the conditions of Theorem 2 are tabulated in Table 3. The proof of Theorem 2 relies mainly on Euler's formula, whereas the proofs of the sharp results for planar and toroidal graphs do not use Euler's formula at all. Also, the method of proof used for Theorem 2 cannot work unless $d \geqslant 9$, 4 or 2 when $k = 3$, 4 or 5 respectively, and so it is not surprising that the lower bound obtained for d is far from sharp for high values of ϵ, close to $\epsilon = 2$, especially when $k = 3$. In contrast, if $k \geqslant 6$ then the result for $d = 0$, for both colorability and choosability, is simply the Heawood bound, which is well known to be sharp (except for the Klein bottle). It is not clear how sharp the result is for low (i.e., large negative) values of ϵ when $d > 0$, for any value of k.

We will prove Theorem 1 in Section 2, the technical lemmas needed for Theorem 2 in Section 3, and Theorem 2 itself in Section 4.

2. $(4,1)^*$-colorings

If U is a set of vertices of a graph G, we say that a coloring of G is *U-proper* if no vertex in U has any neighbor outside U with the same color as itself; this does not rule out the possibility that two adjacent vertices of U may have the same color as each other.

In order to prove that every planar graph is $(4,1)^*$-choosable, Cushing and Kierstead proved a stronger and more technical result ([4], Theorem 2). To state this result in full would require the introduction of terminology that is otherwise unnecessary here. However, we can state the following easy consequence of their result.

Theorem 2.1. *Let G be a 2-connected plane triangulation, let each vertex v of G be assigned a list $L(v)$ of at least four colors, let U be a set of three vertices bounding a face of G, and suppose that the vertices of U are all precolored from their lists but not all with the same color. Then this coloring*

Proof. There is no loss of generality in taking the face bounded by U to be the outside face. Label the vertices of U as b_1, b_2, b_3 in such a way that b_3 does not have the same color as either b_1 or b_2, which is possible since the colors of these three vertices are not all equal. Then the result is exactly Theorem 2(A) of [4]. ■

We will need also the following two theorems, which both follow immediately from characterizations proved by Wagner [15]. Here V_8, a *Möbius ladder*, is the graph obtained from a circuit of length eight by joining each pair of diagonally opposite vertices by a new edge.

Theorem 2.2. *If G is an edge-maximal $K_{3,3}$-minor-free graph, then either G is planar, or $G \cong K_5$, or G has a cutset consisting of two adjacent vertices.*

Theorem 2.3. *If G is an edge-maximal K_5-minor-free graph, then either G is planar, or $G \cong V_8$, or G has a cutset consisting of two or three mutually adjacent vertices.*

Now let $\mathcal{G}$ be the largest hereditary class of graphs (i.e., every subgraph of a graph in $\mathcal{G}$ is in $\mathcal{G}$) such that every edge-maximal graph in $\mathcal{G}$ either is planar, or has maximum degree at most four, or has a cutset consisting of two or three mutually adjacent vertices. It follows from Theorems 2.2 and 2.3 that $\mathcal{G}$ contains all $K_{3,3}$-minor-free and K_5-minor-free graphs. Thus the following theorem implies Theorem 1.

Theorem 2.4. *Let G be a graph in $\mathcal{G}$, let each vertex v of G be assigned a list $L(v)$ of at least four colors, let U be a set of at most three mutually adjacent vertices of G, and suppose that the vertices of U are all precolored from their lists with at most two of them having the same color. Then this coloring of U can be extended to a U-proper $(L,1)^*$-coloring of G.*

Proof. There is no loss of generality in assuming that G is an edge-maximal member of $\mathcal{G}$. Thus G either is a maximal planar graph, or has maximum degree at most four, or has a cutset of at most three mutually adjacent vertices.

Suppose first that G is a maximal planar graph (a 2-connected triangulation). There is no loss of generality in assuming that $|U| = 3$. Let C be the circuit whose vertex-set is U. If C is a face boundary then the result is just Theorem 2.1, otherwise it follows by applying Theorem 2.1 twice, extending the given coloring of U first to all vertices inside C, and then to all vertices outside C.

Suppose next that G has maximum degree at most four. In each component of $G - U$, color the vertices sequentially in such an order that the graph of the uncolored vertices remains connected throughout; let the last vertex to be colored be w. Then each vertex except w has at most three colored neighbors at the time of its coloring, and so can be colored properly from its list. The vertex w can also be colored properly unless it has four neighbors, one with each of the four different colors in $L(w)$; in this case it can be given the same color as a neighbor that is not in U. The result is the required $(L,1)^*$-coloring of G.

Suppose finally that G has a cutset X consisting of at most three mutually adjacent vertices. Let G_1 and G_2 be subgraphs of G, each with at least $|X|+1$ vertices, such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ is the subgraph induced by X. Since all vertices in U are mutually adjacent, we may assume that $U \subseteq V(G_1)$. We may assume inductively that the given coloring of U can be extended to a U-proper $(L,1)^*$-coloring of G_1, and that the resulting coloring of X can be extended to an X-proper $(L,1)^*$-coloring of G_2, and the union of these two colorings is the required U-proper $(L,1)^*$-coloring of G. ■

3. Technical Lemmas

The proof of Theorem 2(a) uses only Lemmas 3.1 and 3.2. The remaining lemmas build up to Lemma 3.7, which is needed for the proof of Theorem 2(b). Throughout this section, k and d will denote integers with $k \geqslant 1$ and $d \geqslant 0$.

Lemma 3.1. *Let G be a graph with maximum degree $\Delta(G) < k(d+1)$. Then G is $(k,d)^*$-choosable, and hence $(k,d)^*$-colorable.*

Proof. Suppose every vertex $v \in V(G)$ is assigned a list $L(v)$ of k colors. Let f be an L-coloring of G with as few bad edges as possible, where an edge is *bad* if it joins two vertices of the same color. For each vertex v, since $\deg(v) < k(d+1)$, there is at least one color $c(v) \in L(v)$ that is used on at most d neighbors of v. If v has defect greater than d in f, then changing the color of v from $f(v)$ to $c(v)$ will reduce the number of bad edges, which contradicts the choice of f. Thus f is an $(L,d)^*$-coloring, as required. ■

Lemma 3.2. *Let G be a graph that is not $(k,d)^*$-colorable.*

(a) *Let the vertices of G be $v_0, v_1, \ldots,$ listed in nonincreasing order of degree. Then $\deg(v_{i(d+1)}) \geqslant (k-i)d + k$ for each $i \in \{0,1,\ldots,k\}$.*

(b) *There is a set V of $(k-1)(d+1)+1$ vertices of G, all with degree at least $d+k$, such that $\sum_{v \in V} \deg(v) \geqslant (\frac{1}{2}k(k-1)d + k^2)(d+1)$.*

Proof. Note first that $\deg(v_0) \geqslant k(d+1)$ by Lemma 3.1, and so (a) holds if $i = 0$, and all the vertices referred to in (a) do exist in G. Suppose that (a) fails for some $i \in \{1,\ldots,k\}$. We will construct a $(k,d)^*$-coloring of G in four steps.

Step 1. Color vertices $v_0, \ldots, v_{i(d+1)-1}$ with colors $1, \ldots, i$, with $d+1$ vertices receiving each color. (Since (a) fails, each vertex that is still uncolored now has degree less than $(k-i)d+k$ in G.)

Step 2. While there remains an uncolored vertex v and a color $c \in \{1, \ldots, i\}$ such that v has no neighbor with color c, color v with c. (Now every uncolored vertex has at least one neighbor with each of colors $1, \ldots, i$. Thus there are no uncolored vertices left if $i = k$, when every vertex that is uncolored at the end of Step 1 has degree less than k in G.) If there are no uncolored vertices left, then stop; otherwise, proceed to Step 3.

Step 3. Delete all the colored vertices from G to leave a graph G' in which each vertex has degree less than $((k-i)d+k) - i = (k-i)(d+1)$.

Step 4. Construct a $(k-i, d)^*$-coloring of G' using colors $i+1, \ldots, k$, which is possible by Lemma 3.1.

The result is a $(k, d)^*$-coloring of G, and this contradiction proves (a).

We now prove (b), using (a). For $i \in \{1, \ldots, k-1\}$ let

$$V_i := \{v_j : (i-1)(d+1)+1 \leqslant j \leqslant i(d+1)\},$$

so that V_i contains $d+1$ vertices which all have degree at least $(k-i)d+k$, by (a). Let $V := V_1 \cup \ldots \cup V_{k-1} \cup \{v_0\}$, so that $|V| = (k-1)(d+1)+1$. The sum of the degrees of all vertices in V is at least

$$\begin{aligned}\frac{1}{2}k(k-1)d(d+1) &+ k(k-1)(d+1) + k(d+1) \\ &= \Big(\frac{1}{2}k(k-1)d + k^2\Big)(d+1),\end{aligned}$$

as required. ∎

We now seek an analogue of Lemma 3.2 for choosability. For the rest of this section, let G be a minimal graph that is not $(k, d)^*$-choosable, so that every proper subgraph of G is $(k, d)^*$-choosable; let L be a list-assignment to G such that $|L(v)| \geqslant k$ for each vertex v and such that G is not $(L, d)^*$-colorable; and let $C := \bigcup_{v \in V(G)} L(v)$. If $X \subseteq C$ then an *X-set* is a set T_X of $|X|(d+1)$ vertices that has an *X-coloring*, defined to be an L-coloring of T_X such that each color in X is used to color exactly $d+1$ vertices of T_X.

Lemma 3.3. (a) *For every subset $X \subseteq C$ there exists an X-set.*

(b) *If $X \subset Y \subseteq C$ and T_X, f_X are an X-set and an associated X-coloring of T_X, then there exists a Y-set T_Y and an associated Y-coloring f_Y of T_Y such that* (i) *$T_X \subset T_Y$ and* (ii) *for any vertex $v \in T_X$ such that $f_X(v) \neq f_Y(v)$, there is no vertex $w \in V(G) \setminus T_Y$ such that $f_Y(v) \in L(w)$.*

Proof. For each color $c \in C$, let $V_c := \{v \in V(G) : c \in L(v)\}$, and if $X \subseteq C$ let $V_X := \bigcup_{c \in X} V_c$. Suppose the result of (a) is not true. Then, by the 'harem' version of Hall's marriage theorem, there is a subset $X \subseteq C$ such that $|V_X| < (d+1)|X|$. Choose X minimal with this property, say $X = \{c_1, \ldots, c_r\}$. Define $p_i := d+1$ if $i \in \{1, \ldots, r-1\}$ and $p_r := |V_X \setminus V_{X \setminus \{c_r\}}| < d+1$. Then, for each subset $S \subseteq X$, $|V_S| \geqslant \sum_{c_i \in S} p_i$. It follows from the harem theorem that there is an L-coloring f of V_X in which each color c_i is used to color p_i vertices. Evidently each vertex has defect at most d. Let $G' := G - V_X$. By the choice of G, G has no $(L,d)^*$-coloring, but every proper subgraph of G is $(k,d)^*$-choosable. It follows that G' is nonempty and has an $(L,d)^*$-coloring f'. But no color in X is in the list of any vertex in $V(G')$, and so f and f' together give an $(L,d)^*$-coloring of G. This contradiction proves (a).

We now use (a) to prove (b). Let $V := V(G)$. Starting with the X-coloring f_X of T_X, as long as there is an uncolored vertex $v \in V$ that has in its list a color $c \in Y$ that has not yet been used on $d+1$ vertices, color v with c. When there is no such uncolored vertex v, if not every color in Y has been used on $d+1$ vertices, we carry out the following procedure to color one more vertex at a time. Let D be the digraph with vertex set $V(D) = V$ in which there is an arc from u to v whenever u is currently colored with a color that is in $L(v)$. (The digraph D is reconstructed afresh after each new vertex is colored.) Let c_0 be a color in Y that is currently used on fewer than $d+1$ vertices, let W be the set of all vertices of D that can be reached by directed paths from V_{c_0}, let C' comprise c_0 together with all colors that are used to color vertices in W, and let U be the set of uncolored vertices. By the definitions of W and C', $W = V_{C'}$, and so $|W| \geqslant (d+1)|C'|$ by (a). But there are fewer than $(d+1)|C'|$ colored vertices in W, since c_0 is used on fewer than $d+1$ vertices and every other color in C' is used on at most $d+1$ vertices. Thus $U \cap W \neq \emptyset$. Let $P = v_1 v_2 \ldots v_s$ be a shortest directed path in D from V_{c_0} to U, where v_i is colored with $c_i \in L(v_{i+1})$ $(i = 1, \ldots, s-1)$. Give color c_{i-1} to v_i $(i = 1, \ldots, s)$. The set of colored vertices of V thereby increases by the addition of v_s, which is colored with c_{s-1}, and each vertex

v_i $(i = 1, \ldots, s-1)$ changes color from c_i to c_{i-1}. However, there is no uncolored vertex of V that has any of $c_0, \ldots, c_{s-2}$ in its list, because if there were, then there would be a shorter path than P from V_{c_0} to U. So if the current coloring is f, and v is a vertex of T_X such that $f(v) \neq f_X(v)$, then $f(v) \notin L(w)$ for each uncolored vertex w. Continue in this way until every color in Y is used on $d+1$ vertices, let f_Y be the final coloring, and let T_Y be the set of vertices of V that are now colored. Then f_Y and T_Y have the required properties. ■

Lemma 3.4. *Let G, L and C be as defined before Lemma* 3.3. *Suppose there is an oracle that, given sets $V_{j-1} \subsetneqq V(G)$ and $C_{j-1} \subset C$ with $|C_{j-1}| < k$, will deliver a vertex $u_j \in V(G) \setminus V_{j-1}$ and a color $c_j \in L(u_j) \setminus C_{j-1}$. Then we can construct sets $C_0 \subset C_1 \subset \cdots \subset C_k$, $U_0 \subset U_1 \subset \cdots \subset U_k$ and $V_0 \subset V_1 \subset \cdots \subset V_k$ such that $C_0 = U_0 = V_0 = \emptyset$ and, for each $i \in \{1, \ldots, k\}$, the following hold: $C_i = \{c_1, \ldots, c_i\}$ and $U_i = \{u_1, \ldots, u_i\}$, where each vertex u_j and color c_j are chosen by the oracle as described above, after the sets C_{j-1}, U_{j-1} and V_{j-1} have been defined; $U_i \cup V_{i-1} \subset V_i$, V_i is a C_i-set* ($|V_i| = i(d+1)$)*, and there is a C_i-coloring f_i of V_i such that, for any vertex $v \in V_{i-1}$ such that $f_{i-1}(v) \neq f_i(v)$, there is no vertex $w \in V(G) \setminus V_i$ such that $f_i(v) \in L(w)$.*

Proof. This follows by k applications of Lemma 3.3. At each stage, given the C_{i-1}-set V_{i-1} and the C_{i-1}-coloring f_{i-1} of V_{i-1}, set $X := C_{i-1}$, $T_X := V_{i-1}$, $f_X := f_{i-1}$ and $Y := C_i = C_{i-1} \cup \{c_i\}$. The first step in the construction of V_i and f_i from V_{i-1} and f_{i-1} is to add the vertex u_i with color c_i. Thereafter, the construction of V_i and f_i proceeds as described in the proof of Lemma 3.3. ■

Lemma 3.5. *In the situation described in Lemma* 3.4, *we can choose the sets $V_1, \ldots, V_k$ so that, for each $i \in \{1, \ldots, k\}$ and each vertex $w \in V(G) \setminus V_i$, there are at least $d|L(w) \cap C_i|$ vertices in $V_i \setminus U_i$ that have degree at least as large as* $\deg(w)$.

Proof. We follow the algorithm as described in Lemmas 3.3 and 3.4, with the additional specification that at each stage we color a vertex whose degree is as large as possible; in particular, whenever we have a choice of shortest directed paths in the proof of Lemma 3.3(b), we always choose one of them that ends in an uncolored vertex whose degree is as large as possible. Note that if $w \in V(G) \setminus V_i$ and $c_j \in L(w) \cap C_i$ then no vertex in V_i can have been recolored with c_j, since by (ii) in the statement of Lemma 3.3, we never recolor a vertex with a color that is in the list of an uncolored vertex w; in particular, the only vertex in U_i that can have color c_j is u_j. (Each vertex $u_q \in U_i$ was originally colored with c_q. It may subsequently have been recolored, but not with color c_j.) Thus for each color $c_j \in L(w) \cap C_i$, at least d of the $d+1$ vertices $v \in V_i$ such that $f_i(v) = c_j$ are in $V_i \setminus U_i$; and they all have degree at least $\deg(w)$, since each such vertex v was uncolored at the point when we colored it with c_j, and if $\deg(v) < \deg(w)$ then we would have chosen to color w instead of v. This proves the result. ■

Lemma 3.6. *Let G be a minimal graph that is not $(k,d)^*$-choosable. Then there is a list $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_{k-1}$ of integers such that $a_0 = k$ and $a_i \geqslant k - i$ for each i, and a list of indices $l_0 \leqslant l_1 \leqslant \cdots \leqslant l_{k-1}$ such that $l_0 = 0$ and $l_i - l_{i-1} = (a_{i-1} - a_i)d + 1$, and a list of vertices $v_0, v_1, \ldots, v_{l_{k-1}}$ of G, such that $\deg(v_0) \geqslant k(d+1)$ and $\deg(v_j) \geqslant a_i d + k$ if $l_{i-1} < j \leqslant l_i$ $(i = 1, \ldots, k-1)$.*

Proof. We follow the procedure in Lemmas 3.3–3.5, starting with the sets $C_0 = U_0 = V_0 = \emptyset$. We must describe the behavior of the oracle. Let v_0 be any vertex with degree at least $k(d+1)$, which exists by Lemma 3.1. Initially, the oracle delivers the vertex $u_1 = v_0$ and an arbitrary color $c_1 \in L(u_1)$. Then we define $C_1 := \{c_1\}$ and $U_1 := \{u_1\}$, and, as indicated in Lemma 3.5, we define the C_1-set V_1 to consist of u_1 together with d other vertices with the largest degree possible that have color c_1 in their lists; and we define the C_1-coloring f_1 of V_1 such that $f_1(v) = c_1$ for each $v \in V_1$. This completes Step 1; the lists currently consist of the numbers $a_0 = k$ and $l_0 = 0$, and the single vertex $v_0 = u_1$.

At the end of Step i we will have constructed the C_i-set V_i and the C_i-coloring f_i of V_i, and the numbers a_{i-1} and l_{i-1}, and the current list of vertices $v_0, \ldots, v_{l_{i-1}}$ will include the vertices $u_1, \ldots, u_i$ in U_i and exactly $(k - a_{i-1})d$ vertices of $V_i \setminus U_i$. (We cannot just add all vertices of V_i to the list, because we do not know that their degrees are large enough.) The oracle then chooses u_{i+1} and c_{i+1} as follows. For each uncolored vertex w, let $L'(w)$ be formed from $L(w)$ by removing every color that has been used by f_i on a neighbor of w in V_i. Among all L'-colorings of $G - V_i$, let f' be one that minimizes the number of bad edges, as in the proof of Lemma 3.1. If there is no vertex with defect greater than d then f' and f_i together give an $(L,d)^*$-coloring of G, which is impossible. Thus the oracle can choose a vertex w with defect greater than d. If there is a color $c \in L'(w)$ such that w has at most d neighbors with color c, then changing the color of w from $f(w)$ to c will reduce the number of bad edges, a contradiction. Thus for each color $c \in L(w)$, either $c \in L'(w)$ and w has at least $d+1$ neighbors with color c in $G - V_i$, or $c \notin L'(w)$ and w has a neighbor of color c in V_i. So if $t := |L(w) \cap C_i| \leqslant |C_i| = i$, then $|L'(w)| \geqslant k - t$ and $\deg(w) \geqslant (k-t)(d+1) + t = (k-t)d + k$. The oracle returns the vertex $u_{i+1} = w$ and an arbitrary color $c_{i+1} \in L(w) \setminus C_i$.

If $k - t \geqslant a_{i-1}$ then let $a_i := a_{i-1}$, $l_i := l_{i-1} + 1$ and $v_{l_i} := w$; this works since $\deg(w) \geqslant (k-t)d + k \geqslant a_i d + k$. Otherwise, let $a_i := k - t < a_{i-1}$ and $l_i := l_{i-1} + (a_{i-1} - a_i)d + 1$. Then $a_i \geqslant k - i$ since $t \leqslant i$. By Lemma 3.5 there are at least $td = (k - a_i)d$ vertices of $V_i \setminus U_i$ that have degree at least as large as that of w, and so far we have added at most $(k - a_{i-1})d$ of these to the list. So we add a further $(a_{i-1} - a_i)d$ of them, and w itself, as vertices v_j with $l_{i-1} < j \leqslant l_i$, which all have degree at least $a_i d + k$.

If $i = k - 1$ then we stop at this point; the proof is complete. Otherwise we define $C_{i+1} := C_i \cup \{c_{i+1}\}$ and $U_{i+1} := U_i \cup \{u_{i+1}\}$, and we carry out the next stage in the algorithm described in Lemmas 3.3–3.5 to construct the C_{i+1}-set V_{i+1} and the C_{i+1}-coloring f_{i+1} of V_{i+1}. This completes Step $i + 1$. ∎

Lemma 3.7. *Let G be a minimal graph that is not $(k,d)^*$-choosable, and let d' be a real number such that $d' \leqslant d + k$ if $d \geqslant 2$ and $d' \leqslant d + k - 1$ if $d = 1$. Let n_i denote the number of vertices of degree i in G, and let*

$$g(k,d,d') :=$$
$$:= \begin{cases} k(k(d+1) - d') & \text{if } d' \leqslant d + k - 1, \\ (k-1)(k(d+1) - d') + ((k-1)d + 1)p & \text{if } d' = d + k - p,\ 0 \leqslant p \leqslant 1. \end{cases}$$

Then $\sum_{i \geqslant d+k} (i - d')n_i \geqslant g(k,d,d')$.

Proof. Note that if $d' = d + k - p$ then the second definition of $g(k,d,d')$ rearranges to
$$k(k(d+1) - d') - (k-1)d(1-p),$$
which is equal to the first definition if $p = 1$, and otherwise is smaller. We will prove that
$$\sum_{i=0}^{l_{k-1}} (\deg(v_i) - d') \geqslant g(k,d,d'), \tag{3.1}$$

where $v_0, \ldots, v_{l_{k-1}}$ is the list of vertices whose existence was proved in Lemma 3.6; note that these vertices all have degree at least $d+k$. There is no loss of generality in assuming that their degrees are all equal to the lower bounds given in Lemma 3.6, so that the sum in (3.1) is determined by the numbers $a_0, \ldots, a_{k-1}$.

The result holds, with equality if $d' \leqslant d+k-1$, if the numbers $a_0, \ldots, a_{k-1}$ are all equal, since then they are all equal to $a_0 = k$, and the list consists of exactly k vertices, each with degree exactly $k(d+1)$. So we may suppose that there are at least two different values among the numbers a_i. Let the two smallest values be s and t, where $k \geqslant s > t = a_{k-1} \geqslant 1$, and let the number of i's such that $a_i = t$ be r. Since $a_i \geqslant k-i$ for each i, it follows that $a_{k-1-t} \geqslant t+1$, and so at most t of the numbers, namely $a_{k-t}, \ldots, a_{k-1}$, can equal t; that is, $r \leqslant t$.

Suppose that we raise the r smallest numbers a_i so that they equal s. Then the degrees of the r vertices u_i of smallest degree each increase by $(s-t)d$, but also $(s-t)d$ vertices disappear from the list altogether, each with degree $td+k$. The result is that the sum in (3.1) decreases by

$$(s-t)d(td+k-d'-r) \geqslant (s-t)d(td+k-d'-t). \tag{3.2}$$

Suppose first that $d' \leqslant d+k-1$. Then the last bracket in (3.2) is at least $td-d+1-t = (t-1)(d-1) \geqslant 0$. Thus if we raise all the numbers a_i with the smallest value so that they equal the second-smallest value, then the sum in (3.1) does not increase. We can do this until all the numbers a_i are equal to k, at which point, as we have seen, (3.1) holds. It follows that (3.1) always holds, as required.

Suppose now that $d' = d+k-p$, where $0 \leqslant p < 1$ and $d \geqslant 2$. Then the last bracket in (3.2) is $(t-1)(d-1)-1+p$, which can be negative, but only if $t=1$, which implies $r=1$ since $1 \leqslant r \leqslant t$. Thus if $a_{k-1} \geqslant 2$ then the sum in (3.1) is smallest when all the numbers a_i are equal to k, and this is also the situation in which lowering a_{k-1} to 1 causes the sum to reduce by the largest amount, since then $s=k$ and by (3.2) the reduction is

$$(s-t)d(1-p) = (k-1)d(1-p).$$

So the sum is minimized by taking $a_0 = \cdots = a_{k-2} = k$ and $a_{k-1} = 1$, when the list of vertices consists of $k-1$ vertices with degree $k(d+1)$, followed by $(k-1)d+1$ vertices with degree $d+k = d'+p$, which each contribute p to the sum. Thus (3.1) holds with equality in this case. ∎

4. Graphs in Surfaces

We now prove Theorem 2. The start of the proof is the same for both parts of the theorem. By way of contradiction, let S be a surface for which either part fails, whose characteristic is as high (i.e., as close to 2) as possible, and let G be a minimal counterexample drawn in G. Let each vertex $v \in V(G)$ be assigned a list $L(v)$ of k colors, in such a way that G has no $(L,d)^*$-coloring, where in proving Theorem 2(a) we assume that the lists are all identical. Let

$$V_1 := \{v \in V(G) : k \leqslant \deg(v) \leqslant d+k-1\},$$
$$V_2 := \{v \in V(G) : \deg(v) \geqslant d+k\}.$$

(The sets V_1 and V_2 are used only when $k \leqslant 5$.)

We now list some simple properties of G and S.

P1. G is connected, since if each component of G is $(L,d)^*$-colorable, then so is G.

P2. Each region into which G divides S is an open 2-cell, since otherwise G embeds in a surface of higher characteristic, which contradicts the choice of S.

P3. Every face boundary contains at least three edges, since G is simple.

P4. G has no vertex with degree less than k, since if $\deg(v) < k$ then an $(L,d)^*$-coloring of G is easily obtained from one of $G - v$; hence $V(G) = V_1 \cup V_2$.

P5. V_1 is an independent set in G. For, suppose e is an edge joining two vertices $u, v \in V_1$. By the minimality of G, there is an $(L,d)^*$-coloring f of $G - e$. Now, u and v have degree at most $d+k-2$ in $G-e$, and so if either of them has d neighbors with the same color as itself, then there is a color in its list that is not used on any of its neighbors, and we can recolor it with that color. So we may assume that u and v both have defect at most $d-1$ in $G-e$, so that f is also an $(L,d)^*$-coloring of G. This contradiction proves P5.

Archdeacon [1] describes how to construct a pseudograph G' (allowing loops and multiple edges) that contains G, triangulates S, has the same vertex-set as G, and such that no edge in $E(G') \setminus E(G)$ has an endvertex in V_1. Then V_1 is still an independent set in G', and $\deg_{G'}(v) \leqslant d+k-1$ if $v \in V_1$ and $\deg_{G'}(v) \geqslant d+k$ if $v \in V_2$.

Let E_1 be the set of edges of G' incident with a vertex in V_1 and let E_2 be the set of edges that join vertices in V_2. Since G' is a triangulation, $3|F(G')| = 2|E(G')| = 2(|E_1| + |E_2|)$, where $F(G')$ is the set of faces. But each face is incident with an edge in E_2, because there is no triangle whose edges are all in E_1; thus $|F(G')| \leqslant 2|E_2|$, which with the previous equation gives $|E_1| \leqslant 2|E_2|$.

Let n_i denote the number of vertices of G' with degree i, and define

$$\Sigma_1 := \sum_{i=k}^{d+k-1} i n_i = \sum_{v \in V_1} \deg(v) = |E_1|$$

and

$$\Sigma_2 := \sum_{i \geqslant d+k} i n_i = \sum_{v \in V_2} \deg(v) = |E_1| + 2|E_2|.$$

It follows that $2\Sigma_1 \leqslant \Sigma_2$, i.e.,

$$2 \sum_{i=k}^{d+k-1} i n_i - \sum_{i \geqslant d+k} i n_i \leqslant 0. \tag{4.3}$$

Also, Euler's formula applied to 2-cell triangulations implies that

$$\sum_{i \geqslant k} (i-6) n_i = -6\epsilon. \tag{4.4}$$

We now consider several different cases. Since each vertex has degree at least as large in G' as in G, the results of Lemmas 3.2 and 3.7 hold for G' as well as for G.

Case (i). $k = 3$ and $d \geqslant 9$.
If G is not $(3, d)^*$-colorable then Lemma 3.2(b) gives

$$\begin{aligned}[(3d+9)(d+1)] - 12[2(d+1)+1] &\leqslant \sum_{i \geqslant d+k} (i-12) n_i \\ &\leqslant \sum_{i=3}^{d+k-1} (4i-12) n_i + \sum_{i \geqslant d+k} (i-12) n_i \leqslant -12\epsilon,\end{aligned}$$

where the last inequality is obtained by adding twice (4.4) to (4.3). Dividing by 3 and rearranging gives $d^2 - 4d - (9 - 4\epsilon) \leqslant 0$; but this contradicts the hypothesis of Theorem 2(a)(i), and this contradiction shows that G is $(3, d)^*$-colorable whenever the hypothesis of Theorem 2(a)(i) holds.

If G is not $(3, d)^*$-choosable and $d \geqslant 10$ (so that $12 \leqslant d + k - 1$) then Lemma 3.7 with $d' = 12$ gives

$$3(3(d+1) - 12) \leqslant \sum_{i \geqslant d+k} (i - 12)n_i \leqslant -12\epsilon,$$

i.e., $d \leqslant 3 - \frac{4}{3}\epsilon$, while if $d = 9$ then Lemma 3.7 with $p = 0$ gives $2(3 \cdot 10 - 12) \leqslant -12\epsilon$, i.e., $\epsilon \leqslant -3$, and both of these contradict the hypothesis of Theorem 2(b)(i).

Case (ii). $k = 4$ and $d \geqslant 4$.
If G is not $(4, d)^*$-colorable then Lemma 3.2(b) gives

$$\begin{aligned}[(6d+16)(d+1)] - 8[3(d+1)+1] &\leqslant \sum_{i \geqslant d+k} (i-8)n_i \\ &\leqslant \sum_{i=4}^{d+k-1} (2i-8)n_i + \sum_{i \geqslant d+k} (i-8)n_i \leqslant -8\epsilon,\end{aligned}$$

where the last inequality is obtained by adding four times (4.4) to (4.3) and dividing by 3. Halving and rearranging gives $3d^2 - d - (8 - 4\epsilon) \leqslant 0$, and this contradicts the hypothesis of Theorem 2(a)(ii).

If G is not $(4, d)^*$-choosable and $d \geqslant 5$ (so that $8 \leqslant d + k - 1$) then Lemma 3.7 with $d' = 8$ gives

$$16(d+1) - 32 \leqslant \sum_{i \geqslant d+k} (i - 8)n_i \leqslant -8\epsilon,$$

i.e., $d \leqslant 1 - \frac{1}{2}\epsilon$, while if $d = 4$ then Lemma 3.7 with $p = 0$ gives $12 \cdot 5 - 24 \leqslant -8\epsilon$, i.e., $2\epsilon \leqslant -9$, and both of these contradict the hypothesis of Theorem 2(b)(ii).

Case (iii). $k = 5$ and $d \geqslant 2$.
If G is not $(5, d)^*$-colorable then Lemma 3.2(b) gives

$$\begin{aligned}3[(10d+25)(d+1)] - 20[4(d+1)+1] &\leqslant \sum_{i \geqslant d+k} (3i-20)n_i \\ &\leqslant \sum_{i=5}^{d+k-1} (4i-20)n_i + \sum_{i \geqslant d+k} (3i-20)n_i \leqslant -20\epsilon,\end{aligned}$$

where the last inequality is obtained by adding ten times (4.4) to (4.3) and dividing by 3. Dividing by 5 and rearranging gives $6d^2 + 5d - (5 - 4\epsilon) \leqslant 0$, and this contradicts the hypothesis of Theorem 2(a)(iii).

If G is not $(5, d)^*$-choosable and $d \geqslant 3$ (so that $\frac{20}{3} < d + k - 1$) then Lemma 3.7 with $d' = \frac{20}{3}$ gives

$$75(d+1) - 100 \leqslant \sum_{i \geqslant d+k} (3i - 20)n_i \leqslant -20\epsilon,$$

i.e., $d \leqslant \frac{1}{3} - \frac{4}{15}\epsilon$, while if $d = 2$ then $\frac{20}{3} = d + k - \frac{1}{3}$, and Lemma 3.7 with $p = \frac{1}{3}$ gives $60 \cdot 3 - 80 + 9 \leqslant -20\epsilon$, i.e., $20\epsilon \leqslant -109$; both of these contradict the hypothesis of Theorem 2(b)(iii).

Case (iv). $k \geqslant 6$.

In this case we ignore (4.3) and use only (4.4), together with the fact that, by Lemma 3.1, G has at least $k(d+1) + 1 = kd + k + 1$ vertices, all with degree at least k (property P4). Since $i - 6 = (k - 6) + (i - k)$, it follows that

$$(4.5) \qquad (kd + k + 1)(k - 6) + \sum_{i \geqslant d+k} (i - k)n_i \leqslant \sum_{i \geqslant k} (i - 6)n_i = -6\epsilon.$$

If G is not $(k, d)^*$-colorable then (4.5) and Lemma 3.2(b) give

$$(kd + k + 1)(k - 6) + [\tfrac{1}{2}k(k-1)d + k^2)(d+1)] - k[(k-1)(d+1) + 1] \leqslant -6\epsilon,$$

i.e.,

$$\tfrac{1}{2}k(k-1)d^2 + (\tfrac{3}{2}k(k-1) - 4k)d + (k^2 - 5k - 6 + 6\epsilon) \leqslant 0,$$

and this contradicts the hypothesis of Theorem 2(a)(iv).

If G is not $(k, d)^*$-choosable then (4.5) and Lemma 3.7 with $d' = k$ (if $d \geqslant 1$), or (4.5) alone (if $d = 0$), gives

$$(kd + k + 1)(k - 6) + k^2 d \leqslant -6\epsilon,$$

i.e., $2k(k-3)d + (k^2 - 5k - 6 + 6\epsilon) \leqslant 0$, and this contradicts the hypothesis of Theorem 2(b)(iv).

In every case we have obtained a contradiction, and so the proof of Theorem 2 is complete.

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