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ADJACENT VERTEX DISTINGUISHING EDGE-COLORINGS OF PLANAR GRAPHS WITH GIRTH AT LEAST SIX

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Abstract

An adjacent vertex distinguishing edge-coloring of a graph G is a proper edge-coloring of G such that any pair of adjacent vertices are incident to distinct sets of colors. The minimum number of colors required for an adjacent vertex distinguishing edge-coloring of G is denoted by $\chi'_a(G)$. We prove that $\chi'_a(G)$ is at most the maximum degree plus 2 if G is a planar graph without isolated edges whose girth is at least 6. This gives new evidence to a conjecture proposed in [Z. Zhang, L. Liu, and J. Wang, *Adjacent strong edge coloring of graphs*, Appl. Math. Lett., **15** (2002) 623–626.]

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1. Introduction

In this paper we only consider simple graphs, i.e., graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *proper k-edge-coloring* is a mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two incident edges e and e'. Let $C_\phi(v) = \{\phi(xv) \mid xv \in E(G)\}$ denote the set of colors assigned to edges incident to the vertex v. A proper k-edge-coloring ϕ is *vertex distinguishing* if $C_\phi(u) \neq C_\phi(v)$ for any pair of distinct vertices u and v. This concept has been studied in papers such as [1, 4, 7], and [9].

We are concerned with a closely related concept in this paper. A proper k-edge-coloring ϕ of G is *adjacent vertex distinguishing*, or a *k-avd-coloring*, if $C_\phi(u) \neq C_\phi(v)$ whenever u and v are adjacent vertices. The *adjacent vertex distinguishing chromatic index*, denoted $\chi'_a(G)$, is the smallest integer k such that G has a k-avd-coloring. Adjacent vertex distinguishing colorings are variously known as adjacent strong edge coloring [12] and 1-strong edge coloring [2]. Note that an isolated edge has no avd-coloring and a k-avd-coloring can be regarded as an m-avd-coloring for any $m \geqslant k$.

The *chromatic index* $\chi'(G)$ of a graph G is the smallest integer k such that G has a proper k-edge-coloring. Evidently, $\chi'_a(G) \geqslant \chi'(G)$. Let $\Delta(G)$ denote the maximum degree of G. The well-known Vizing Theorem [11] asserts that $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G)+1$ for every graph G. In contrast, there exist infinitely many graphs G such that $\chi'_a(G) > \Delta(G) + 1$. For instance, it is proved in [12] that, if $n \not\equiv 0 \pmod 3$ and $n \neq 5$, then the cycle C_n satisfies $\chi'_a(C_n) = 4 = \Delta(C_n) + 2$. However, $\chi'_a(C_5) = 5 = \Delta(C_5) + 3$.

Zhang, Liu, and Wang [12] completely determined the adjacent vertex distinguishing chromatic indices for paths, cycles, trees, complete graphs, and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

Conjecture 1. If G is a connected graph with at least 6 vertices, then $\chi'_a(G) \leqslant \Delta(G) + 2$.

Balister, Győri, Lehel, and Schelp [3] established the following three theorems.

Theorem 2. *If G is a graph without isolated edges and $\Delta(G) = 3$, then $\chi'_a(G) \leqslant 5$.*

Theorem 3. *If G is a bipartite graph without isolated edges, then $\chi'_a(G) \leqslant \Delta(G) + 2$.*

Theorem 4. *If G is a graph without isolated edges and the chromatic number of G is k, then $\chi'_a(G) \leqslant \Delta(G) + O(\log k)$.*

The following bound proved by Hatami [10] is better than Theorem 4 for graphs with extremely large chromatic numbers.

Theorem 5. *If G is a graph without isolated edges and $\Delta(G) > 10^{20}$, then $\chi'_a(G) \leqslant \Delta(G) + 300$.*

The better bound $\chi'_a(G) \leqslant \Delta(G) + 1$ has been established for any planar bipartite graph G with $\Delta(G) \geqslant 12$ in [8] and for the multidimensional meshes and the hypercubes in [5]. Conjecture 1 has also been extended to the general case for multigraphs in [6]. In the following statement, $\mu(G)$ denotes the maximum number of parallel edges between two adjacent vertices.

Conjecture 6. For any connected multigraph G of at least 3 vertices, $G \neq C_5$, and of multiplicity $\mu(G)$, $\chi'_a(G) \leqslant \Delta(G) + \mu(G) + 1$.

In this paper, we prove Conjecture 1 for planar graphs with girth at least 6. In this case, the upper bound $\Delta(G) + 2$ is tight for infinitely many graphs, e.g., cycles of length at least six and not a multiple of 3. The assumption on girth cannot be decreased further as it can be attested by the cycle on five vertices.

2. Notation

A *plane graph* is a particular drawing of a planar graph in the Euclidean plane. For a plane graph G, we denote its set of faces by $F(G)$. The *degree* of a vertex v in G, denoted $d_G(v)$, is the number of vertices in G that are adjacent to v. Those vertices are also called the *neighbors* of v. A *k-vertex* is a vertex of degree k. A 1-vertex is also said to be a *leaf.* Let $D_G(v)$ denote the number of neighbors of v in G that are not leaves. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1 u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are all the vertices of $b(f)$ traversed once in cyclic order. Thus repeated occurrences of a vertex are allowed. The *degree* of a face is the number of edge-steps in its boundary walk. Note that each cut-edge is counted twice. Let $n_2(f)$ denote the number of occurrences of 2-vertices in $b(f)$. When $v \in V(G)$ is a k-vertex, we say that there are k faces incident to v. However, these faces are not required to be distinct when v occurs more than once on a boundary walk. The girth $g(G)$ of a graph G is the length of a shortest cycle of G. The girth is defined to equal infinity when the graph has no cycles, i.e., it is a forest. A path $x_0x_1 \cdots x_kx_{k+1}$ of length $k+1$ in G is called a *k-chain* if $d_G(x_0) \geqslant 3$, $d_G(x_{k+1}) \geqslant 3$, and $d_G(x_i) = 2$ for all $i = 1, 2, \ldots, k$.

3. Main Result

Theorem 7. *If G is a plane graph without isolated edges and with girth $g(G) \geqslant 6$, then $\chi'_a(G) \leqslant \Delta(G) + 2$.*

Proof. We note that the theorem encompasses the case for G being a forest without isolated edges since $g(G) = \infty > 6$. In this case the theorem is already known to be true [12].

Our proof proceeds by *reductio ad absurdum.* Assume that G is a counterexample to the theorem whose $|V(G)| + |E(G)|$ is the least possible. Since $\chi'_a(G) = \max\{\chi'_a(G_i)\}$ and $\Delta(G) = \max\{\Delta(G_i)\}$, both maxima being taken over all components G_i of G, we know that G is a connected plane graph such that $\infty > g(G) \geqslant 6$ and $\chi'_a(G) > \Delta(G) + 2$. Any proper subgraph H of G without isolated edges satisfies $g(H) \geqslant g(G)$, and hence $\chi'_a(H) \leqslant \Delta(H) + 2 \leqslant \Delta(G) + 2$. We observe that $\Delta(G) \geqslant 4$ by Theorem 2 and G is distinct from any star $K_{1,n}$, for otherwise $\chi'_a(K_{1,n}) = \Delta(K_{1,n})$ when $n \geqslant 2$.

We are going to analyze the structure of G with a sequence of auxiliary claims. Then we will derive a contradiction using the discharging method.

In the subsequent proofs, we routinely construct appropriate proper edge-colorings without verifying in detail that they are adjacent vertex distinguishing because that usually can be supplied in a straightforward manner.

Claim 8. No 2-vertex is adjacent to a leaf.

Proof. Assume to contrary that G contains a 2-vertex v adjacent to a leaf u. Let $w \neq u$ be the second neighbor of v. Since G is not a star, there exists a neighbor $x \neq v$ of w. Let $H = G - u$. Then H is a connected proper subgraph of G, hence there is a $(\Delta(G) + 2)$-avd-coloring ϕ of H

with the color set $C = \{1, 2, \dots, \Delta(G) + 2\}$. We color uv with a color $a \in C \setminus \{\phi(vw), \phi(wx)\}$. Since $|C| = \Delta(G) + 2 \geqslant 6$, the color a exists. The extended coloring is a $(\Delta(G) + 2)$-avd-coloring of G, contradicting the choice of G. □

Claim 9. If $d_G(v) \geqslant 3$, then $D_G(v) \geqslant 3$.

Proof. Assume to the contrary that there is a k-vertex v, $k \geqslant 3$, adjacent to $k - 2$ leaves. Since G is not a star, we may suppose that $v_1, v_2, \dots, v_k$ are the neighbors of v such that $d_G(v_1) \geqslant 2$, $d_G(v_2) \geqslant 1$, and $d_G(v_3) = d_G(v_4) = \cdots = d_G(v_k) = 1$. Let $H = G - \{v_3, v_4, \dots, v_k\}$. Then H is a connected proper subgraph of G, hence there is a $(\Delta(G)+2)$-avd-coloring ϕ of H with the color set $C = \{1, 2, \dots, \Delta(G) + 2\}$. Suppose that $\phi(vv_1) = 1$ and $\phi(vv_2) = 2$. We choose a color $a \in C_\phi(v_1) \setminus \{1, 2\}$ if there is any such a, otherwise let $a = 2$. Then we choose a color $b \in C_\phi(v_2) \setminus \{1, 2\}$ if there is any such b, otherwise let $b = 1$. Now we color the edges $vv_3, vv_4, \dots, vv_k$ with distinct colors in $C \setminus \{1, 2, a, b\}$. Since $|C \setminus \{1, 2, a, b\}| \geqslant |C| - 4 = \Delta(G) - 2 \geqslant d_G(v) - 2 = k - 2$, such a coloring is possible. Since neither a nor b appears on an edge incident to v, the extended coloring is a $(\Delta(G) + 2)$-avd-coloring of G, a contradiction. □

Claim 10. There does not exist any k-chain if $k \geqslant 3$.

Proof. Assume to the contrary that $v_0 v_1 \cdots v_{k+1}$ is a k-chain for some $k \geqslant 3$. Let $H = G - v_2$. Then H is a proper subgraph of G without isolated edges, hence H admits a $(\Delta(G) + 2)$-avd-coloring ϕ using the color set $C = \{1, 2, \dots, \Delta(G) + 2\}$.

If $k = 3$, i.e., $d_G(v_4) \geqslant 3$, we color $v_1 v_2$ with $c_1 \in C \setminus \{\phi(v_0 v_1), \phi(v_3 v_4)\}$ and $v_2 v_3$ with $c_2 \in C \setminus \{c_1, \phi(v_0 v_1), \phi(v_3 v_4)\}$. If $k \geqslant 4$, i.e., $d_G(v_4) = 2$, we color $v_2 v_3$ with $c_3 \in C \setminus \{\phi(v_0 v_1), \phi(v_3 v_4), \phi(v_4 v_5)\}$ and $v_1 v_2$ with $c_4 \in C \setminus \{c_3, \phi(v_0 v_1), \phi(v_3 v_4)\}$. Since $|C| = \Delta(G) + 2 \geqslant 6$, all the colors c_i are available. The extended coloring is a $(\Delta(G) + 2)$-avd-coloring of G, a contradiction. □

Claim 11. There exists no edge xy with $d_G(x) = 2$ and $D_G(y) = 3$.

Proof. Assume to the contrary that there is an edge xy such that $d_G(x) = 2$ and $D_G(y) = 3$. Let $z \neq y$ be the second neighbor of x. In addition to x, let y_1, y_2 be the neighbors of y having degree at least 2. Denote by

$y_3, y_4, \ldots, y_n$ all neighbors of y that are leaves if any exists. By Claim 8, $d_G(z) \geqslant 2$.

Case 1. $d_G(z) \geqslant 3$.
Let $H_1 = G - xy$. Then H_1 is a proper subgraph of G without isolated edges, hence H_1 admits a $(\Delta(G)+2)$-avd-coloring ϕ using the color set $C = \{1, 2, \ldots, \Delta(G)+2\}$. We may assume that $\phi(yy_i) = i$ for $i = 1, 2, \ldots, n$. We see that $\{n+1, n+2, n+3\} \subseteq C$ since $|C| = \Delta(G)+2 \geqslant d_G(y)+2 = n+3$. Without loss of generality, we may further assume that $\phi(xz) \notin \{n+2, n+3\}$.

If $n+2 \in C_\phi(y_1) \cap C_\phi(y_2)$, we color xy with $n+3$. If $n+2 \notin C_\phi(y_1) \cup C_\phi(y_2)$, we color xy with $n+2$. If $n+3 \in C_\phi(y_1) \cap C_\phi(y_2)$, we color xy with $n+2$. If $n+3 \notin C_\phi(y_1) \cup C_\phi(y_2)$, we color xy with $n+3$. If $\{n+2, n+3\} \subseteq C_\phi(y_1) \setminus C_\phi(y_2)$, or $\{n+2, n+3\} \subseteq C_\phi(y_2) \setminus C_\phi(y_1)$, we color xy with $n+2$.

Finally, we may suppose that $n+2 \in C_\phi(y_1) \setminus C_\phi(y_2)$ and $n+3 \in C_\phi(y_2) \setminus C_\phi(y_1)$. If $n \geqslant 3$, we color xy with $n+2$ and recolor yy_3 with $n+3$. If $n = 2$, then $D_G(y) = d_G(y) = 3$. We color xy with $a \in \{3, 6\} \setminus \{\phi(xz)\}$. The color a exists since $|C| = \Delta(G)+2 \geqslant 6$.

Case 2. $d_G(z) = 2$.
Let $u \neq x$ be the second neighbor of z. By Claim 10, $d_G(u) \geqslant 3$. Let $H_2 = G - xz$. Then H_2 is a proper subgraph of G without isolated edges, hence H_2 admits a $(\Delta(G)+2)$-avd-coloring ψ using the color set $C = \{1, 2, \ldots, \Delta(G)+2\}$. We may assume that $\psi(yy_i) = i$ for $i = 1, 2, \ldots, n$ and $\psi(xy) = n+1$. If $\psi(zu) \neq n+1$, we color xz with a color different from $n+1$ and $\psi(zu)$. Suppose that $\psi(zu) = n+1$.

If $n \geqslant 3$, we interchange colors of yy_3 and xy, and then color xz with a color different from $n+1$ and 3.

If $n = 2$, then $D_G(y) = d_G(y) = 3$. If $4 \in C_\psi(y_1) \cap C_\psi(y_2)$, we recolor xy with 5 and color xz with 4. If $4 \notin C_\psi(y_1) \cup C_\psi(y_2)$, we recolor xy with 4 and color xz with 5. If $5 \in C_\psi(y_1) \cap C_\psi(y_2)$, we recolor xy with 4 and color xz with 5. If $5 \notin C_\psi(y_1) \cup C_\psi(y_2)$, we recolor xy with 5 and color xz with 4. If $\{4, 5\} \subseteq C_\psi(y_1) \setminus C_\psi(y_2)$ or $\{4, 5\} \subseteq C_\psi(y_2) \setminus C_\psi(y_1)$, we recolor xy with 4 and color xz with 5.

Finally, we may suppose that $4 \in C_\psi(y_1) \setminus C_\psi(y_2)$ and $5 \in C_\psi(y_2) \setminus C_\psi(y_1)$. We recolor xy with 6 and color xz with 4. □

Claim 12. There does not exist a vertex v with neighbors $v_1, v_2, \ldots, v_k$, $k \geqslant 4$, such that $d_G(v_1) = d_G(v_2) = 2$, $d_G(v_3) \geqslant 2$, $d_G(v_4) \geqslant 2$, and

$d_G(v_i) = 1$ for all $i = 5, 6, \dots, k$.

Proof. Assume to the contrary that G contains such a vertex v. For $i = 1, 2$, let $u_i \neq v$ be the second neighbor of v_i. By symmetry between v_1 and v_2, it suffices to prove the following two cases.

Case 1. $d_G(u_1) = 2$.
Let $y \neq v_1$ be the second neighbor of u_1. By Claim 10, $d_G(y) \geqslant 3$. Let $H_1 = G - v_1u_1$. Then H_1 is a proper subgraph of G without isolated edges, hence H_1 admits a $(\Delta(G) + 2)$-avd-coloring ϕ using the color set $C = \{1, 2, \dots, \Delta(G)+2\}$. We may assume that $\phi(vv_i) = i$ for $i = 1, 2, \dots, k$. If $\phi(u_1y) \neq 1$, we color v_1u_1 with a color different from 1 and $\phi(u_1y)$. Suppose that $\phi(u_1y) = 1$. Note that $\{k+1, k+2\} \subseteq C$ since $|C| = \Delta(G)+2 \geqslant d_G(v) + 2 = k + 2$. We first color v_1u_1 with 2. Then we argue as follows.

If $k + 1 \in C_\phi(v_3) \cap C_\phi(v_4)$, we recolor vv_1 with $k + 2$. If $k + 1 \notin C_\phi(v_3) \cup C_\phi(v_4)$, we recolor vv_1 with $k + 1$. If $k + 2 \in C_\phi(v_3) \cap C_\phi(v_4)$, we recolor vv_1 with $k+1$. If $k+2 \notin C_\phi(v_3) \cup C_\phi(v_4)$, we recolor vv_1 with $k+2$. If $\{k + 1, k + 2\} \subseteq C_\phi(v_3) \setminus C_\phi(v_4)$ or $\{k + 1, k + 2\} \subseteq C_\phi(v_4) \setminus C_\phi(v_3)$, we recolor vv_1 with $k + 1$.

Finally, we may suppose that $k+1 \in C_\phi(v_3) \setminus C_\phi(v_4)$ and $k+2 \in C_\phi(v_4) \setminus C_\phi(v_3)$. If $d_G(u_2) \geq 3$, we recolor vv_2 with a color $a \in \{k+1, k+2\} \setminus \{\phi(v_2u_2)\}$ and vv_1 with the only remaining color in $\{k + 1, k + 2\} \setminus \{a\}$. Assume that $d_G(u_2) = 2$, and let $z \neq v_2$ be the second neighbor of u_2. If there exists $b \in \{k+1, k+2\} \setminus \{\phi(v_2u_2), \phi(u_2z)\}$, we recolor vv_2 with b and vv_1 with the only remaining color in $\{k + 1, k + 2\} \setminus \{b\}$. If $\{k + 1, k + 2\} = \{\phi(v_2u_2), \phi(u_2z)\}$, then we exchange the colors of vv_1 and vv_2 and recolor v_1u_1 with a color different from 1 and 2.

Case 2. $d_G(u_1) \geqslant 3$ *and* $d_G(u_2) \geqslant 3$.
Let $H_2 = G - vv_1$. Then H_2 is a proper subgraph of G without isolated edges, hence H_2 admits a $(\Delta(G) + 2)$-avd-coloring ψ using the color set $C = \{1, 2, \dots, \Delta(G)+2\}$. We may assume that $\psi(vv_i) = i$ for $i = 2, 3, \dots, k$. We may also assume that $\psi(v_1u_1) \notin \{k + 1, k + 2\}$, for otherwise we just exchange the color $\psi(v_1u_1)$ with color 1 everywhere. The rest of the proof goes exactly like the previous case. □

Claim 13. There does not exist a vertex v with neighbors $v_1, v_2, \dots, v_k$, $k \geqslant 5$, such that $d_G(v_1) = d_G(v_2) = d_G(v_3) = 2$, $d_G(v_4) \geqslant 2$, $d_G(v_5) \geqslant 2$, and $d_G(v_i) = 1$ for all $i = 6, 7, \dots, k$.

The proof is omitted because it is similar to that of Claim 12.

Claim 14. There does not exist a face $f = [x_1x_2 \cdots x_6]$ such that $d_G(x_i) = 2$ for all x_i except x_1 and x_4.

Proof. Assume to the contrary that G contains such a 6-face f. By Claim 10, $d_G(x_1) \geqslant 3$ and $d_G(x_4) \geqslant 3$. Let $H = G - x_2x_3$. Then H is a proper subgraph of G without isolated edges, hence H admits a $(\Delta(G) + 2)$-avd-coloring ϕ using the color set $C = \{1, 2, \ldots, \Delta(G)+2\}$. If $\phi(x_1x_2) \neq \phi(x_3x_4)$, we color x_2x_3 with a color different from $\phi(x_1x_2)$ and $\phi(x_3x_4)$. Otherwise, assume $\phi(x_1x_2) = \phi(x_3x_4) = 1$. We observe that $\phi(x_1x_6) \neq \phi(x_4x_5)$ and both colors are different from color 1. We interchange the colors of x_1x_2 and x_1x_6, and then properly color x_2x_3 and recolor x_5x_6 if necessary. □

Now we resume the proof of Theorem 7.

Let H be the graph obtained from G by removing all leaves of G. Then H is a connected subgraph of G. It follows from Claims 8 and 9 that, for every $v \in V(H)$, $d_H(v) \geqslant 2$, $d_H(v) = 2$ if $d_G(v) = 2$, and $D_H(v) = D_G(v)$. Furthermore, Claims 10 to 14 hold for H.

Using $\sum_{v \in V(H)} d_H(v) = \sum_{f \in F(H)} d_H(f) = 2|E(H)|$ and Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$, we can derive the following identity.

$$\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12. \tag{1}$$

We define a weight function w by $w(v) = 2d_H(v) - 6$ for $v \in V(H)$ and $w(f) = d_H(f) - 6$ for $f \in F(H)$. It follows from identity (1) that the sum of all weights is equal to -12. We will design appropriate discharging rules and then redistribute weights accordingly. In the redistribution process, we say that x sends 1 to y if we decrease the weight of x by 1 and increase the weight of y by 1. Once the discharging is finished, a new weight function w' is produced. The sum of all weights is kept fixed while the discharging is in progress since no weights are going to be created or destroyed. However, after the redistribution is done, we can show that the outcome $w'(x)$ is nonnegative for all $x \in V(H) \cup F(H)$. This leads to the following obvious contradiction.

$$0 \leqslant \sum_{x \in V(H) \cup F(H)} w'(x) = \sum_{x \in V(H) \cup F(H)} w(x) = -12. \tag{2}$$

There are two discharging rules.

(R1) If v is a 2-vertex incident to a face f, then f sends 1 to v for each occurrence of v in $b(f)$.

A face $f = [uvw\cdots]$ of H is called a *light face belonging to* v if $d_H(v) \geqslant 4$ and either $d_H(u) = 2$ or $d_H(w) = 2$.

(R2) If $d_H(v) \geqslant 4$, then v sends 1 to each light face belonging to v.

Now we are going to verify that $w'(v) \geqslant 0$ for any vertex v of H.

If $d_H(v) = 2$, then $w(v) = -2$ and $w'(v) = -2 + 1 + 1 = 0$ by (R1).

If $d_H(v) = 3$, then $w'(v) = w(v) = 0$.

If $d_H(v) = 4$, then $w(v) = 2$. By Claim 12, v is adjacent to at most one 2-vertex in G, hence there are at most two light faces belonging to v. Therefore, $w'(v) \geqslant 2 - 2 = 0$ by (R2).

If $d_H(v) = 5$, then $w(v) = 4$. By Claim 13, v is adjacent to at most two 2-vertices, hence there are at most four light faces belonging to v. Therefore, $w'(v) \geqslant 4 - 4 = 0$ by (R2).

If $d_H(v) \geqslant 6$, then there are at most $d_H(v)$ light faces belonging to v. Therefore, $w'(v) \geqslant 2d_H(v) - 6 - d_H(v) \geqslant 0$ by (R2).

Next we are going to verify that $w'(f) \geqslant 0$ for any face f of H.

Since $g(H) \geqslant 6$, we have $d_H(f) \geqslant 6$. A vertex $v \in b(f)$ is called f-*good* if it gives 1 to f during discharging. Let $\sigma(f)$ denote the number of f-good vertices in $b(f)$. An immediate consequence of Claim 10 is $n_2(f) \leqslant \lfloor \frac{2}{3} d(f) \rfloor$. It is also easy to show by Claims 10 and 11 that there are at least two f-good vertices if $n_2(f) \geqslant 1$, and $\sigma(f) \geqslant \lceil \frac{1}{3} n_2(f) \rceil$ when $n_2(f) \geqslant 1$.

It follows that

$$
\begin{aligned}
w'(f) &\geqslant d_H(f) - 6 - n_2(f) + \sigma(f) \\
&\geqslant d_H(f) - 6 - n_2(f) + \left\lceil \frac{1}{3} n_2(f) \right\rceil \\
&= d_H(f) - 6 - \left\lfloor \frac{2}{3} n_2(f) \right\rfloor \\
&\geqslant d_H(f) - 6 - \left\lfloor \frac{2}{3} \left\lfloor \frac{2}{3} d_H(f) \right\rfloor \right\rfloor \\
&\geqslant \left\lceil \frac{5}{9} d_H(f) \right\rceil - 6.
\end{aligned}
$$

Thus $w'(f) \geqslant 0$ if $d_H(f) \geqslant 10$.

Assume that $d_H(f) \leqslant 9$. If $n_2(f) = 0$, then trivially $w'(f) \geqslant w(f) = d_H(f) - 6 \geqslant 0$. So assume $n_2(f) \geqslant 1$.

If $n_2(f) \leqslant d_H(f) - 4$, then $w'(f) \geqslant w(f) + \sigma(f) - n_2(f) \geqslant d_H(f) - 6 + 2 - n_2(f) \geqslant 0$ by (R1) and (R2).

Suppose that $n_2(f) \geqslant d_H(f) - 3$. If $n_2(f) > d_H(f) - 3$, then the following two possibilities may occur.

(1) For some $k \geqslant 3$, $b(f)$ contains a k-chain, or

(2) $d_H(f) = 6$ and there exist two 2-chains.

However, (1) contradicts Claim 10 and (2) contradicts Claim 14. It follows that $n_2(f) = d_H(f) - 3$.

Each of the three vertices in $b(f)$ whose degree is at least 3 must have a neighbor of degree 2 in H, otherwise the three vertices form a subpath of $b(f)$, and hence there would exist a k-chain for some $k \geqslant 3$ in $b(f)$, contradicting Claim 10. By Claim 11, these three vertices have degree at least 4, and hence they are all f-good. Thus $w'(f) = w(f) + 3 - n_2(f) = d_H(f) - 6 + 3 - d_H(f) + 3 = 0$. ■

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