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SIGNED DOMINATION AND SIGNED DOMATIC NUMBERS OF DIGRAPHS

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Abstract

Let D be a finite and simple digraph with the vertex set $V(D)$, and let $f : V(D) \rightarrow \{-1, 1\}$ be a two-valued function. If $\sum_{x \in N^-[v]} f(x) \geq 1$ for each $v \in V(D)$, where $N^-[v]$ consists of v and all vertices of D from which arcs go into v, then f is a signed dominating function on D. The sum $f(V(D))$ is called the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed dominating functions f on D, is the signed domination number $\gamma_S(D)$ of D. A set $\{f_1, f_2, \ldots, f_d\}$ of signed dominating functions on D with the property that $\sum_{i=1}^{d} f_i(x) \leq 1$ for each $x \in V(D)$, is called a signed dominating family (of functions) on D. The maximum number of functions in a signed dominating family on D is the signed domatic number of D, denoted by $d_S(D)$.

In this work we show that $4 - n \leq \gamma_S(D) \leq n$ for each digraph D of order $n \geq 2$, and we characterize the digraphs attending the lower bound as well as the upper bound. Furthermore, we prove that $\gamma_S(D) + d_S(D) \leq n + 1$ for any digraph D of order n, and we characterize the digraphs D with $\gamma_S(D) + d_S(D) = n + 1$. Some of our theorems imply well-known results on the signed domination number of graphs.

Keywords: digraph, oriented graph, signed dominating function, signed domination number, signed domatic number.

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In this paper all digraphs are finite without loops or multiple arcs. A digraph without directed cycles of length 2 is an *oriented graph*. The vertex set and arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The *order* $n = n(D)$ of a digraph D is the number of its vertices. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. If A and B are two disjoint vertex sets of a digraph D such that $a \to b$ for each $a \in A$ and each $b \in B$, then we use the symbol $A \to B$. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N_D^-(v)$ and $N^+(v) = N_D^+(v)$, respectively. Furthermore, $N^-[v] = N_D^-[v] = N^-(v) \cup \{v\}$. The numbers $d_D^-(v) = d^-(v) = |N^-(v)|$ and $d_D^+(v) = d^+(v) = |N^+(v)|$ are the *indegree* and *outdegree* of v, respectively. The *minimum indegree*, *maximum indegree*, *minimum outdegree* and *maximum outdegree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. A digraph D is *strongly connected* if, for each pair of vertices u and v in D, there is a directed path from u to v in D. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v. The *complete digraph* of order n is denoted by K_n^*. If $X \subseteq V(D)$ and f is a mapping from $V(D)$ into some set of numbers, then $f(X) = \sum_{x \in X} f(x)$.

A *signed dominating function* of a digraph D is defined in [6] as a two-valued function $f : V(D) \to \{-1, 1\}$ such that $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq 1$ for each $v \in V(D)$. The sum $f(V(D))$ is called the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed dominating functions f on D, is called the *signed domination number* of D, denoted by $\gamma_S(D)$. Signed domination in digraphs has been studied in [3] and [6].

A set $\{f_1, f_2, \ldots, f_d\}$ of signed dominating functions on D with the property that $\sum_{i=1}^{d} f_i(x) \leq 1$ for each vertex $x \in V(D)$, is called a *signed dominating family* (of functions) on D. The maximum number of functions in a signed dominating family on D is the *signed domatic number* of D, denoted by $d_S(D)$. The signed domatic number of digraphs was introduced by Sheikholeslami and Volkmann [4]. We start with a simple observation.

Observation 1. *Let D be a digraph of order n. If $1 \leq n \leq 2$, then $\gamma_S(D) = n$, and if $n \geq 3$, then*

$$4 - n \leq \gamma_S(D) \leq n.$$

Proof. It is easy to see that $\gamma_S(D) = n$ when $1 \leq n \leq 2$. Assume now that $n \geq 3$. The upper bound $\gamma_S(D) \leq n$ is immediate. If f is a signed dominating function on D, then the condition $n \geq 3$ implies that there are at least two distinct vertices u and v such that $f(u) = f(v) = 1$, and thus $\gamma_S(D) \geq 2 - (n - 2) = 4 - n$. ■

Let $\mathcal{F}$ be the family of digraphs of order $n \geq 3$ such that there exist two vertices u and v such $\{u,v\} \rightarrow x$ for each $x \in V(D) \setminus \{u,v\}$, the set $V(D) \setminus \{u,v\}$ is independent, and there are at most two arcs from $V(D) \setminus \{u,v\}$ to $\{u,v\}$. If there are two arcs from $V(D) \setminus \{u,v\}$ to $\{u,v\}$, then the end-vertices of these arcs are different. In addition,

if there is no arc from $V(D) \setminus \{u,v\}$ to $\{u,v\}$, then $\{u,v\}$ is an independent set or there are one or two arcs between u and v,

if there is exactly one arc from $V(D) \setminus \{u,v\}$ to $\{u,v\}$, say $w \rightarrow u$, then $v \rightarrow u$,

if there are exactly two arcs from $V(D) \setminus \{u,v\}$ to $\{u,v\}$, say $w \rightarrow u$ and $z \rightarrow v$, where $w = z$ is admissible, then $v \rightarrow u$ as well as $u \rightarrow v$.

Theorem 2. *Let D be a digraph of order $n \geq 3$. Then $\gamma_S(D) = 4 - n$ if and only if D is a member of $\mathcal{F}$.*

Proof. If D is a member of $\mathcal{F}$, then it is a simple matter to verify that the function $f : V(D) \rightarrow \{-1, 1\}$ such that $f(u) = f(v) = 1$ and $f(x) = -1$ for $x \in V(D) \setminus \{u,v\}$ is a signed dominating function on D of weight $4 - n$. Applying Observation 1, we obtain $\gamma_S(D) = 4 - n$.

Conversely, assume that $\gamma_S(D) = 4-n$, and let f be a signed dominating function on D of weight $4 - n$. Then there exist exactly two vertices, say u and v, such that $f(u) = f(v) = 1$ and $f(x) = -1$ for $x \in V(D) \setminus \{u,v\}$. Because of $\sum_{y \in N^-[x]} f(y) \geq 1$ for each $x \in V(D) \setminus \{u,v\}$, we deduce that $\{u,v\} \rightarrow x$ for every $x \in V(D) \setminus \{u,v\}$ and that $V(D) \setminus \{u,v\}$ is an independent set. If there are at least three arcs from $V(D) \setminus \{u,v\}$ to $\{u,v\}$, then u or v, say u, has at least two in-neighbors in $V(D) \setminus \{u,v\}$, and we obtain the contradiction $\sum_{x \in N^-[u]} f(x) \leq 0$. Thus there are at most two arcs from $V(D) \setminus \{u,v\}$ to $\{u,v\}$. Now it is straightforward to verify that D is a member of $\mathcal{F}$. ■

Corollary 3 (Karami, Sheikholeslami, Khodar [3] 2009)**.** *If D is an oriented graph of order $n \geq 3$, then $\gamma_S(D) \geq 4 - n$ with equality if and only if there exist two vertices u and v such $\{u,v\} \rightarrow x$ for each $x \in V(D) \setminus \{u,v\}$, the set $V(D) \setminus \{u,v\}$ is independent, and $\{u,v\}$ is independent or there is exactly one arc between u and v.*

Corollary 4. *If D is a strongly connected digraph of order $n \geq 5$, then $\gamma_S(D) \geq 6 - n$.*

Let H be the digraph of order $n \geq 5$ with vertex set $V(D) = \{u, v, w, x_1, x_2, \ldots, x_{n-3}\}$ such that $\{u, v, w\} \to \{x_1, x_2, \ldots, x_{n-3}\}$, $x_1 \to x_2 \to \cdots \to x_{n-3} \to w$ and $w \to v \to u \to w$. Then H is strongly connected, and the function $f : V(H) \to \{-1, 1\}$ such that $f(u) = f(v) = f(w) = 1$ and $f(x_i) = -1$ for $1 \leq i \leq n-3$ is a signed dominating function on D of weight $6 - n$. Therefore the bound given in Corollary 4 is best possible.

Let Q be the digraph of order $n = 4$ with vertex set $V(D) = \{u, v, x_1, x_2\}$ such that $\{u, v\} \to \{x_1, x_2\}$, $x_1 \to u$, $x_2 \to v$, $u \to v$ and $v \to u$. Then Q is strongly connected, and the function $f : V(Q) \to \{-1, 1\}$ such that $f(u) = f(v) = 1$ and $f(x_1) = f(x_2) = -1$ is a signed dominating function on Q of weight 0. This example demonstrates that Corollary 4 does not hold for $n = 4$.

Theorem 5. *If D is a strongly connected oriented graph of order $n \geq 7$, then $\gamma_S(D) \geq 8 - n$, and this bound is sharp.*

Proof. According to Corollary 4, we have $\gamma_S(D) \geq 6 - n$. Suppose to the contrary that $\gamma_S(D) = 6 - n$, and let f be a signed dominating function on D of weight $6 - n$. Then there exist exactly three vertices, say u, v and w, such that $f(u) = f(v) = f(w) = 1$ and $f(x) = -1$ for $x \in V(D) \setminus \{u, v, w\}$. Because of $\sum_{y \in N^-[x]} f(y) \geq 1$ for each $x \in V(D) \setminus \{u, v, w\}$, each such vertex has at least two in-neighbors in $\{u, v, w\}$. Let $V(D) \setminus \{u, v, w\} = \{x_1, x_2, \ldots, x_{n-3}\}$.

First we show that $V(D) \setminus \{u, v, w\}$ is an independent set. Suppose to the contrary that there exists an arc, say x_1x_2, in $V(D) \setminus \{u, v, w\}$. Then $\{u, v, w\} \to x_2$, and since D is a strongly connected oriented graph, x_2 dominates a further vertex, say x_3, in $V(D) \setminus \{u, v, w\}$. Thus $\{u, v, w\} \to x_3$, and since D is a strongly connected oriented graph, x_3 dominates a further vertex of $V(D) \setminus \{u, v, w\}$. If we continue this process we arrive at a directed cycle C_1, say $C_1 = x_1x_2 \ldots x_kx_1$ with $k \geq 3$. This implies that $\{u, v, w\} \to V(C_1)$. Since D is an oriented graph, there is no arc from C_1 to $\{u, v, w\}$. If $k = n - 3$, then D is not strongly connected, a contradiction. Otherwise, as D is strongly connected, there exists an arc az from C_1 to $V(D) \setminus (V(C_1) \cup \{u, v, w\})$. This implies $\{u, v, w\} \to z$. As above the vertex z is contained in a cycle C_2 such that $V(C_2) \subseteq (V(D) \setminus (V(C_1) \cup \{u, v, w\}))$. But this leads to the contradiction $\sum_{x \in N^-[z]} f(x) \leq 0$, and thus $V(D) \setminus \{u, v, w\}$ is an independent set.

Since D is strongly connected, we deduce that each vertex of $V(D) \setminus \{u, v, w\}$ has an out-neighbor in $\{u, v, w\}$. The hypothesis $n \geq 7$ implies that at least one vertex in $\{u, v, w\}$, say u, has at least two in-neighbors in $V(D) \setminus \{u, v, w\}$. If u has at least three in-neighbors in $V(D) \setminus \{u, v, w\}$, then we obtain the contradiction $\sum_{x \in N^-[u]} f(x) \le 0$. If u has exactly two in-neighbors in $V(D) \setminus \{u, v, w\}$, then it follows that $\{v, w\} \to u$. If v or w, say v, has two in-neighbors in $V(D) \setminus \{u, v, w\}$, then it follows that $\{u, w\} \to v$, a contradiction to the fact that D is an oriented graph. Finally, if v and w have exactly one in-neighbor in $V(D) \setminus \{u, v, w\}$, then $w \to v$, and we obtain the contradition $u \to w$ or $v \to w$. This contradiction implies that $\gamma_S(D) \ge 8 - n$.

In order to prove that this bound is sharp, let H be the digraph of order $n \ge 7$ with vertex set $V(H) = \{u, v, w, z, x_1, x_2, \ldots, x_{n-4}\}$ such that $\{v, w, z\} \to \{x_1, x_2, \ldots, x_{n-4}\}$, $x_1 \to u \to \{x_2, x_3, \ldots, x_{n-4}\}$, $x_1 \to x_2 \to \cdots \to x_{n-4} \to x_1$ and $u \to v \to w \to z \to u$. Then H is a strongly connected oriented graph, and the function $f : V(H) \to \{-1, 1\}$ such that $f(u) = f(v) = f(w) = f(z) = 1$ and $f(x_i) = -1$ for $1 \le i \le n-4$ is a signed dominating function on H of weight $8 - n$. Therefore $\gamma_S(H) \le 8 - n$, and thus $\gamma_S(H) = 8 - n$. ■

Let Q be the digraph of order $n = 6$ with vertex set $V(Q) = \{u, v, w, x_1, x_2, x_3\}$ such that $u \to \{x_2, x_3\}$, $v \to \{x_1, x_3\}$, $w \to \{x_1, x_2\}$, $x_1 \to u$, $x_2 \to v$, $x_3 \to w$ and $u \to v \to w \to u$. Then Q is a strongly connected oriented graph, and the function $f : V(Q) \to \{-1, 1\}$ such that $f(u) = f(v) = f(w) = 1$ and $f(x_1) = f(x_2) = f(x_3) = -1$ is a signed dominating function on Q of weight 0. This example demonstrates that Theorem 5 does not hold for $n = 6$.

Theorem 6. *Let $r \ge 0$ be an integer, and let D be an oriented graph of order n such that $d^-(x) = r$ for every vertex $x \in V(D)$. Then*

$$\gamma_S(D) \ge 2r + 2 - n \text{ if } r \text{ is even}$$

and

$$\gamma_S(D) \ge 2r + 4 - n \text{ if } r \text{ is odd.}$$

Proof. Let f be an arbitrary signed dominating function on D, and let V^+ be the set of vertices with $f(x) = 1$ for $x \in V^+$ and $V^- = V(D) \setminus V^+$. Furthermore, define $|V^+| = t$.

First, let $r = 2k$ be even. Because of $\sum_{x \in N^-[u]} f(x) \ge 1$ for each vertex u, every vertex $x \in V^+$ has at most k in-neighbors in V^-. It follows that

$$2kt = \sum_{x \in V^+} d^-(x) \le kt + \frac{t(t-1)}{2}$$

and thus $t \ge 2k+1$. Since f was chosen arbitrary, this implies the desired bound $\gamma_S(D) \ge 2k+1-(n-(2k+1)) = 4k+2-n = 2r+2-n$.

Second, let $r = 2k-1$ be odd. Because of $\sum_{x \in N^-[u]} f(x) \ge 1$ for each vertex u, every vertex $x \in V^+$ has at most $k-1$ in-neighbors in V^-. It follows that

$$(2k-1)t = \sum_{x \in V^+} d^-(x) \le t(k-1) + \frac{t(t-1)}{2}$$

and thus $t \ge 2k+1$. This implies that $\gamma_S(D) \ge 2k+1-(n-(2k+1)) = 4k+2-n = 2r+4-n$, and the proof is complete. ■

Theorem 7. *If D is a digraph of order n, then*

$$\gamma_S(D) \ge \frac{\delta^+ + 2 - \Delta^+}{\delta^+ + 2 + \Delta^+} \cdot n.$$

Proof. Let f be an arbitrary signed dominating function on D, and let V^+ be the set of vertices with $f(x) = 1$ for $x \in V^+$ and $V^- = V(D) \setminus V^+$. Then

$$\begin{aligned} n &\le \sum_{x \in V(D)} f(N^-[x]) = \sum_{x \in V(D)} (d^+(x)+1)f(x) \\ &= \sum_{x \in V^+} (d^+(x)+1) - \sum_{x \in V^-} (d^+(x)+1) \\ &\le |V^+|(\Delta^+ + 1) - |V^-|(\delta^+ + 1) \\ &= |V^+|(\Delta^+ + \delta^+ + 2) - n(\delta^+ + 1). \end{aligned}$$

This implies

$$|V^+| \ge \frac{n(\delta^+ + 2)}{\delta^+ + 2 + \Delta^+},$$

and hence we obtain the desired bound as follows

$$\gamma_S(D) \ge |V^+| - |V^-| = 2|V^+| - n$$

$$\geq \frac{2n(\delta^+ + 2)}{\delta^+ + 2 + \Delta^+} - n$$

$$= \frac{\delta^+ + 2 - \Delta^+}{\delta^+ + 2 + \Delta^+} \cdot n. \qquad \blacksquare$$

Corollary 8. *If D is a digraph of order n such that $d^+(x) = k$ for all $x \in V(D)$, then*

$$\gamma_S(D) \geq \frac{n}{k+1}.$$

Corollary 9 (Karami, Sheikholeslami, Khodar [3] 2009)**.** *If D is a digraph of order n such that $d^-(x) = d^+(x) = k$ for all $x \in V(D)$, then*

$$\gamma_S(D) \geq \frac{n}{k+1}.$$

If f is a signed dominating function on D, and $d^-(v)$ is odd, then it follows that $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq 2$. Using this inequality, we obtain the next result analogously to the proof of Theorem 7.

Theorem 10. *If D is a digraph of order n such that $d^-(v)$ is odd for all $v \in V(D)$, then*

$$\gamma_S(D) \geq \frac{\delta^+ + 4 - \Delta^+}{\delta^+ + 2 + \Delta^+} \cdot n.$$

Corollary 11. *Let D be a digraph of order n such that $d^-(x) = d^+(x) = k$ for all $x \in V(D)$. If k is odd, then*

$$\gamma_S(D) \geq \frac{2n}{k+1}.$$

Theorem 12. *If D is a digraph of order n, then*

$$\gamma_S(D) \geq \frac{n + |A(D)| - n\Delta^+}{\Delta^+ + 1}.$$

Proof. Let f be an arbitrary signed dominating function on D, and let V^+ be the set of vertices with $f(x) = 1$ for $x \in V^+$ and $V^- = V(D) \setminus V^+$. Then

$$n \leq \sum_{x \in V(D)} f(N^-[x]) = \sum_{x \in V(D)} (d^+(x) + 1) f(x)$$

$$
\begin{aligned}
&= \sum_{x\in V^+}(d^+(x)+1) - \sum_{x\in V^-}(d^+(x)+1)\\
&= |V^+| - |V^-| + \sum_{x\in V^+} d^+(x) - \sum_{x\in V^-} d^+(x)\\
&= 2|V^+| - n + 2\sum_{x\in V^+} d^+(x) - \sum_{x\in V(D)} d^+(x)\\
&= 2|V^+| - n + 2\sum_{x\in V^+} d^+(x) - |A(D)|\\
&\le 2|V^+| - n + 2|V^+|\Delta^+ - |A(D)|\\
&= 2|V^+|(\Delta^+ + 1) - n - |A(D)|.
\end{aligned}
$$

This implies

$$|V^+| \ge \frac{2n + |AD)|}{2(\Delta^+ + 1)},$$

and hence we obtain the desired bound as follows

$$
\begin{aligned}
\gamma_S(D) &\ge |V^+| - |V^-| = 2|V^+| - n\\
&\ge \frac{2n + |A(D)|}{\Delta^+ + 1} - n\\
&= \frac{n + |A(D)| - n\Delta^+}{\Delta^+ + 1}.
\end{aligned}
$$

■

Theorem 12 also implies Corollary 8 immediately. In the special case that $d^-(v)$ is odd for all $v \in V(D)$, we obtain $\gamma_S(D) \ge (2n + |A(D)| - n\Delta^+)/(\Delta^+ + 1)$ instead of the bound in Theorem 12.

The *signed dominating function* of a graph G is defined in [1] as a function $f : V(G) \longrightarrow \{-1, 1\}$ such that $\sum_{x\in N_G[v]} f(x) \ge 1$ for all $v \in V(G)$. The sum $\sum_{x\in V(G)} f(x)$ is the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed dominating functions f on G is called the *signed domination number* of G, denoted by $\gamma_S(G)$.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 13. *If $D(G)$ is the associated digraph of a graph G, then* $\gamma_S(D(G)) = \gamma_S(G)$.

There are a lot of interesting applications of Observation 13, as for example the following three results.

Corollary 14 (Zhang, Xu, Li, Liu [7] 1999)**.** *If G is a graph of order n, maximum degree $\Delta(G)$ and minimum degree $\delta(G)$, then*

$$\gamma_S(G) \geq \frac{\delta(G) + 2 - \Delta(G)}{\delta(G) + 2 + \Delta(G)} \cdot n.$$

Proof. Since $\delta(G) = \delta^+(D(G))$, $\Delta(G) = \Delta^+(D(G))$ and $n = n(D(G))$, it follows from Theorem 7 and Observation 13 that

$$\gamma_S(G) = \gamma_S(D(G)) \geq \frac{\delta^+(D(G)) + 2 - \Delta^+(D(G))}{\delta^+(D(G)) + 2 + \Delta^+(D(G))} n = \frac{\delta(G) + 2 - \Delta(G)}{\delta(G) + 2 + \Delta(G)} n.$$

■

Corollary 15 (Dunbar, Hedetniemi, Henning, Slater [1] 1995)**.** *If G is a k-regular graph of order n, then $\gamma_S(G) \geq n/(k+1)$.*

Corollary 16 (Henning, Slater [2] 1996)**.** *For every k-regular graph G of order n with k odd, $\gamma_S(G) \geq 2n/(k+1)$.*

Proof. Since k is odd and $d_G(x) = d^-_{D(G)}(x) = d^+_{D(G)}(x) = k$ for all $x \in V(G)$ and $n = n(D(G))$, it follows from Corollary 11 and Observation 13 that

$$\gamma_S(G) = \gamma_S(D(G)) \geq \frac{2n(D(G))}{k+1} = \frac{2n(G)}{k+1}.$$

■

Theorem 17. *If D is a digraph of order n, then*

$$\gamma_S(D) \geq n \left(\frac{2 \left\lceil \frac{\delta^-(D)}{2} \right\rceil + 1 - \Delta^+(D)}{\Delta^+(D) + 1} \right).$$

Proof. Let f be a signed dominating function on D such that $w(f) = \gamma_S(D)$, and let V^+ be the set of vertices with $f(x) = 1$ for $x \in V^+$ and $V^- = V(D) \setminus V^+$. In addition, let s be the number of arcs from V^+ to V^-.

The condition $f(N^-[x]) \geq 1$ implies that $|E(V^+, x)| \geq |E(V^-, x)|$ for $x \in V^+$ and $|E(V^+, x)| \geq |E(V^-, x)| + 2$ for $x \in V^-$. Thus we obtain

$$\delta^-(D) \leq d^-(x) = |E(V^+, x)| + |E(V^-, x)| \leq 2|E(V^+, x)| - 2$$

and so $|E(V^+, x)| \geq \left\lceil \frac{\delta^-(D)+2}{2} \right\rceil$ for each vertex $x \in V^-$. Hence we deduce that

$$s = \sum_{x \in V^-} |E(V^+, x)| \geq \sum_{x \in V^-} \left\lceil \frac{\delta^-(D) + 2}{2} \right\rceil = |V^-| \left\lceil \frac{\delta^-(D) + 2}{2} \right\rceil . \tag{1}$$

Since $|E(V^+, x)| \geq \left\lceil \frac{\delta^-(D)}{2} \right\rceil$ for $x \in V^+$, it follows that

$$|E(D[V^+])| = \sum_{y \in V^+} |E(V^+, y)| \geq |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil .$$

This implies that

$$\begin{aligned} s &= \sum_{y \in V^+} d^+(y) - |E(D[V^+])| \\ &\leq \sum_{y \in V^+} d^+(y) - |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil \\ &\leq |V^+|\Delta^+(D) - |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil . \end{aligned} \tag{2}$$

Inequalities (1) and (2) lead to

$$|V^-| \leq \frac{|V^+|\Delta^+(D) - |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil}{\left\lceil \frac{\delta^-(D)+2}{2} \right\rceil}.$$

Since $\gamma_S(D) = |V^+| - |V^-|$ and $n = |V^+| + |V^-|$, it follows from the last inequality that

$$\gamma_S(D) \geq |V^+| - \frac{|V^+|\Delta^+(D) - |V^+| \left\lceil \frac{\delta^-(D)}{2} \right\rceil}{\left\lceil \frac{\delta^-(D)+2}{2} \right\rceil}$$

$$= \left(\frac{n+\gamma_S(D)}{2}\right) \frac{2\left\lceil \frac{\delta^-(D)}{2} \right\rceil + 1 - \Delta^+(D)}{\left\lceil \frac{\delta^-(D)}{2} \right\rceil + 1}$$

and this yields to the desired bound. ■

Note that Observation 13 and Theorem 17 also imply Corollaries 15 and 16 immediately.

Theorem 18. *For any digraph D, $\gamma_S(D) = n(D)$ if and only if every vertex has either indegree less or equal one or is an in-neighbor of a vertex of indegree one.*

Proof. Assume that every vertex has either indegree less or equal one or is an in-neighbor of a vertex of indegree one. Let f be an arbitrary signed dominating function on D. If v is vertex such that $d^-(v) \leq 1$, then the definition of the signed dominating function implies that $f(v) = 1$. If v is an in-neighbor of a vertex y such that $d^-(y) = 1$, then the condition $\sum_{x\in N^-[y]} f(x) \geq 1$ leads to $f(v) = 1$. Hence $f(v) = 1$ for each $v \in V(D)$ and we deduce that $\gamma_S(D) = n(D)$.

The necessity follows from the observation that if we have a vertex v that is neither of indegree less or equal one nor an in-neighbor of a vertex of indegree one, then we can assign the value -1 to v and the value 1 to each other vertex to produce a signed dominating function on D of weight $n(D) - 2$. ■

The following known results are useful for the proof of our last theorem.

Theorem A (Sheikholeslami, Volkmann [4])**.** *For any digraph D,*

$$\gamma_S(D) \cdot d_S(D) \leq n(D).$$

Theorem B (Sheikholeslami, Volkmann [4])**.** *For any digraph D,*

$$1 \leq d_S(D) \leq \delta^-(D) + 1.$$

Theorem C (Sheikholeslami, Volkmann [4])**.** *The signed domatic number of a digraph is an odd integer.*

Theorem D (Sheikholeslami, Volkmann [4] and Volkmann, Zelinka [5])**.** *Let K_n^* be the complete digraph of order n. Then $d_S(K_n^*) = n$ if n is odd,* *and if $n = 2p$ is even, then $d_S(K_n^*) = p$ if p is odd and $d_S(K_n^*) = p - 1$ if p is even.*

Theorem 19. *If D is a digraph of order n, then*

$$\gamma_S(D) + d_S(D) \le n + 1 \tag{3}$$

with equality if and only if n is odd and $D = K_n^*$ or every vertex of D has either indegree less or equal one or is an in-neighbor of a vertex of indegree one.

Proof. According to Theorem A, we obtain

$$\gamma_S(D) + d_S(D) \le \frac{n}{d_S(D)} + d_S(D).$$

Using the fact that $g(x) = x + n/x$ is decreasing for $1 \le x \le \sqrt{n}$ and increasing for $\sqrt{n} \le x \le n$, this inequality leads to (3) immediately.

If n is odd and $D = K_n^*$, then $\gamma_S(D) = 1$ and Theorem D implies $d_S(D) = n$, and we obtain equality in (3). If every vertex of D has either indegree less or equal one or is an in-neighbor of a vertex of indegree one, then Theorems B, C and 18 yield that $\gamma_S(D) = n$ and $d_S(D) = 1$, and so we have equality in (3) too.

Conversely, assume that D is neither complete of odd order nor that every vertex of D has either indegree less or equal one or is an in-neighbor of a vertex of indegree one. First we note that every digraph of order $1 \le n \le 3$ is complete of odd order or every vertex of D has either indegree less or equal one or is an in-neighbor of a vertex of indegree one, and hence $\gamma_S(D) + d_S(D) = n + 1$ for $n \in \{1, 2, 3\}$.

Assume now that $n \ge 4$. If D is not complete, then $\delta^-(D) \le n - 2$, and thus Theorem B leads to $d_S(D) \le n-1$. If D is complete and n is even, then Theorem D implies $d_S(D) \le n/2 \le n - 1$. Thus, in view of Theorem 18, we observe that $d_S(D) \le n - 1$ and $\gamma_S(G) \le n - 1$ if D is neither complete of odd order nor that every vertex of D has either indegree less or equal one or is an in-neighbor of a vertex of indegree one. If $d_S(D) = 1$, then we deduce that $\gamma_S(D) + d_S(D) \le 1 + n - 1 = n$. If $d_S(D) \ge 2$, then as above and since $n \ge 4$, we obtain

$$\gamma_S(D) + d_S(D) \le \frac{n}{d_S(D)} + d_S(D) \le \max\left\{\frac{n}{2} + 2, \frac{n}{n-1} + n - 1\right\} < n + 1.$$

Hence the equality $\gamma_S(D) + d_S(D) = n + 1$ is impossible in this case, and the proof of Theorem 19 is complete. ■

Note that the inequality (3) was proved in [4], however, the characterization of the digraphs D with $\gamma_S(D) + d_S(D) = n + 1$ is new.

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