

GRAPHS WITH EQUAL DOMINATION AND 2-DISTANCE DOMINATION NUMBERS

JOANNA RACZEK

Department of Applied Physics and Mathematics
Gdansk University of Technology
Narutowicza 11/12, 80-233 Gdańsk, Poland

e-mail: Joanna.Raczek@pg.gda.pl

Abstract

Let $G = (V, E)$ be a graph. The distance between two vertices u and v in a connected graph G is the length of the shortest $(u - v)$ path in G . A set $D \subseteq V(G)$ is a dominating set if every vertex of G is at distance at most 1 from an element of D . The domination number of G is the minimum cardinality of a dominating set of G . A set $D \subseteq V(G)$ is a 2-distance dominating set if every vertex of G is at distance at most 2 from an element of D . The 2-distance domination number of G is the minimum cardinality of a 2-distance dominating set of G . We characterize all trees and all unicyclic graphs with equal domination and 2-distance domination numbers.

Keywords: domination number, trees, unicyclic graphs.

2010 Mathematics Subject Classification: 05C05, 05C69.

1. DEFINITIONS

Here we consider simple undirected graphs $G = (V, E)$ with $|V| = n(G)$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $(u - v)$ path in G . If D is a set and $u \in V(G)$, then $d_G(u, D) = \min\{d_G(u, v) : v \in D\}$. The *k-neighbourhood* $N_G^k[v]$ of a vertex $v \in V(G)$ is the set of all vertices at distance at most k from v . For a set $D \subseteq V$, the *k-neighbourhood* $N_G^k[D]$ is defined to be $\bigcup_{v \in D} N_G^k[v]$. A

subset D of V is k -distance dominating in G if every vertex of $V(G) - D$ is at distance at most k from at least one vertex of D . Let $\gamma^k(G)$ be the minimum cardinality of a k -distance dominating set of G . This kind of domination was defined by Borowiecki and Kuzak [1]. Note that the 1-distance domination number is the *domination number*, denoted $\gamma(G)$.

The degree of a vertex v is $d_G(v) = |N_G^1(v)|$ and a vertex of degree 1 is called a *leaf*. A vertex which is a neighbour of a leaf is called a *support vertex*. Denote by $S(G)$ the set of all support vertices of G . If a support vertex is adjacent to more than one leaf, then we call it a *strong support vertex*. We denote a path on n vertices by $P_n = (v_0, \dots, v_{n-1})$ and the cycle on n vertices by C_n . For example, P_2 contains two leaves and two support vertices. For any unexplained terms and symbols see [2].

In this paper we study trees and unicyclic graphs for which the domination number and the 2-distance domination number are the same.

2. GENERAL RESULTS

First we give some general results for graphs with equal domination and 2-distance domination numbers. Obviously, for any graph G if $\gamma(G) = 1$, then $\gamma^2(G) = 1$ and thus $\gamma(G) = \gamma^2(G)$. We start with a necessary condition for a graph G with $1 < \gamma(G) = \gamma^2(G)$. A set $D \subseteq V(G)$ is a *2-packing* in G if $d_G(u, v) \geq 3$ for every $u, v \in D$.

Proposition 1. *If G is a connected graph with $\gamma(G) = \gamma^2(G)$ and $\gamma(G) > 1$, then every minimum dominating set of G is a 2-packing of G .*

Proof. Suppose D is a minimum dominating set of G such that $|D| \geq 2$ and D is not a 2-packing. Then there exist $u, v \in D$ in G such that $d_G(u, v) \leq 2$. Denote by x a vertex which belongs to $N_G[u] \cap N_G[v]$ (if u and v are adjacent, then possibly $x = u$ or $x = v$) and let $D' = (D - \{u, v\}) \cup \{x\}$. Then $N_G[u] \subseteq N_G^2[x]$ and $N_G[v] \subseteq N_G^2[x]$. Hence D' is a 2-distance dominating set of G of smaller cardinality than $\gamma(G)$, a contradiction. ■

The condition in Proposition 1 is not sufficient. Consider, for example the cycle C_9 . Next result gives a sufficient condition for a graph G to have equal domination and 2-distance domination numbers.

Proposition 2. *Let G be the graph obtained from a graph H and $n(H)$ copies of P_2 , where the i th vertex of H is adjacent to exactly one vertex of the i th copy of P_2 . Then $\gamma(G) = \gamma^2(G)$.*

Proof. Let G be the graph obtained from a graph H and $n(H)$ copies of P_2 , where the i th vertex of H is adjacent to exactly one vertex of the i th copy of P_2 . Denote by D a $\gamma^2(G)$ -set. Observe that the distance between any two leaves adjacent to two different support vertices in G is greater than or equal to 5. For this reason, if u and v are two leaves adjacent to two different support vertices, then u and v cannot be 2-dominated by the same element of D . This implies that $\gamma^2(G) \geq |S(G)|$. Since $\gamma^2(G) \leq \gamma(G)$, it follows that $\gamma(G) = \gamma^2(G)$. ■

3. TREES

In what follows, we constructively characterize all trees T for which $\gamma(T) = \gamma^2(T)$.

Let \mathcal{T} be the family of all trees T that can be obtained from sequence T_1, \dots, T_j ($j \geq 1$) of trees such that T_1 is the path P_2 and $T = T_j$, and, if $j > 1$, then T_{i+1} can be obtained recursively from T_i by the operation \mathcal{T}_1 , \mathcal{T}_2 or \mathcal{T}_3 :

- **Operation \mathcal{T}_1 .** The tree T_{i+1} is obtained from T_i by adding a vertex x_1 and the edge x_1y where $y \in V(T_i)$ is a support vertex of T_i .
- **Operation \mathcal{T}_2 .** The tree T_{i+1} is obtained from T_i by adding a path (x_1, x_2, x_3) and the edge x_1y where $y \in V(T_i)$ is neither a leaf nor a support vertex in T_i .
- **Operation \mathcal{T}_3 .** The tree T_{i+1} is obtained from T_i by adding a path (x_1, x_2, x_3, x_4) and the edge x_1y where $y \in V(T_i)$ is a support vertex in T_i .

Additionally, let P_1 belong to \mathcal{T} .

The following observation follows immediately from the way in which each tree in the family \mathcal{T} is constructed.

Observation 3. *If a tree T belonging to the family \mathcal{T} has at least 2 vertices, then:*

1. *If $u, v \in S(T)$, then $d_T(u, v) \geq 3$, that is, if $u, v \in S(T)$, then $S(T)$ is a 2-packing in T ;*
2. *If $u \in V(T)$, then $|N_T[u] \cap S(T)| = 1$;*
3. *$S(T)$ is a minimum dominating set of T .* ■

We show first that each tree T belonging to the family \mathcal{T} is a tree with $\gamma(T) = \gamma^2(T)$. To this aim we prove the following lemma.

Lemma 4. *If a tree T of order at least 2 belongs to the family \mathcal{T} , then $\gamma^2(T) = |S(T)|$.*

Proof. Let T be a tree belonging to the family \mathcal{T} and let D be a $\gamma^2(T)$ -set. Since $S(T)$ is a 2-packing in T , the distance between any two leaves adjacent to different support vertices is greater than or equal to 5. For this reason, if u and v are two leaves adjacent to different support vertices in T , then u and v cannot be 2-distance dominated by the same element of D . This implies that $|D| \geq |S|$. On the other hand, since $S(T)$ is a dominating set of T , it is also a 2-distance dominating set of T . We conclude that $\gamma^2(T) = |S(T)|$. ■

By Lemma 4 and Observation 3 we obtain immediately.

Corollary 5. *If a tree T belongs to the family \mathcal{T} , then $\gamma(T) = \gamma^2(T)$.*

Before we prove our next Lemma, observe that for any tree T with at least 3 vertices, $\gamma(T) \geq |S(T)|$.

Lemma 6. *If T is a tree with $\gamma^2(T) = \gamma(T)$, then T belongs to the family \mathcal{T} .*

Proof. Let T be a tree with $\gamma^2(T) = \gamma(T)$. Let (v_0, v_1, \dots, v_k) be a longest path in T . If $k \in \{1, 2\}$, then T is P_1 or a star $K_{1,p}$, for a positive integer p , and clearly T is in \mathcal{T} .

If $k \in \{3, 4\}$, then $\gamma^2(T) = 1$, but $\gamma(T) > 1$. For this reason now we assume $k \geq 5$. We proceed by induction on the number $n(T)$ of vertices of a tree T with $\gamma^2(T) = \gamma(T)$. If $n(T) = 6$, then $T = P_6$ and T belongs to the family \mathcal{T} . (Observe that P_6 may be obtained from P_2 by operation \mathcal{T}_3). Now let T be a tree with $\gamma^2(T) = \gamma(T)$ and $n(T) \geq 7$, and assume that each tree T' with $n(T') < n(T)$, $k \geq 5$ and $\gamma^2(T') = \gamma(T')$ belongs to the family \mathcal{T} .

If there exists $v \in S(T)$ such that v is adjacent to at least two leaves, say x_1 and x_2 , then clearly $\gamma(T') = \gamma(T)$ and $\gamma^2(T') = \gamma^2(T)$, where $T' = T - x_1$. Thus, $\gamma^2(T') = \gamma(T')$ and by the induction, T' belongs to the family \mathcal{T} . Moreover, T may be obtained from T' by operation \mathcal{T}_1 and we conclude that T also belongs to the family \mathcal{T} .

Now assume that each support vertex of T is adjacent to exactly one leaf. For this reason $d_T(v_1) = 2$. If $d_T(v_2) > 2$, then v_2 is adjacent to a leaf or $|N_T(v_2) \cap S(T)| \geq 2$. In both cases v_2 2-distance dominates all support vertices and leaves at distance at most 2 from v_2 , while $\gamma(T) \geq |S(T)|$. Hence $\gamma(T) > \gamma^2(T)$, which is impossible. Thus, $d_T(v_2) = 2$.

Observe that either v_0 or v_1 is in every minimum dominating set of T . Assume $d_T(v_3) > 2$. If v_3 belongs to some minimum dominating set of T , say D , then $(D \cup \{v_2\}) - \{v_0, v_1, v_3\}$ is a 2-distance dominating set of T of cardinality smaller than $\gamma(T)$, which is impossible. Hence v_3 does not belong to any minimum dominating set of T and this reason together with $n(T) \geq 7$ imply that v_3 is not a support vertex of T . Denote $T' = T - \{v_0, v_1, v_2\}$. Since $d_T(v_3) > 2$, v_3 is not a leaf in T' and since $k \geq 5$, v_3 is not a support vertex in T' . Moreover, it is no problem to verify that $\gamma(T') = \gamma(T) - 1$ and $\gamma^2(T') \geq \gamma^2(T) - 1$. Hence

$$\gamma^2(T) - 1 \leq \gamma^2(T') \leq \gamma(T') = \gamma(T) - 1 = \gamma^2(T) - 1.$$

Thus, $\gamma^2(T') = \gamma(T')$ and by the induction, T' belongs to the family \mathcal{T} . Moreover, T may be obtained from T' by operation \mathcal{T}_2 and we conclude that T also belongs to the family \mathcal{T} .

Thus assume $d_T(v_1) = d_T(v_2) = d_T(v_3) = 2$. Without loss of generality, denote by D a minimum dominating set of T containing v_1 . In this situation v_2, v_3 or v_4 belong to D to dominate v_3 . If v_2 or v_3 is in D , then $D' = (D \cup \{v_2\}) - \{v_1, v_3\}$ is a 2-distance dominating set of T of cardinality smaller than $\gamma(T)$, which is impossible. Hence $v_4 \in D$. Observe that D' , defined as above, 2-distance dominates v_4 . Moreover, if w is a neighbour of v_4 and $d_T(w, D - \{v_4\}) \leq 2$, then w is 2-distance dominated by D' and again $\gamma^2(T') < \gamma(T)$. Thus v_4 has a neighbour, say u , such that $d_T(u, D - \{v_4\}) \geq 3$. Since T is a tree and each neighbour of u is dominated by D , we conclude that u is a leaf and for this reason v_4 is a support vertex. Denote $T' = T - \{v_0, v_1, v_2, v_4\}$. Since u is a leaf in T' , v_4 is a support vertex in T' . Moreover, it is no problem to verify that $\gamma(T') + 1 = \gamma(T)$. Further, since $d_T(u, v_0) = 5$, $\gamma^2(T') + 1 = \gamma^2(T)$. Thus, $\gamma^2(T') = \gamma(T')$ and by the induction, T' belongs to the family \mathcal{T} . Moreover, T may be obtained from T' by operation \mathcal{T}_3 and we conclude that T also belongs to the family \mathcal{T} . ■

The following Theorem is an immediate consequence of Lemma 6 and Corollary 5.

Theorem 7. *Let T be a tree. Then $\gamma(T) = \gamma^2(T)$ if and only if T belongs to the family \mathcal{T} .* ■

4. UNICYCLIC GRAPHS

A unicyclic graph is a graph that contains precisely one cycle. Our next results consider graphs with cycles.

Lemma 8. *Let G be a connected graph with $\gamma(G) = \gamma^2(G)$. If u, v are two leaves of G adjacent to the same support vertex, then $\gamma(G+uv) = \gamma^2(G+uv)$.*

Proof. Let G be a connected graph with $\gamma(G) = \gamma^2(G)$ and let u, v be two leaves of G such that $d_G(u, v) = 2$ and let w be the neighbour of u and v . By our assumptions and some immediate properties of the domination number of a graph,

$$\gamma^2(G + uv) \leq \gamma(G + uv) \leq \gamma(G) = \gamma^2(G).$$

Hence it suffices to justify that $\gamma^2(G + uv) \geq \gamma^2(G)$. Clearly, $N_{G+uv}^2[x] = N_G^2[x]$ for each $x \in V(G)$. Thus, every minimum 2-distance dominating set of $G + uv$ is also a minimum 2-distance dominating set of G . Therefore, $\gamma^2(G + uv) \geq \gamma^2(G)$ and hence $\gamma(G + uv) = \gamma^2(G + uv)$. ■

By Theorem 7 and recursively using Lemma 8 we may obtain graphs G with $\gamma(G) = \gamma^2(G)$ and containing any number of induced cycles C_3 .

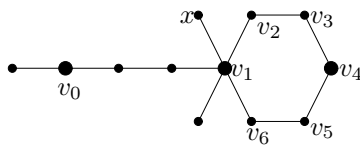
Now we characterize all connected unicyclic graphs G with $\gamma(G) = \gamma^2(G)$. To this aim we introduce some additional notations. Let T be a tree belonging to the family \mathcal{T} . We call $v \in V(T)$ an *active vertex*, if v is a leaf adjacent to a strong support vertex or $v \in V(T) - (S(T) \cup \Omega(T))$. Further, let \mathcal{C}_6^+ be the family of all unicyclic graphs that may be obtained from a tree T belonging to the family \mathcal{T} and the cycle C_6 by identifying one vertex of C_6 with a support vertex of T . In addition, let C_6 belong to \mathcal{C}_6^+ .

Define \mathcal{C} to be the family of all unicyclic graphs that belong to \mathcal{C}_6^+ or may be obtained from a tree T belonging to the family \mathcal{T} by adding an edge between two active vertices of T .

The following two lemmas prove that $\gamma(G) = \gamma^2(G)$ for every graph G belonging to the family \mathcal{C} .

Lemma 9. *Each graph belonging to the family \mathcal{C}_6^+ has equal domination and 2-distance domination numbers.*

Proof. Let $G \in \mathcal{C}_6^+$. Obviously $\gamma(C_6) = \gamma^2(C_6)$. Thus let G be obtained from a tree T belonging to the family \mathcal{T} and the cycle $C_6 = (v_1, \dots, v_6, v_1)$ by identifying the vertex v_1 with a support vertex of T .


 Figure 1. Graph $G \in \mathcal{C}_6^+$. $\{v_0, v_1, v_4\}$ is the $\gamma(G)$ -set.

Since G is unicyclic and connected, $G - v_5v_6$ is a tree. It is no problem to observe, that $G - v_5v_6$ may be obtained from T by adding to T first the path $P_4 = (v_2, v_3, v_4, v_5)$ and the edge v_1v_2 , and then v_6 and the edge v_1v_6 . Since $T \in \mathcal{T}$ and $G - v_5v_6$ may be obtained from T by operations \mathcal{T}_3 and \mathcal{T}_1 , we conclude that $G - v_5v_6 \in \mathcal{T}$. Thus by Lemma 4, $\gamma^2(G - v_5v_6) = |S(G - v_5v_6)|$ and by Lemma 5, $\gamma(G - v_5v_6) = \gamma^2(G - v_5v_6)$.

Let D be a $\gamma^2(G)$ -set. Since G is obtained from T and C_6 by identifying v_1 with a support vertex of T and $\gamma^2(T) = |S(T)|$, $|D| \geq |S(T)|$. Denote by x a leaf adjacent to v_1 in G . Then there exists a vertex y such that $y \in N_G^2[x] \cap D$. In any choice of y , at least one vertex belonging to $\{v_1, \dots, v_6\} - \{y\}$ belongs also to D (because D is 2-distance dominating). Thus $|D| \geq |S(T)| + 1$. On the other hand, $S(G) \cup \{v_4\}$ is a 2-distance dominating set of G of cardinality $|S(G)| + 1$. Thus

$$\begin{aligned} |S(G)| + 1 = \gamma^2(G) &\leq \gamma(G) \leq \gamma(G - v_5v_6) \\ &= \gamma^2(G - v_5v_6) = |S(G - v_5v_6)|. \end{aligned} \quad (1)$$

Since $|S(G)| = |S(G - v_5v_6)| - 1$, we have equalities throughout the inequality chain (1). In particular, $\gamma^2(G) = \gamma(G)$. ■

Lemma 10. *If G is a graph obtained from a tree T belonging to the family \mathcal{T} by adding an edge between two active vertices of T , then $\gamma(G) = \gamma^2(G)$.*

Proof. Let T be a tree belonging to the family \mathcal{T} . Denote by u and v two active vertices of T and let D be a $\gamma^2(G)$ -set, where $G = T + uv$. If u and v are leaves adjacent to the same support vertex, then the result follows from Lemma 8.

Thus assume u and v are adjacent to different support vertices of T or at most one of u and v is a leaf. In both cases, $S(T) = S(G)$ and similarly like in T , the distance between any two leaves adjacent to different support vertices in G is greater than or equal to 5. For this reason, if u and v

are two leaves adjacent to different support vertices in G , then u and v cannot be 2-distance dominated by the same element of D . This implies that $\gamma^2(G) \geq |S(G)|$. Hence

$$|S(G)| \leq \gamma^2(G) \leq \gamma(G) \leq \gamma(T) = \gamma^2(T) = |S(T)| = |S(G)|.$$

Therefore $\gamma(G) = \gamma^2(G)$. ■

For a cycle C_n on $n \geq 3$ vertices it is no problem to see that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ and $\gamma^2(C_n) = \lceil \frac{n}{5} \rceil$.

Lemma 11. *If G is a connected unicyclic graph with $\gamma(G) = \gamma^2(G)$, then G belongs to the family \mathcal{C} .*

Proof. Let G be a unicyclic graph, where $C_k = (v_1, \dots, v_k, v_1)$ is the unique cycle of G . If $d_G(v_i) > 2$ for some $v_i \in V(C_k)$, then let $T(v_i)$ be the tree attached to the vertex v_i and let v_i be the root of $T(v_i)$. Let D be a minimum dominating set of G containing all support vertices of G .

By Proposition 1, at most $\lfloor \frac{k}{3} \rfloor$ vertices of C_k belong to D and the distance between any two elements of D is at least 3. Thus there exists an edge, without loss of generality say v_2v_3 (where $v_2, v_3 \in V(C_k)$), such that $v_2 \notin D$ and $v_3 \notin D$. Note that neither v_2 nor v_3 is a support vertex. Since G is unicyclic and connected, $G - v_2v_3$ is a tree. Moreover, by our assumptions and some immediate properties of the domination number of a graph,

$$\gamma(G) = \gamma^2(G) \leq \gamma^2(G - v_2v_3) \leq \gamma(G - v_2v_3). \quad (2)$$

However, since $v_2, v_3 \notin D$, D is also a dominating set in $G - v_2v_3$. Therefore, $\gamma(G) = \gamma(G - v_2v_3)$ and thus we have equalities throughout the inequality chain (2). In particular, $\gamma^2(G - v_2v_3) = \gamma(G - v_2v_3)$ and since $G - v_2v_3$ is a tree, Theorem 7 implies that $G - v_2v_3$ belongs to the family \mathcal{T} . By Observation 3, each vertex of $G - v_2v_3$ is a support vertex or is a neighbour of exactly one support vertex. Of course $v_2, v_3 \notin S(G - v_2v_3)$. Hence denote by s_2 and s_3 the support vertices adjacent in $G - v_2v_3$ to v_2 and v_3 , respectively. Observe that s_2 and s_3 may not be support vertices in G .

If $s_2 = s_3$, then $v_1 = s_2$. If v_1 is a support vertex in G , then G may be obtained from the tree $G - v_2v_3$ by adding an edge between two active vertices adjacent to the same support vertex and thus $G \in \mathcal{C}$. If $v_1 \notin S(G)$, then at least one of v_2, v_3 is of degree 2 in G . Assume first $d_G(v_2) = d_G(v_3) = 2$. Then v_2 and v_3 are leaves in $G - v_2v_3$ and for

this reason G again may be obtained from the tree $G - v_2v_3$ by adding an edge between two active vertices. Thus assume, without loss of generality, $d_G(v_2) = 2$ and $d_G(v_3) \geq 3$. Observe that since $v_1 \notin S(G)$, every element of $V(G) - \{v_1, v_2\}$ is within distance 2 from a vertex belonging to $D - \{v_1\}$. Thus, $D - \{v_1\}$ 2-distance dominates $V(G) - \{v_1, v_2\}$. Denote by x an element of $D \cap V(T(v_3))$, which is at distance 3 from v_1 and let (x, y, v_3, v_1) be the shortest path from x to v_1 . Define $D' = (D - \{x, v_1\}) \cup \{y\}$. Now every element of $V(G)$ is within distance 2 from an element of D' , so D' is a 2-distance dominating set of G smaller than $\gamma(G)$, which contradicts that $\gamma(G) = \gamma^2(G)$.

In what follows we assume $s_2 \neq s_3$ and we consider three cases.

1. If $s_2 \in S(G)$ and $s_3 \in S(G)$, then v_2 and v_3 are both active vertices in $G - v_2v_3$. Therefore G may be obtained from the tree $G - v_2v_3$ by adding the edge v_2v_3 and thus G belongs to the family \mathcal{C} .

2. Without loss of generality, assume that $s_2 \notin S(G)$ and $s_3 \in S(G)$. Then v_2 is the unique leaf adjacent to s_2 in $G - v_2v_3$. Therefore $d_G(v_2) = 2$ and $s_2 = v_1$. Observe, that since $v_1 \notin S(G)$, each element of $V(G) - \{v_1\}$ is within distance 2 from an element of $D - \{v_1\}$. Thus, $D - \{v_1\}$ 2-distance dominates $V(G) - \{v_1\}$.

If $d_G(v_1) \geq 3$, then since v_1 is not a support vertex in G , $D \cap V(T(v_1)) \neq \emptyset$. Denote by x an element of $D \cap V(T(v_1))$, which is at distance 3 from v_1 and let (x, y, z, v_1) be the shortest path from x to v_1 . Define $D' = (D - \{x, v_1\}) \cup \{y\}$. It is no problem to see that D' is a 2-distance dominating set of G , which contradicts that $\gamma(G) = \gamma^2(G)$. We conclude that $d_G(v_1) = 2$.

If $s_3 \neq v_4$, then $d_G(v_3) \geq 3$. Define $D' = (D - \{s_3\}) \cup \{v_3\}$. Then, since $d_G(v_1, v_3) = 2$, $D' - \{v_1\}$ is a 2-distance dominating set of G , contradicting that $\gamma(G) = \gamma^2(G)$. We conclude that $s_3 = v_4$ and since v_4 is a support vertex, $d_G(v_4) \geq 3$ and $v_1 \neq v_4$. Moreover, $v_5, v_6 \notin D$ and for this reason $v_5, v_6 \notin S(G)$. Denote by v_0 a vertex belonging to D and at distance 2 from v_k . If $v_0 \neq v_k$, then $(D - \{v_1, v_4\}) \cup \{v_3\}$ is a 2-distance dominating set of G of smaller cardinality than $\gamma(G)$, a contradiction. Therefore, $v_0 = v_4$ and since $d_G(v_4, v_k) = 2$ we obtain $v_k = v_6$.

We have already proven, that under our conditions $d_G(v_1) = d_G(v_2) = 2$ and v_4 is a support vertex. Suppose $d_G(v_6) \geq 3$. Then since v_6 is not a support vertex in G , $D \cap V(T(v_6)) \neq \emptyset$. Denote by x an element of $D \cap V(T(v_6))$, which is at distance 3 from v_1 and let (x, y, v_6, v_1) be the shortest path from x to v_1 . Define $D' = (D - \{x, v_1\}) \cup \{y\}$. Now D' is

a 2-distance dominating set of G , which contradicts that $\gamma(G) = \gamma^2(G)$. Therefore $d_G(v_6) = 2$.

Suppose $d_G(v_5) \geq 3$. Then since v_5 is not a support vertex in G , $D \cap V(T(v_5)) \neq \emptyset$. Denote by x an element of $D \cap V(T(v_5))$, which is at distance 3 from v_4 and let (x, y, v_5, v_4) be the shortest path from x to v_4 . Define $D' = (D - \{x, v_1, v_4\}) \cup \{y, v_3\}$. Now D' is a 2-distance dominating set of G , which contradicts that $\gamma(G) = \gamma^2(G)$. Therefore $d_G(v_5) = 2$. Similarly we prove that $d_G(v_3) = 2$.

Therefore, $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_5) = d_G(v_6) = 2$ and v_4 is a support vertex. Hence G may be obtained from a tree T and the cycle C_6 by identifying one vertex of C_6 with a support vertex of T . Clearly, $D - \{v_1\}$ is a dominating set of T , so

$$\gamma^2(T) \leq \gamma(T) \leq \gamma(G) - 1 = \gamma^2(G) - 1. \quad (3)$$

On the other hand, any 2-distance dominating set of T may be extended to a dominating set of G by adding to it v_1 . Thus $\gamma^2(G) \leq \gamma^2(T) + 1$ and we have equalities through the inequality chain (3). In particular, $\gamma^2(T) = \gamma(T)$. By Theorem 7, T belongs to the family \mathcal{T} . Hence G may be obtained from $T \in \mathcal{T}$ and the cycle C_6 by identifying one vertex of C_6 with a support vertex of T . Thus $G \in \mathcal{C}_6^+$.

3. If $s_2 \notin S(G)$ and $s_3 \notin S(G)$, then $d_G(v_2) = 2$ and $d_G(v_3) = 2$. Moreover, $v_1 = s_2$ and $v_4 = s_3$. Since v_1 is not a support vertex, each element of $V(G) - \{v_1\}$ is within distance 2 from an element of $D - \{v_1\}$. Thus, $D - \{v_1\}$ 2-distance dominates $V(G) - \{v_1\}$. By the same reasoning, $D - \{v_4\}$ 2-distance dominates $V(G) - \{v_4\}$. Similarly as in previous case, we deduce that $d_G(v_1) = d_G(v_4) = 2$. Since $v_1 \neq v_4$, the unique cycle contains at least 6 vertices, $v_5, v_6 \notin D$ and $v_5, v_6 \notin S(G)$.

If $d_G(v_5) \geq 3$, then since v_5 is not a support vertex, $D \cap V(T(v_5)) \neq \emptyset$. Denote by x an element of $D \cap V(T(v_5))$, which is at distance 3 from v_4 and let (x, y, v_5, v_4) be the shortest path from x to v_4 . Define $D' = (D - \{x, v_4\}) \cup \{y\}$. Now D' is a 2-distance dominating set of G , which contradicts that $\gamma(G) = \gamma^2(G)$. Therefore $d_G(v_5) = 2$.

Since D is dominating, v_6 has a neighbour in D . If there exists $x \in N_G(v_6) \cap D$ such that $x \neq v_1$, then $(D - \{v_1, v_4\}) \cup \{v_3\}$ is a 2-distance dominating set of G , which contradicts that $\gamma(G) = \gamma^2(G)$. Thus we conclude that $\{v_1\} = N_G(v_6) \cap D$. Therefore the unique cycle of G contains exactly 6 vertices. By similar reasoning as for v_5 , we obtain that $d_G(v_6) = 2$. Hence

each vertex of the unique cycle is of degree 2 and $G = C_2$. Therefore G belongs to the family \mathcal{C} . ■

The following results are consequences of Theorem 7 and Lemmas 9 and 11.

Theorem 12. *Let G be a connected unicyclic graph. Then $\gamma(G) = \gamma^2(G)$ if and only if G belongs to the family \mathcal{C} .* ■

Theorem 13. *Let G be a unicyclic graph. Then $\gamma(G) = \gamma^2(G)$ if and only if exactly one connected component of G is a unicyclic graph belonging to the family \mathcal{C} and each other connected component of G is a tree belonging to the family \mathcal{T} .* ■

REFERENCES

- [1] M. Borowiecki and M. Kuzak, On the k -stable and k -dominating sets of graphs, in: Graphs, Hypergraphs and Block Systems. Proc. Symp. Zielona Góra 1976, ed. by M. Borowiecki, Z. Skupień, L. Szamkołowicz, (Zielona Góra, 1976).
- [2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker Inc., 1998).

Received 18 December 2009

Revised 15 June 2010

Accepted 25 August 2010