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GRAPHS WITH EQUAL DOMINATION AND 2-DISTANCE DOMINATION NUMBERS

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Abstract

Let $G = (V, E)$ be a graph. The distance between two vertices u and v in a connected graph G is the length of the shortest $(u-v)$ path in G. A set $D \subseteq V(G)$ is a dominating set if every vertex of G is at distance at most 1 from an element of D. The domination number of G is the minimum cardinality of a dominating set of G. A set $D \subseteq V(G)$ is a 2-distance dominating set if every vertex of G is at distance at most 2 from an element of D. The 2-distance domination number of G is the minimum cardinality of a 2-distance dominating set of G. We characterize all trees and all unicyclic graphs with equal domination and 2-distance domination numbers.

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1. Definitions

Here we consider simple undirected graphs $G = (V, E)$ with $|V| = n(G)$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $(u - v)$ path in G. If D is a set and $u \in V(G)$, then $d_G(u, D) = \min\{d_G(u, v) : v \in D\}$. The *k-neighbourhood* $N_G^k[v]$ of a vertex $v \in V(G)$ is the set of all vertices at distance at most k from v. For a set $D \subseteq V$, the *k-neighbourhood* $N_G^k[D]$ is defined to be $\bigcup_{v \in D} N_G^k[v]$. A subset D of V is k-*distance dominating* in G if every vertex of $V(G)-D$ is at distance at most k from at least one vertex of D. Let $\gamma^k(G)$ be the minimum cardinality of a k-distance dominating set of G. This kind of domination was defined by Borowiecki and Kuzak [1]. Note that the 1-distance domination number is the *domination number*, denoted $\gamma(G)$.

The degree of a vertex v is $d_G(v) = |N_G^1(v)|$ and a vertex of degree 1 is called *a leaf*. A vertex which is a neighbour of a leaf is called *a support vertex*. Denote by $S(G)$ the set of all support vertices of G. If a support vertex is adjacent to more than one leaf, then we call it a *strong support vertex*. We denote a path on n vertices by $P_n = (v_0, \ldots, v_{n-1})$ and the cycle on n vertices by C_n. For example, P_2 contains two leaves and two support vertices. For any unexplained terms and symbols see [2].

In this paper we study trees and unicyclic graphs for which the domination number and the 2-distance domination number are the same.

2. General results

First we give some general results for graphs with equal domination and 2-distance domination numbers. Obviously, for any graph G if $\gamma(G) = 1$, then $\gamma^2(G) = 1$ and thus $\gamma(G) = \gamma^2(G)$. We start with a necessary condition for a graph G with $1 < \gamma(G) = \gamma^2(G)$. A set $D \subseteq V(G)$ is a 2-*packing* in G if $d_G(u, v) \geq 3$ for every $u, v \in D$.

Proposition 1. *If G is a connected graph with $\gamma(G) = \gamma^2(G)$ and $\gamma(G) > 1$, then every minimum dominating set of G is a 2-packing of G.*

Proof. Suppose D is a minimum dominating set of G such that $|D| \geq 2$ and D is not a 2-packing. Then there exist $u, v \in D$ in G such that $d_G(u, v) \leq 2$. Denote by x a vertex which belongs to $N_G[u] \cap N_G[v]$ (if u and v are adjacent, then possibly $x = u$ or $x = v$) and let $D' = (D - \{u, v\}) \cup \{x\}$. Then $N_G[u] \subseteq N_G^2[x]$ and $N_G[v] \subseteq N_G^2[x]$. Hence D' is a 2-distance dominating set of G of smaller cardinality than $\gamma(G)$, a contradiction. ∎

The condition in Proposition 1 it not sufficient. Consider, for example the cycle C_9. Next result gives a sufficient condition for a graph G to have equal domination and 2-distance domination numbers.

Proposition 2. *Let G be the graph obtained from a graph H and $n(H)$ copies of P_2, where the* ith *vertex of H is adjacent to exactly one vertex of the* ith *copy of P_2. Then $\gamma(G) = \gamma^2(G)$.*

Proof. Let G be the graph obtained from a graph H and $n(H)$ copies of P_2, where the ith vertex of H is adjacent to exactly one vertex of the ith copy of P_2. Denote by D a $\gamma^2(G)$-set. Observe that the distance between any two leaves adjacent to two different support vertices in G is greater than or equal to 5. For this reason, if u and v are two leaves adjacent to two different support vertices, then u and v cannot be 2-dominated by the same element of D. This implies that $\gamma^2(G) \geq |S(G)|$. Since $\gamma^2(G) \leq \gamma(G)$, it follows that $\gamma(G) = \gamma^2(G)$. ■

3. Trees

In what follows, we constructively characterize all trees T for which $\gamma(T) = \gamma^2(T)$.

Let $\mathcal{T}$ be the family of all trees T that can be obtained from sequence $T_1, \ldots, T_j$ ($j \geq 1$) of trees such that T_1 is the path P_2 and $T = T_j$, and, if $j > 1$, then T_{i+1} can be obtained recursively from T_i by the operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$:

- **Operation** $\mathcal{T}_1$. The tree T_{i+1} is obtained from T_i by adding a vertex x_1 and the edge x_1y where $y \in V(T_i)$ is a support vertex of T_i.
- **Operation** $\mathcal{T}_2$. The tree T_{i+1} is obtained from T_i by adding a path (x_1, x_2, x_3) and the edge x_1y where $y \in V(T_i)$ is neither a leaf nor a support vertex in T_i.
- **Operation** $\mathcal{T}_3$. The tree T_{i+1} is obtained from T_i by adding a path (x_1, x_2, x_3, x_4) and the edge x_1y where $y \in V(T_i)$ is a support vertex in T_i.

Additionally, let P_1 belong to $\mathcal{T}$.

The following observation follows immediately from the way in which each tree in the family $\mathcal{T}$ is constructed.

Observation 3. *If a tree T belonging to the family $\mathcal{T}$ has at least 2 vertices, then:*

1. *If $u, v \in S(T)$, then $d_T(u, v) \geq 3$, that is, if $u, v \in S(T)$, then $S(T)$ is a 2-packing in T;*
2. *If $u \in V(T)$, then $|N_T[u] \cap S(T)| = 1$;*
3. *$S(T)$ is a minimum dominating set of T.* ■

We show first that each tree T belonging to the family $\mathcal{T}$ is a tree with $\gamma(T) = \gamma^2(T)$. To this aim we prove the following lemma.

Lemma 4. *If a tree T of order at least 2 belongs to the family $\mathcal{T}$, then $\gamma^2(T) = |S(T)|$.*

Proof. Let T be a tree belonging to the family $\mathcal{T}$ and let D be a $\gamma^2(T)$-set. Since $S(T)$ is a 2-packing in T, the distance between any two leaves adjacent to different support vertices is greater than or equal to 5. For this reason, if u and v are two leaves adjacent to different support vertices in T, then u and v cannot be 2-distance dominated by the same element of D. This implies that $|D| \geq |S|$. On the other hand, since $S(T)$ is a dominating set of T, it is also a 2-distance dominating set of T. We conclude that $\gamma^2(T) = |S(T)|$. ∎

By Lemma 4 and Observation 3 we obtain immediately.

Corollary 5. *If a tree T belongs to the family $\mathcal{T}$, then $\gamma(T) = \gamma^2(T)$.*

Before we prove our next Lemma, observe that for any tree T with at least 3 vertices, $\gamma(T) \geq |S(T)|$.

Lemma 6. *If T is a tree with $\gamma^2(T) = \gamma(T)$, then T belongs to the family $\mathcal{T}$.*

Proof. Let T be a tree with $\gamma^2(T) = \gamma(T)$. Let $(v_0, v_1, \ldots, v_k)$ be a longest path in T. If $k \in \{1, 2\}$, then T is P_1 or a star $K_{1,p}$, for a positive integer p, and clearly T is in $\mathcal{T}$.

If $k \in \{3, 4\}$, then $\gamma^2(T) = 1$, but $\gamma(T) > 1$. For this reason now we assume $k \geq 5$. We proceed by induction on the number $n(T)$ of vertices of a tree T with $\gamma^2(T) = \gamma(T)$. If $n(T) = 6$, then $T = P_6$ and T belongs to the family $\mathcal{T}$. (Observe that P_6 may be obtained from P_2 by operation $\mathcal{T}_3$). Now let T be a tree with $\gamma^2(T) = \gamma(T)$ and $n(T) \geq 7$, and assume that each tree T' with $n(T') < n(T)$, $k \geq 5$ and $\gamma^2(T') = \gamma(T')$ belongs to the family $\mathcal{T}$.

If there exists $v \in S(T)$ such that v is adjacent to at least two leaves, say x_1 and x_2, then clearly $\gamma(T') = \gamma(T)$ and $\gamma^2(T') = \gamma^2(T)$, where $T' = T - x_1$. Thus, $\gamma^2(T') = \gamma(T')$ and by the induction, T' belongs to the family $\mathcal{T}$. Moreover, T may be obtained from T' by operation $\mathcal{T}_1$ and we conclude that T also belongs to the family $\mathcal{T}$.

Now assume that each support vertex of T is adjacent to exactly one leaf. For this reason $d_T(v_1) = 2$. If $d_T(v_2) > 2$, then v_2 is adjacent to a leaf or $|N_T(v_2) \cap S(T)| \geq 2$. In both cases v_2 2-distance dominates all support vertices and leaves at distance at most 2 from v_2, while $\gamma(T) \geq |S(T)|$. Hence $\gamma(T) > \gamma^2(T)$, which is impossible. Thus, $d_T(v_2) = 2$.

Observe that either v_0 or v_1 is in every minimum dominating set of T. Assume $d_T(v_3) > 2$. If v_3 belongs to some minimum dominating set of T, say D, then $(D \cup \{v_2\}) - \{v_0, v_1, v_3\}$ is a 2-distance dominating set of T of cardinality smaller than $\gamma(T)$, which is impossible. Hence v_3 does not belong to any minimum dominating set of T and this reason together with $n(T) \geq 7$ imply that v_3 is not a support vertex of T. Denote $T' = T - \{v_0, v_1, v_2\}$. Since $d_T(v_3) > 2$, v_3 is not a leaf in T' and since $k \geq 5$, v_3 is not a support vertex in T'. Moreover, it is no problem to verify that $\gamma(T') = \gamma(T) - 1$ and $\gamma^2(T') \geq \gamma^2(T) - 1$. Hence

$$\gamma^2(T) - 1 \leq \gamma^2(T') \leq \gamma(T') = \gamma(T) - 1 = \gamma^2(T) - 1.$$

Thus, $\gamma^2(T') = \gamma(T')$ and by the induction, T' belongs to the family $\mathcal{T}$. Moreover, T may be obtained from T' by operation $\mathcal{T}_2$ and we conclude that T also belongs to the family $\mathcal{T}$.

Thus assume $d_T(v_1) = d_T(v_2) = d_T(v_3) = 2$. Without loss of generality, denote by D a minimum dominating set of T containing v_1. In this situation v_2, v_3 or v_4 belong to D to dominate v_3. If v_2 or v_3 is in D, then $D' = (D \cup \{v_2\}) - \{v_1, v_3\}$ is a 2-distance dominating set of T of cardinality smaller than $\gamma(T)$, which is impossible. Hence $v_4 \in D$. Observe that D', defined as above, 2-distance dominates v_4. Moreover, if w is a neighbour of v_4 and $d_T(w, D - \{v_4\}) \leq 2$, then w is 2-distance dominated by D' and again $\gamma^2(T') < \gamma(T)$. Thus v_4 has a neighbour, say u, such that $d_T(u, D - \{v_4\}) \geq 3$. Since T is a tree and each neighbour of u is dominated by D, we conclude that u is a leaf and for this reason v_4 is a support vertex. Denote $T' = T - \{v_0, v_1, v_2, v_4\}$. Since u is a leaf in T', v_4 is a support vertex in T'. Moreover, it is no problem to verify that $\gamma(T') + 1 = \gamma(T)$. Further, since $d_T(u, v_0) = 5$, $\gamma^2(T') + 1 = \gamma^2(T)$. Thus, $\gamma^2(T') = \gamma(T')$ and by the induction, T' belongs to the family $\mathcal{T}$. Moreover, T may be obtained from T' by operation $\mathcal{T}_3$ and we conclude that T also belongs to the family $\mathcal{T}$. ∎

The following Theorem is an immediate consequence of Lemma 6 and Corollary 5.

Theorem 7. *Let T be a tree. Then $\gamma(T) = \gamma^2(T)$ if and only if T belongs to the family $\mathcal{T}$.* ∎

4. Unicyclic Graphs

A unicyclic graph is a graph that contains precisely one cycle. Our next results consider graphs with cycles.

Lemma 8. *Let G be a connected graph with $\gamma(G) = \gamma^2(G)$. If u, v are two leaves of G adjacent to the same support vertex, then $\gamma(G+uv) = \gamma^2(G+uv)$.*

Proof. Let G be a connected graph with $\gamma(G) = \gamma^2(G)$ and let u, v be two leaves of G such that $d_G(u, v) = 2$ and let w be the neighbour of u and v. By our assumptions and some immediate properties of the domination number of a graph,

$$\gamma^2(G + uv) \leq \gamma(G + uv) \leq \gamma(G) = \gamma^2(G).$$

Hence it suffices to justify that $\gamma^2(G + uv) \geq \gamma^2(G)$. Clearly, $N^2_{G+uv}[x] = N^2_G[x]$ for each $x \in V(G)$. Thus, every minimum 2-distance dominating set of $G + uv$ is also a minimum 2-distance dominating set of G. Therefore, $\gamma^2(G + uv) \geq \gamma^2(G)$ and hence $\gamma(G + uv) = \gamma^2(G + uv)$. ■

By Theorem 7 and recursively using Lemma 8 we may obtain graphs G with $\gamma(G) = \gamma^2(G)$ and containing any number of induced cycles C_3.

Now we characterize all connected unicyclic graphs G with $\gamma(G) = \gamma^2(G)$. To this aim we introduce some additional notations. Let T be a tree belonging to the family $\mathcal{T}$. We call $v \in V(T)$ *an active vertex*, if v is a leaf adjacent to a strong support vertex or $v \in V(T) - (S(T) \cup \Omega(T))$. Further, let $\mathcal{C}_6^+$ be the family of all unicyclic graphs that may be obtained from a tree T belonging to the family $\mathcal{T}$ and the cycle C_6 by identifying one vertex of C_6 with a support vertex of T. In addition, let C_6 belong to $\mathcal{C}_6^+$.

Define $\mathcal{C}$ to be the family of all unicyclic graphs that belong to $\mathcal{C}_6^+$ or may be obtained from a tree T belonging to the family $\mathcal{T}$ by adding an edge between two active vertices of T.

The following two lemmas prove that $\gamma(G) = \gamma^2(G)$ for every graph G belonging to the family $\mathcal{C}$.

Lemma 9. *Each graph belonging to the family $\mathcal{C}_6^+$ has equal domination and 2-distance domination numbers.*

Proof. Let $G \in \mathcal{C}_6^+$. Obviously $\gamma(C_6) = \gamma^2(C_6)$. Thus let G be obtained from a tree T belonging to the family $\mathcal{T}$ and the cycle $C_6 = (v_1, \ldots, v_6, v_1)$ by identifying the vertex v_1 with a support vertex of T.

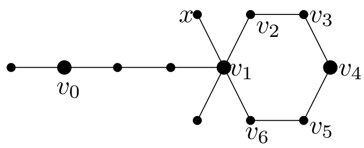

Figure 1. Graph $G \in \mathcal{C}_6^+$. $\{v_0, v_1, v_4\}$ is the $\gamma(G)$-set.

Since G is unicyclic and connected, $G - v_5v_6$ is a tree. It is no problem to observe, that $G - v_5v_6$ may be obtained from T by adding to T first the path $P_4 = (v_2, v_3, v_4, v_5)$ and the edge v_1v_2, and then v_6 and the edge v_1v_6. Since $T \in \mathcal{T}$ and $G - v_5v_6$ may be obtained from T by operations $\mathcal{T}_3$ and $\mathcal{T}_1$, we conclude that $G - v_5v_6 \in \mathcal{T}$. Thus by Lemma 4, $\gamma^2(G - v_5v_6) = |S(G - v_5v_6)|$ and by Lemma 5, $\gamma(G - v_5v_6) = \gamma^2(G - v_5v_6)$.

Let D be a $\gamma^2(G)$-set. Since G is obtained from T and C_6 by identifying v_1 with a support vertex of T and $\gamma^2(T) = |S(T)|$, $|D| \geq |S(T)|$. Denote by x a leaf adjacent to v_1 in G. Then there exists a vertex y such that $y \in N_G^2[x] \cap D$. In any choice of y, at least one vertex belonging to $\{v_1, \ldots, v_6\} - \{y\}$ belongs also to D (because D is 2-distance dominating). Thus $|D| \geq |S(T)| + 1$. On the other hand, $S(G) \cup \{v_4\}$ is a 2-distance dominating set of G of cardinality $|S(G)| + 1$. Thus

$$\begin{aligned} |S(G)| + 1 &= \gamma^2(G) \leq \gamma(G) \leq \gamma(G - v_5v_6) \\ &= \gamma^2(G - v_5v_6) = |S(G - v_5v_6)|. \end{aligned} \tag{1}$$

Since $|S(G)| = |S(G - v_5v_6)| - 1$, we have equalities throughout the inequality chain (1). In particular, $\gamma^2(G) = \gamma(G)$. ∎

Lemma 10. *If G is a graph obtained from a tree T belonging to the family $\mathcal{T}$ by adding an edge between two active vertices of T, then $\gamma(G) = \gamma^2(G)$.*

Proof. Let T be a tree belonging to the family $\mathcal{T}$. Denote by u and v two active vertices of T and let D be a $\gamma^2(G)$-set, where $G = T + uv$. If u and v are leaves adjacent to the same support vertex, then the result follows from Lemma 8.

Thus assume u and v are adjacent to different support vertices of T or at most one of u and v is a leaf. In both cases, $S(T) = S(G)$ and similarly like in T, the distance between any two leaves adjacent to different support vertices in G is greater than or equal to 5. For this reason, if u and v

are two leaves adjacent to different support vertices in G, then u and v cannot be 2-distance dominated by the same element of D. This implies that $\gamma^2(G) \geq |S(G)|$. Hence

$$|S(G)| \leq \gamma^2(G) \leq \gamma(G) \leq \gamma(T) = \gamma^2(T) = |S(T)| = |S(G)|.$$

Therefore $\gamma(G) = \gamma^2(G)$. ■

For a cycle C_n on $n \geq 3$ vertices it is no problem to see that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ and $\gamma^2(C_n) = \lceil \frac{n}{5} \rceil$.

Lemma 11. *If G is a connected unicyclic graph with $\gamma(G) = \gamma^2(G)$, then G belongs to the family $\mathcal{C}$.*

Proof. Let G be a unicyclic graph, where $C_k = (v_1, \ldots, v_k, v_1)$ is the unique cycle of G. If $d_G(v_i) > 2$ for some $v_i \in V(C_k)$, then let $T(v_i)$ be the tree attached to the vertex v_i and let v_i be the root of $T(v_i)$. Let D be a minimum dominating set of G containing all support vertices of G.

By Proposition 1, at most $\lfloor \frac{k}{3} \rfloor$ vertices of C_k belong to D and the distance between any two elements of D is at least 3. Thus there exists an edge, without loss of generality say v_2v_3 (where $v_2, v_3 \in V(C_k)$), such that $v_2 \notin D$ and $v_3 \notin D$. Note that neither v_2 nor v_3 is a support vertex. Since G is unicyclic and connected, $G - v_2v_3$ is a tree. Moreover, by our assumptions and some immediate properties of the domination number of a graph,

$$\gamma(G) = \gamma^2(G) \leq \gamma^2(G - v_2v_3) \leq \gamma(G - v_2v_3). \tag{2}$$

However, since $v_2, v_3 \notin D$, D is also a dominating set in $G - v_2v_3$. Therefore, $\gamma(G) = \gamma(G - v_2v_3)$ and thus we have equalities throughout the inequality chain (2). In particular, $\gamma^2(G - v_2v_3) = \gamma(G - v_2v_3)$ and since $G - v_2v_3$ is a tree, Theorem 7 implies that $G - v_2v_3$ belongs to the family $\mathcal{T}$. By Obsevation 3, each vertex of $G - v_2v_3$ is a support vertex or is a neighbour of exactly one support vertex. Of course $v_2, v_3 \notin S(G - v_2v_3)$. Hence denote by s_2 and s_3 the support vertices adjacent in $G - v_2v_3$ to v_2 and v_3, respectively. Observe that s_2 and s_3 may not be support vertices in G.

If $s_2 = s_3$, then $v_1 = s_2$. If v_1 is a support vertex in G, then G may be obtained from the tree $G - v_2v_3$ by adding an edge between two active vertices adjacent to the same support vertex and thus $G \in \mathcal{C}$. If $v_1 \notin S(G)$, then at least one of v_2, v_3 is of degree 2 in G. Assume first $d_G(v_2) = d_G(v_3) = 2$. Then v_2 and v_3 are leaves in $G - v_2v_3$ and for this reason G again may be obtained from the tree $G - v_2v_3$ by adding an edge between two active vertices. Thus assume, without loss of generality, $d_G(v_2) = 2$ and $d_G(v_3) \geq 3$. Observe that since $v_1 \notin S(G)$, every element of $V(G) - \{v_1, v_2\}$ is within distance 2 from a vertex belonging to $D - \{v_1\}$. Thus, $D - \{v_1\}$ 2-distance dominates $V(G) - \{v_1, v_2\}$. Denote by x an element of $D \cap V(T(v_3))$, which is at distance 3 from v_1 and let (x, y, v_3, v_1) be the shortest path from x to v_1. Define $D' = (D - \{x, v_1\}) \cup \{y\}$. Now every element of $V(G)$ is within distance 2 from an element of D', so D' is a 2-distance dominating set of G smaller than $\gamma(G)$, which contradicts that $\gamma(G) = \gamma^2(G)$.

In what follows we assume $s_2 \neq s_3$ and we consider three cases.

1. If $s_2 \in S(G)$ and $s_3 \in S(G)$, then v_2 and v_3 are both active vertices in $G - v_2v_3$. Therefore G may be obtained from the tree $G - v_2v_3$ by adding the edge v_2v_3 and thus G belongs to the family $\mathcal{C}$.

2. Without loss of generality, assume that $s_2 \notin S(G)$ and $s_3 \in S(G)$. Then v_2 is the unique leaf adjacent to s_2 in $G - v_2v_3$. Therefore $d_G(v_2) = 2$ and $s_2 = v_1$. Observe, that since $v_1 \notin S(G)$, each element of $V(G) - \{v_1\}$ is within distance 2 from an element of $D - \{v_1\}$. Thus, $D - \{v_1\}$ 2-distance dominates $V(G) - \{v_1\}$.

If $d_G(v_1) \geq 3$, then since v_1 is not a support vertex in G, $D \cap V(T(v_1)) \neq \emptyset$. Denote by x an element of $D \cap V(T(v_1))$, which is at distance 3 from v_1 and let (x, y, z, v_1) be the shortest path from x to v_1. Define $D' = (D - \{x, v_1\}) \cup \{y\}$. It is no problem to see that D' is a 2-distance dominating set of G, which contradicts that $\gamma(G) = \gamma^2(G)$. We conclude that $d_G(v_1) = 2$.

If $s_3 \neq v_4$, then $d_G(v_3) \geq 3$. Define $D' = (D - \{s_3\}) \cup \{v_3\}$. Then, since $d_G(v_1, v_3) = 2$, $D' - \{v_1\}$ is a 2-distance dominating set of G, contradicting that $\gamma(G) = \gamma^2(G)$. We conclude that $s_3 = v_4$ and since v_4 is a support vertex, $d_G(v_4) \geq 3$ and $v_1 \neq v_4$. Moreover, $v_5, v_6 \notin D$ and for this reason $v_5, v_6 \notin S(G)$. Denote by v_0 a vertex belonging to D and at distance 2 from v_k. If $v_0 \neq v_k$, then $(D - \{v_1, v_4\}) \cup \{v_3\}$ is a 2-distance dominating set of G of smaller cardinality than $\gamma(G)$, a contradiction. Therefore, $v_0 = v_4$ and since $d_G(v_4, v_k) = 2$ we obtain $v_k = v_6$.

We have already proven, that under our conditions $d_G(v_1) = d_G(v_2) = 2$ and v_4 is a support vertex. Suppose $d_G(v_6) \geq 3$. Then since v_6 is not a support vertex in G, $D \cap V(T(v_6)) \neq \emptyset$. Denote by x an element of $D \cap V(T(v_6))$, which is at distance 3 from v_1 and let (x, y, v_6, v_1) be the shortest path from x to v_1. Define $D' = (D - \{x, v_1\}) \cup \{y\}$. Now D' is a 2-distance dominating set of G, which contradicts that $\gamma(G) = \gamma^2(G)$. Therefore $d_G(v_6) = 2$.

Suppose $d_G(v_5) \geq 3$. Then since v_5 is not a support vertex in G, $D \cap V(T(v_5)) \neq \emptyset$. Denote by x an element of $D \cap V(T(v_5))$, which is at distance 3 from v_4 and let (x, y, v_5, v_4) be the shortest path from x to v_4. Define $D' = (D - \{x, v_1, v_4\}) \cup \{y, v_3\}$. Now D' is a 2-distance dominating set of G, which contradicts that $\gamma(G) = \gamma^2(G)$. Therefore $d_G(v_5) = 2$. Similarly we prove that $d_G(v_3) = 2$.

Therefore, $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_5) = d_G(v_6) = 2$ and v_4 is a support vertex. Hence G may be obtained from a tree T and the cycle C_6 by identifying one vertex of C_6 with a support vertex of T. Clearly, $D - \{v_1\}$ is a dominating set of T, so

$$\gamma^2(T) \leq \gamma(T) \leq \gamma(G) - 1 = \gamma^2(G) - 1. \tag{3}$$

On the other hand, any 2-distance dominating set of T may be extended to a dominating set of G by adding to it v_1. Thus $\gamma^2(G) \leq \gamma^2(T) + 1$ and we have equalities throught the inequality chain (3). In particular, $\gamma^2(T) = \gamma(T)$. By Theorem 7, T belongs to the family $\mathcal{T}$. Hence G may be obtained from $T \in \mathcal{T}$ and the cycle C_6 by identifying one vertex of C_6 with a support vertex of T. Thus $G \in \mathcal{C}_6^+$.

3. If $s_2 \notin S(G)$ and $s_3 \notin S(G)$, then $d_G(v_2) = 2$ and $d_G(v_3) = 2$. Moreover, $v_1 = s_2$ and $v_4 = s_3$. Since v_1 is not a support vertex, each element of $V(G) - \{v_1\}$ is within distance 2 from an element of $D - \{v_1\}$. Thus, $D - \{v_1\}$ 2-distance dominates $V(G) - \{v_1\}$. By the same reasoning, $D - \{v_4\}$ 2-distance dominates $V(G) - \{v_4\}$. Similarly as in previous case, we deduce that $d_G(v_1) = d_G(v_4) = 2$. Since $v_1 \neq v_4$, the unique cycle contains at least 6 vertices, $v_5, v_6 \notin D$ and $v_5, v_6 \notin S(G)$.

If $d_G(v_5) \geq 3$, then since v_5 is not a support vertex, $D \cap V(T(v_5)) \neq \emptyset$. Denote by x an element of $D \cap V(T(v_5))$, which is at distance 3 from v_4 and let (x, y, v_5, v_4) be the shortest path from x to v_4. Define $D' = (D - \{x, v_4\}) \cup \{y\}$. Now D' is a 2-distance dominating set of G, which contradicts that $\gamma(G) = \gamma^2(G)$. Therefore $d_G(v_5) = 2$.

Since D is dominating, v_6 has a neighbour in D. If there exists $x \in N_G(v_6) \cap D$ such that $x \neq v_1$, then $(D - \{v_1, v_4\}) \cup \{v_3\}$ is a 2-distance dominating set of G, which contradicts that $\gamma(G) = \gamma^2(G)$. Thus we conclude that $\{v_1\} = N_G(v_6) \cap D$. Therefore the unique cycle of G contains exactly 6 vertices. By similar reasoning as for v_5, we obtain that $d_G(v_6) = 2$. Hence each vertex of the unique cycle is of degree 2 and $G = C_2$. Therefore G belongs to the family $\mathcal{C}$. ■

The following results are consequences of Theorem 7 and Lemmas 9 and 11.

Theorem 12. *Let G be a connected unicyclic graph. Then $\gamma(G) = \gamma^2(G)$ if and only if G belongs to the family $\mathcal{C}$.* ■

Theorem 13. *Let G be a unicyclic graph. Then $\gamma(G) = \gamma^2(G)$ if and only if exactly one connected component of G is a unicyclic graph belonging to the family $\mathcal{C}$ and each other connected compoment of G is a tree belonging to the family $\mathcal{T}$.* ■

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