# GRAPHS WITH EQUAL DOMINATION AND 2-DISTANCE DOMINATION NUMBERS 

Joanna Raczek<br>Department of Applied Physics and Mathematics<br>Gdansk University of Technology<br>Narutowicza 11/12, 80-233 Gdańsk, Poland<br>e-mail: Joanna.Raczek@pg.gda.pl


#### Abstract

Let $G=(V, E)$ be a graph. The distance between two vertices $u$ and $v$ in a connected graph $G$ is the length of the shortest $(u-v)$ path in $G$. A set $D \subseteq V(G)$ is a dominating set if every vertex of $G$ is at distance at most 1 from an element of $D$. The domination number of $G$ is the minimum cardinality of a dominating set of $G$. A set $D \subseteq V(G)$ is a 2-distance dominating set if every vertex of $G$ is at distance at most 2 from an element of $D$. The 2-distance domination number of $G$ is the minimum cardinality of a 2-distance dominating set of $G$. We characterize all trees and all unicyclic graphs with equal domination and 2-distance domination numbers.


Keywords: domination number, trees, unicyclic graphs.
2010 Mathematics Subject Classification: 05C05, 05C69.

## 1. Definitions

Here we consider simple undirected graphs $G=(V, E)$ with $|V|=n(G)$. The distance $d_{G}(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $(u-v)$ path in $G$. If $D$ is a set and $u \in V(G)$, then $d_{G}(u, D)=\min \left\{d_{G}(u, v): v \in D\right\}$. The $k$-neighbourhood $N_{G}^{k}[v]$ of a vertex $v \in V(G)$ is the set of all vertices at distance at most $k$ from $v$. For a set $D \subseteq V$, the $k$-neighbourhood $N_{G}^{k}[D]$ is defined to be $\bigcup_{v \in D} N_{G}^{k}[v]$. A
subset $D$ of $V$ is $k$-distance dominating in $G$ if every vertex of $V(G)-D$ is at distance at most $k$ from at least one vertex of $D$. Let $\gamma^{k}(G)$ be the minimum cardinality of a $k$-distance dominating set of $G$. This kind of domination was defined by Borowiecki and Kuzak [1]. Note that the 1-distance domination number is the domination number, denoted $\gamma(G)$.

The degree of a vertex $v$ is $d_{G}(v)=\left|N_{G}^{1}(v)\right|$ and a vertex of degree 1 is called a leaf. A vertex which is a neighbour of a leaf is called a support vertex. Denote by $S(G)$ the set of all support vertices of $G$. If a support vertex is adjacent to more than one leaf, then we call it a strong support vertex. We denote a path on $n$ vertices by $P_{n}=\left(v_{0}, \ldots, v_{n-1}\right)$ and the cycle on $n$ vertices by $C_{n}$. For example, $P_{2}$ contains two leaves and two support vertices. For any unexplained terms and symbols see [2].

In this paper we study trees and unicyclic graphs for which the domination number and the 2-distance domination number are the same.

## 2. General Results

First we give some general results for graphs with equal domination and 2distance domination numbers. Obviously, for any graph $G$ if $\gamma(G)=1$, then $\gamma^{2}(G)=1$ and thus $\gamma(G)=\gamma^{2}(G)$. We start with a necessary condition for a graph $G$ with $1<\gamma(G)=\gamma^{2}(G)$. A set $D \subseteq V(G)$ is a 2-packing in $G$ if $d_{G}(u, v) \geq 3$ for every $u, v \in D$.
Proposition 1. If $G$ is a connected graph with $\gamma(G)=\gamma^{2}(G)$ and $\gamma(G)>1$, then every minimum dominating set of $G$ is a 2-packing of $G$.
Proof. Suppose $D$ is a minimum dominating set of $G$ such that $|D| \geq 2$ and $D$ is not a 2-packing. Then there exist $u, v \in D$ in $G$ such that $d_{G}(u, v) \leq 2$. Denote by $x$ a vertex which belongs to $N_{G}[u] \cap N_{G}[v]$ (if $u$ and $v$ are adjacent, then possibly $x=u$ or $x=v$ ) and let $D^{\prime}=(D-\{u, v\}) \cup\{x\}$. Then $N_{G}[u] \subseteq N_{G}^{2}[x]$ and $N_{G}[v] \subseteq N_{G}^{2}[x]$. Hence $D^{\prime}$ is a 2-distance dominating set of $G$ of smaller cardinality than $\gamma(G)$, a contradiction.

The condition in Proposition 1 it not sufficient. Consider, for example the cycle $C_{9}$. Next result gives a sufficient condition for a graph $G$ to have equal domination and 2-distance domination numbers.

Proposition 2. Let $G$ be the graph obtained from a graph $H$ and $n(H)$ copies of $P_{2}$, where the $i$ th vertex of $H$ is adjacent to exactly one vertex of the $i$ th copy of $P_{2}$. Then $\gamma(G)=\gamma^{2}(G)$.

Proof. Let $G$ be the graph obtained from a graph $H$ and $n(H)$ copies of $P_{2}$, where the $i$ th vertex of $H$ is adjacent to exactly one vertex of the $i$ th copy of $P_{2}$. Denote by $D$ a $\gamma^{2}(G)$-set. Observe that the distance between any two leaves adjacent to two different support vertices in $G$ is greater than or equal to 5 . For this reason, if $u$ and $v$ are two leaves adjacent to two different support vertices, then $u$ and $v$ cannot be 2 -dominated by the same element of $D$. This implies that $\gamma^{2}(G) \geq|S(G)|$. Since $\gamma^{2}(G) \leq \gamma(G)$, it follows that $\gamma(G)=\gamma^{2}(G)$.

## 3. Trees

In what follows, we constructively characterize all trees $T$ for which $\gamma(T)=$ $\gamma^{2}(T)$.

Let $\mathcal{T}$ be the family of all trees $T$ that can be obtained from sequence $T_{1}, \ldots, T_{j}(j \geq 1)$ of trees such that $T_{1}$ is the path $P_{2}$ and $T=T_{j}$, and, if $j>1$, then $T_{i+1}$ can be obtained recursively from $T_{i}$ by the operation $\mathcal{T}_{1}$, $\mathcal{T}_{2}$ or $\mathcal{T}_{3}$ :

- Operation $\mathcal{T}_{1}$. The tree $T_{i+1}$ is obtained from $T_{i}$ by adding a vertex $x_{1}$ and the edge $x_{1} y$ where $y \in V\left(T_{i}\right)$ is a support vertex of $T_{i}$.
- Operation $\mathcal{T}_{2}$. The tree $T_{i+1}$ is obtained from $T_{i}$ by adding a path $\left(x_{1}, x_{2}, x_{3}\right)$ and the edge $x_{1} y$ where $y \in V\left(T_{i}\right)$ is neither a leaf nor a support vertex in $T_{i}$.
- Operation $\mathcal{T}_{3}$. The tree $T_{i+1}$ is obtained from $T_{i}$ by adding a path $\left(x_{1}\right.$, $\left.x_{2}, x_{3}, x_{4}\right)$ and the edge $x_{1} y$ where $y \in V\left(T_{i}\right)$ is a support vertex in $T_{i}$.
Additionally, let $P_{1}$ belong to $\mathcal{T}$.
The following observation follows immediately from the way in which each tree in the family $\mathcal{T}$ is constructed.
Observation 3. If a tree $T$ belonging to the family $\mathcal{T}$ has at least 2 vertices, then:

1. If $u, v \in S(T)$, then $d_{T}(u, v) \geq 3$, that is, if $u, v \in S(T)$, then $S(T)$ is a 2-packing in $T$;
2. If $u \in V(T)$, then $\left|N_{T}[u] \cap S(T)\right|=1$;
3. $S(T)$ is a minimum dominating set of $T$.

We show first that each tree $T$ belonging to the family $\mathcal{T}$ is a tree with $\gamma(T)=\gamma^{2}(T)$. To this aim we prove the following lemma.

Lemma 4. If a tree $T$ of order at least 2 belongs to the family $\mathcal{T}$, then $\gamma^{2}(T)=|S(T)|$.

Proof. Let $T$ be a tree belonging to the family $\mathcal{T}$ and let $D$ be a $\gamma^{2}(T)$ set. Since $S(T)$ is a 2-packing in $T$, the distance between any two leaves adjacent to different support vertices is greater than or equal to 5 . For this reason, if $u$ and $v$ are two leaves adjacent to different support vertices in $T$, then $u$ and $v$ cannot be 2-distance dominated by the same element of $D$. This implies that $|D| \geq|S|$. On the other hand, since $S(T)$ is a dominating set of $T$, it is also a 2-distance dominating set of $T$. We conclude that $\gamma^{2}(T)=|S(T)|$.

By Lemma 4 and Observation 3 we obtain immediately.
Corollary 5. If a tree $T$ belongs to the family $\mathcal{T}$, then $\gamma(T)=\gamma^{2}(T)$.
Before we prove our next Lemma, observe that for any tree $T$ with at least 3 vertices, $\gamma(T) \geq|S(T)|$.

Lemma 6. If $T$ is a tree with $\gamma^{2}(T)=\gamma(T)$, then $T$ belongs to the family $\mathcal{T}$.
Proof. Let $T$ be a tree with $\gamma^{2}(T)=\gamma(T)$. Let $\left(v_{0}, v_{1}, \ldots, v_{k}\right)$ be a longest path in $T$. If $k \in\{1,2\}$, then $T$ is $P_{1}$ or a star $K_{1, p}$, for a positive integer $p$, and clearly $T$ is in $\mathcal{T}$.

If $k \in\{3,4\}$, then $\gamma^{2}(T)=1$, but $\gamma(T)>1$. For this reason now we assume $k \geq 5$. We proceed by induction on the number $n(T)$ of vertices of a tree $T$ with $\gamma^{2}(T)=\gamma(T)$. If $n(T)=6$, then $T=P_{6}$ and $T$ belongs to the family $\mathcal{T}$. (Observe that $P_{6}$ may be obtained from $P_{2}$ by operation $\mathcal{T}_{3}$ ). Now let $T$ be a tree with $\gamma^{2}(T)=\gamma(T)$ and $n(T) \geq 7$, and assume that each tree $T^{\prime}$ with $n\left(T^{\prime}\right)<n(T), k \geq 5$ and $\gamma^{2}\left(T^{\prime}\right)=\gamma\left(T^{\prime}\right)$ belongs to the family $\mathcal{T}$.

If there exists $v \in S(T)$ such that $v$ is adjacent to at least two leaves, say $x_{1}$ and $x_{2}$, then clearly $\gamma\left(T^{\prime}\right)=\gamma(T)$ and $\gamma^{2}\left(T^{\prime}\right)=\gamma^{2}(T)$, where $T^{\prime}=T-x_{1}$. Thus, $\gamma^{2}\left(T^{\prime}\right)=\gamma\left(T^{\prime}\right)$ and by the induction, $T^{\prime}$ belongs to the family $\mathcal{T}$. Moreover, $T$ may be obtained from $T^{\prime}$ by operation $\mathcal{T}_{1}$ and we conclude that $T$ also belongs to the family $\mathcal{T}$.

Now assume that each support vertex of $T$ is adjacent to exactly one leaf. For this reason $d_{T}\left(v_{1}\right)=2$. If $d_{T}\left(v_{2}\right)>2$, then $v_{2}$ is adjacent to a leaf or $\left|N_{T}\left(v_{2}\right) \cap S(T)\right| \geq 2$. In both cases $v_{2} 2$-distance dominates all support vertices and leaves at distance at most 2 from $v_{2}$, while $\gamma(T) \geq|S(T)|$. Hence $\gamma(T)>\gamma^{2}(T)$, which is impossible. Thus, $d_{T}\left(v_{2}\right)=2$.

Observe that either $v_{0}$ or $v_{1}$ is in every minimum dominating set of $T$. Assume $d_{T}\left(v_{3}\right)>2$. If $v_{3}$ belongs to some minimum dominating set of $T$, say $D$, then $\left(D \cup\left\{v_{2}\right\}\right)-\left\{v_{0}, v_{1}, v_{3}\right\}$ is a 2-distance dominating set of $T$ of cardinality smaller than $\gamma(T)$, which is impossible. Hence $v_{3}$ does not belong to any minimum dominating set of $T$ and this reason together with $n(T) \geq 7$ imply that $v_{3}$ is not a support vertex of $T$. Denote $T^{\prime}=T-\left\{v_{0}, v_{1}, v_{2}\right\}$. Since $d_{T}\left(v_{3}\right)>2, v_{3}$ is not a leaf in $T^{\prime}$ and since $k \geq 5, v_{3}$ is not a support vertex in $T^{\prime}$. Moreover, it is no problem to verify that $\gamma\left(T^{\prime}\right)=\gamma(T)-1$ and $\gamma^{2}\left(T^{\prime}\right) \geq \gamma^{2}(T)-1$. Hence

$$
\gamma^{2}(T)-1 \leq \gamma^{2}\left(T^{\prime}\right) \leq \gamma\left(T^{\prime}\right)=\gamma(T)-1=\gamma^{2}(T)-1 .
$$

Thus, $\gamma^{2}\left(T^{\prime}\right)=\gamma\left(T^{\prime}\right)$ and by the induction, $T^{\prime}$ belongs to the family $\mathcal{T}$. Moreover, $T$ may be obtained from $T^{\prime}$ by operation $\mathfrak{T}_{2}$ and we conclude that $T$ also belongs to the family $\mathcal{T}$.

Thus assume $d_{T}\left(v_{1}\right)=d_{T}\left(v_{2}\right)=d_{T}\left(v_{3}\right)=2$. Without loss of generality, denote by $D$ a minimum dominating set of $T$ containing $v_{1}$. In this situation $v_{2}, v_{3}$ or $v_{4}$ belong to $D$ to dominate $v_{3}$. If $v_{2}$ or $v_{3}$ is in $D$, then $D^{\prime}=\left(D \cup\left\{v_{2}\right\}\right)-\left\{v_{1}, v_{3}\right\}$ is a 2-distance dominating set of $T$ of cardinality smaller than $\gamma(T)$, which is impossible. Hence $v_{4} \in D$. Observe that $D^{\prime}$, defined as above, 2-distance dominates $v_{4}$. Moreover, if $w$ is a neighbour of $v_{4}$ and $d_{T}\left(w, D-\left\{v_{4}\right\}\right) \leq 2$, then $w$ is 2-distance dominated by $D^{\prime}$ and again $\gamma^{2}\left(T^{\prime}\right)<\gamma(T)$. Thus $v_{4}$ has a neighbour, say $u$, such that $d_{T}\left(u, D-\left\{v_{4}\right\}\right) \geq 3$. Since $T$ is a tree and each neighbour of $u$ is dominated by $D$, we conclude that $u$ is a leaf and for this reason $v_{4}$ is a support vertex. Denote $T^{\prime}=T-\left\{v_{0}, v_{1}, v_{2}, v_{4}\right\}$. Since $u$ is a leaf in $T^{\prime}, v_{4}$ is a support vertex in $T^{\prime}$. Moreover, it is no problem to verify that $\gamma\left(T^{\prime}\right)+1=\gamma(T)$. Further, since $d_{T}\left(u, v_{0}\right)=5, \gamma^{2}\left(T^{\prime}\right)+1=\gamma^{2}(T)$. Thus, $\gamma^{2}\left(T^{\prime}\right)=\gamma\left(T^{\prime}\right)$ and by the induction, $T^{\prime}$ belongs to the family $\mathcal{T}$. Moreover, $T$ may be obtained from $T^{\prime}$ by operation $\mathcal{T}_{3}$ and we conclude that $T$ also belongs to the family $\mathcal{T}$.

The following Theorem is an immediate consequence of Lemma 6 and Corollary 5.

Theorem 7. Let $T$ be a tree. Then $\gamma(T)=\gamma^{2}(T)$ if and only if $T$ belongs to the family $\mathcal{T}$.

## 4. Unicyclic Graphs

A unicyclic graph is a graph that contains precisely one cycle. Our next results consider graphs with cycles.

Lemma 8. Let $G$ be a connected graph with $\gamma(G)=\gamma^{2}(G)$. If $u$, $v$ are two leaves of $G$ adjacent to the same support vertex, then $\gamma(G+u v)=\gamma^{2}(G+u v)$.

Proof. Let $G$ be a connected graph with $\gamma(G)=\gamma^{2}(G)$ and let $u, v$ be two leaves of $G$ such that $d_{G}(u, v)=2$ and let $w$ be the neighbour of $u$ and $v$. By our assumptions and some immediate properties of the domination number of a graph,

$$
\gamma^{2}(G+u v) \leq \gamma(G+u v) \leq \gamma(G)=\gamma^{2}(G)
$$

Hence it suffices to justify that $\gamma^{2}(G+u v) \geq \gamma^{2}(G)$. Clearly, $N_{G+u v}^{2}[x]=$ $N_{G}^{2}[x]$ for each $x \in V(G)$. Thus, every minimum 2-distance dominating set of $G+u v$ is also a minimum 2-distance dominating set of $G$. Therefore, $\gamma^{2}(G+u v) \geq \gamma^{2}(G)$ and hence $\gamma(G+u v)=\gamma^{2}(G+u v)$.

By Theorem 7 and recursively using Lemma 8 we may obtain graphs $G$ with $\gamma(G)=\gamma^{2}(G)$ and containing any number of induced cycles $C_{3}$.

Now we characterize all connected unicyclic graphs $G$ with $\gamma(G)=$ $\gamma^{2}(G)$. To this aim we introduce some additional notations. Let $T$ be a tree belonging to the family $\mathcal{T}$. We call $v \in V(T)$ an active vertex, if $v$ is a leaf adjacent to a strong support vertex or $v \in V(T)-(S(T) \cup \Omega(T))$. Further, let $\mathcal{C}_{6}^{+}$be the family of all unicyclic graphs that may be obtained from a tree $T$ belonging to the family $\mathcal{T}$ and the cycle $C_{6}$ by identifying one vertex of $C_{6}$ with a support vertex of $T$. In addition, let $C_{6}$ belong to $\mathcal{C}_{6}^{+}$.

Define $\mathcal{C}$ to be the family of all unicyclic graphs that belong to $\mathcal{C}_{6}^{+}$or may be obtained from a tree $T$ belonging to the family $\mathcal{T}$ by adding an edge between two active vertices of $T$.

The following two lemmas prove that $\gamma(G)=\gamma^{2}(G)$ for every graph $G$ belonging to the family $\mathcal{C}$.

Lemma 9. Each graph belonging to the family $\mathcal{C}_{6}^{+}$has equal domination and 2-distance domination numbers.

Proof. Let $G \in \mathcal{C}_{6}^{+}$. Obviously $\gamma\left(C_{6}\right)=\gamma^{2}\left(C_{6}\right)$. Thus let $G$ be obtained from a tree $T$ belonging to the family $\mathcal{T}$ and the cycle $C_{6}=\left(v_{1}, \ldots, v_{6}, v_{1}\right)$ by identifying the vertex $v_{1}$ with a support vertex of $T$.


Figure 1. Graph $G \in \mathcal{C}_{6}^{+} .\left\{v_{0}, v_{1}, v_{4}\right\}$ is the $\gamma(G)$-set.

Since $G$ is unicyclic and connected, $G-v_{5} v_{6}$ is a tree. It is no problem to observe, that $G-v_{5} v_{6}$ may be obtained from $T$ by adding to $T$ first the path $P_{4}=\left(v_{2}, v_{3}, v_{4}, v_{5}\right)$ and the edge $v_{1} v_{2}$, and then $v_{6}$ and the edge $v_{1} v_{6}$. Since $T \in \mathcal{T}$ and $G-v_{5} v_{6}$ may be obtained from $T$ by operations $\mathcal{T}_{3}$ and $\mathcal{T}_{1}$, we conclude that $G-v_{5} v_{6} \in \mathcal{T}$. Thus by Lemma $4, \gamma^{2}\left(G-v_{5} v_{6}\right)=\left|S\left(G-v_{5} v_{6}\right)\right|$ and by Lemma 5, $\gamma\left(G-v_{5} v_{6}\right)=\gamma^{2}\left(G-v_{5} v_{6}\right)$.

Let $D$ be a $\gamma^{2}(G)$-set. Since $G$ is obtained from $T$ and $C_{6}$ by identifying $v_{1}$ with a support vertex of $T$ and $\gamma^{2}(T)=|S(T)|,|D| \geq|S(T)|$. Denote by $x$ a leaf adjacent to $v_{1}$ in $G$. Then there exists a vertex $y$ such that $y \in$ $N_{G}^{2}[x] \cap D$. In any choice of $y$, at least one vertex belonging to $\left\{v_{1}, \ldots, v_{6}\right\}-$ $\{y\}$ belongs also to $D$ (because $D$ is 2-distance dominating). Thus $|D| \geq$ $|S(T)|+1$. On the other hand, $S(G) \cup\left\{v_{4}\right\}$ is a 2 -distance dominating set of $G$ of cardinality $|S(G)|+1$. Thus

$$
\begin{align*}
|S(G)|+1 & =\gamma^{2}(G) \leq \gamma(G) \leq \gamma\left(G-v_{5} v_{6}\right) \\
& =\gamma^{2}\left(G-v_{5} v_{6}\right)=\left|S\left(G-v_{5} v_{6}\right)\right| \tag{1}
\end{align*}
$$

Since $|S(G)|=\left|S\left(G-v_{5} v_{6}\right)\right|-1$, we have equalities throughout the inequality chain (1). In particular, $\gamma^{2}(G)=\gamma(G)$.

Lemma 10. If $G$ is a graph obtained from a tree $T$ belonging to the family $\mathcal{T}$ by adding an edge between two active vertices of $T$, then $\gamma(G)=\gamma^{2}(G)$.

Proof. Let $T$ be a tree belonging to the family $\mathcal{T}$. Denote by $u$ and $v$ two active vertices of $T$ and let $D$ be a $\gamma^{2}(G)$-set, where $G=T+u v$. If $u$ and $v$ are leaves adjacent to the same support vertex, then the result follows from Lemma 8.

Thus assume $u$ and $v$ are adjacent to different support vertices of $T$ or at most one of $u$ and $v$ is a leaf. In both cases, $S(T)=S(G)$ and similarly like in $T$, the distance between any two leaves adjacent to different support vertices in $G$ is greater than or equal to 5 . For this reason, if $u$ and $v$
are two leaves adjacent to different support vertices in $G$, then $u$ and $v$ cannot be 2-distance dominated by the same element of $D$. This implies that $\gamma^{2}(G) \geq|S(G)|$. Hence

$$
|S(G)| \leq \gamma^{2}(G) \leq \gamma(G) \leq \gamma(T)=\gamma^{2}(T)=|S(T)|=|S(G)|
$$

Therefore $\gamma(G)=\gamma^{2}(G)$.
For a cycle $C_{n}$ on $n \geq 3$ vertices it is no problem to see that $\gamma\left(C_{n}\right)=\left\lceil\frac{n}{3}\right\rceil$ and $\gamma^{2}\left(C_{n}\right)=\left\lceil\frac{n}{5}\right\rceil$.
Lemma 11. If $G$ is a connected unicyclic graph with $\gamma(G)=\gamma^{2}(G)$, then $G$ belongs to the family $\mathcal{C}$.

Proof. Let $G$ be a unicyclic graph, where $C_{k}=\left(v_{1}, \ldots, v_{k}, v_{1}\right)$ is the unique cycle of $G$. If $d_{G}\left(v_{i}\right)>2$ for some $v_{i} \in V\left(C_{k}\right)$, then let $T\left(v_{i}\right)$ be the tree attached to the vertex $v_{i}$ and let $v_{i}$ be the root of $T\left(v_{i}\right)$. Let $D$ be a minimum dominating set of $G$ containing all support vertices of $G$.

By Proposition 1, at most $\left\lfloor\frac{k}{3}\right\rfloor$ vertices of $C_{k}$ belong to $D$ and the distance between any two elements of $D$ is at least 3 . Thus there exists an edge, without loss of generality say $v_{2} v_{3}$ (where $v_{2}, v_{3} \in V\left(C_{k}\right)$ ), such that $v_{2} \notin D$ and $v_{3} \notin D$. Note that neither $v_{2}$ nor $v_{3}$ is a support vertex. Since $G$ is unicyclic and connected, $G-v_{2} v_{3}$ is a tree. Moreover, by our assumptions and some immediate properties of the domination number of a graph,

$$
\begin{equation*}
\gamma(G)=\gamma^{2}(G) \leq \gamma^{2}\left(G-v_{2} v_{3}\right) \leq \gamma\left(G-v_{2} v_{3}\right) \tag{2}
\end{equation*}
$$

However, since $v_{2}, v_{3} \notin D, D$ is also a dominating set in $G-v_{2} v_{3}$. Therefore, $\gamma(G)=\gamma\left(G-v_{2} v_{3}\right)$ and thus we have equalities throughout the inequality chain (2). In particular, $\gamma^{2}\left(G-v_{2} v_{3}\right)=\gamma\left(G-v_{2} v_{3}\right)$ and since $G-v_{2} v_{3}$ is a tree, Theorem 7 implies that $G-v_{2} v_{3}$ belongs to the family $\mathcal{T}$. By Obsevation 3, each vertex of $G-v_{2} v_{3}$ is a support vertex or is a neighbour of exactly one support vertex. Of course $v_{2}, v_{3} \notin S\left(G-v_{2} v_{3}\right)$. Hence denote by $s_{2}$ and $s_{3}$ the support vertices adjacent in $G-v_{2} v_{3}$ to $v_{2}$ and $v_{3}$, respectively. Observe that $s_{2}$ and $s_{3}$ may not be support vertices in $G$.

If $s_{2}=s_{3}$, then $v_{1}=s_{2}$. If $v_{1}$ is a support vertex in $G$, then $G$ may be obtained from the tree $G-v_{2} v_{3}$ by adding an edge between two active vertices adjacent to the same support vertex and thus $G \in \mathcal{C}$. If $v_{1} \notin S(G)$, then at least one of $v_{2}, v_{3}$ is of degree 2 in $G$. Assume first $d_{G}\left(v_{2}\right)=d_{G}\left(v_{3}\right)=2$. Then $v_{2}$ and $v_{3}$ are leaves in $G-v_{2} v_{3}$ and for
this reason $G$ again may be obtained from the tree $G-v_{2} v_{3}$ by adding an edge between two active vertices. Thus assume, without loss of generality, $d_{G}\left(v_{2}\right)=2$ and $d_{G}\left(v_{3}\right) \geq 3$. Observe that since $v_{1} \notin S(G)$, every element of $V(G)-\left\{v_{1}, v_{2}\right\}$ is within distance 2 from a vertex belonging to $D-\left\{v_{1}\right\}$. Thus, $D-\left\{v_{1}\right\}$ 2-distance dominates $V(G)-\left\{v_{1}, v_{2}\right\}$. Denote by $x$ an element of $D \cap V\left(T\left(v_{3}\right)\right)$, which is at distance 3 from $v_{1}$ and let $\left(x, y, v_{3}, v_{1}\right)$ be the shortest path from $x$ to $v_{1}$. Define $D^{\prime}=\left(D-\left\{x, v_{1}\right\}\right) \cup\{y\}$. Now every element of $V(G)$ is within distance 2 from an element of $D^{\prime}$, so $D^{\prime}$ is a 2-distance dominating set of $G$ smaller than $\gamma(G)$, which contradicts that $\gamma(G)=\gamma^{2}(G)$.

In what follows we assume $s_{2} \neq s_{3}$ and we consider three cases.

1. If $s_{2} \in S(G)$ and $s_{3} \in S(G)$, then $v_{2}$ and $v_{3}$ are both active vertices in $G-v_{2} v_{3}$. Therefore $G$ may be obtained from the tree $G-v_{2} v_{3}$ by adding the edge $v_{2} v_{3}$ and thus $G$ belongs to the family $\mathcal{C}$.
2. Without loss of generality, assume that $s_{2} \notin S(G)$ and $s_{3} \in S(G)$. Then $v_{2}$ is the unique leaf adjacent to $s_{2}$ in $G-v_{2} v_{3}$. Therefore $d_{G}\left(v_{2}\right)=2$ and $s_{2}=v_{1}$. Observe, that since $v_{1} \notin S(G)$, each element of $V(G)-\left\{v_{1}\right\}$ is within distance 2 from an element of $D-\left\{v_{1}\right\}$. Thus, $D-\left\{v_{1}\right\}$ 2-distance dominates $V(G)-\left\{v_{1}\right\}$.

If $d_{G}\left(v_{1}\right) \geq 3$, then since $v_{1}$ is not a support vertex in $G, D \cap V\left(T\left(v_{1}\right)\right)$ $\neq \emptyset$. Denote by $x$ an element of $D \cap V\left(T\left(v_{1}\right)\right)$, which is at distance 3 from $v_{1}$ and let $\left(x, y, z, v_{1}\right)$ be the shortest path from $x$ to $v_{1}$. Define $D^{\prime}=(D-$ $\left.\left\{x, v_{1}\right\}\right) \cup\{y\}$. It is no problem to see that $D^{\prime}$ is a 2 -distance dominating set of $G$, which contradicts that $\gamma(G)=\gamma^{2}(G)$. We conclude that $d_{G}\left(v_{1}\right)=2$.

If $s_{3} \neq v_{4}$, then $d_{G}\left(v_{3}\right) \geq 3$. Define $D^{\prime}=\left(D-\left\{s_{3}\right\}\right) \cup\left\{v_{3}\right\}$. Then, since $d_{G}\left(v_{1}, v_{3}\right)=2, D^{\prime}-\left\{v_{1}\right\}$ is a 2-distance dominating set of $G$, contradicting that $\gamma(G)=\gamma^{2}(G)$. We conclude that $s_{3}=v_{4}$ and since $v_{4}$ is a support vertex, $d_{G}\left(v_{4}\right) \geq 3$ and $v_{1} \neq v_{4}$. Moreover, $v_{5}, v_{6} \notin D$ and for this reason $v_{5}, v_{6} \notin S(G)$. Denote by $v_{0}$ a vertex belonging to $D$ and at distance 2 from $v_{k}$. If $v_{0} \neq v_{k}$, then $\left(D-\left\{v_{1}, v_{4}\right\}\right) \cup\left\{v_{3}\right\}$ is a 2-distance dominating set of $G$ of smaller cardinality than $\gamma(G)$, a contradiction. Therefore, $v_{0}=v_{4}$ and since $d_{G}\left(v_{4}, v_{k}\right)=2$ we obtain $v_{k}=v_{6}$.

We have already proven, that under our conditions $d_{G}\left(v_{1}\right)=d_{G}\left(v_{2}\right)=2$ and $v_{4}$ is a support vertex. Suppose $d_{G}\left(v_{6}\right) \geq 3$. Then since $v_{6}$ is not a support vertex in $G, D \cap V\left(T\left(v_{6}\right)\right) \neq \emptyset$. Denote by $x$ an element of $D \cap V\left(T\left(v_{6}\right)\right)$, which is at distance 3 from $v_{1}$ and let $\left(x, y, v_{6}, v_{1}\right)$ be the shortest path from $x$ to $v_{1}$. Define $D^{\prime}=\left(D-\left\{x, v_{1}\right\}\right) \cup\{y\}$. Now $D^{\prime}$ is
a 2-distance dominating set of $G$, which contradicts that $\gamma(G)=\gamma^{2}(G)$. Therefore $d_{G}\left(v_{6}\right)=2$.

Suppose $d_{G}\left(v_{5}\right) \geq 3$. Then since $v_{5}$ is not a support vertex in $G$, $D \cap V\left(T\left(v_{5}\right)\right) \neq \emptyset$. Denote by $x$ an element of $D \cap V\left(T\left(v_{5}\right)\right)$, which is at distance 3 from $v_{4}$ and let $\left(x, y, v_{5}, v_{4}\right)$ be the shortest path from $x$ to $v_{4}$. Define $D^{\prime}=\left(D-\left\{x, v_{1}, v_{4}\right\}\right) \cup\left\{y, v_{3}\right\}$. Now $D^{\prime}$ is a 2 -distance dominating set of $G$, which contradicts that $\gamma(G)=\gamma^{2}(G)$. Therefore $d_{G}\left(v_{5}\right)=2$. Similarly we prove that $d_{G}\left(v_{3}\right)=2$.

Therefore, $d_{G}\left(v_{1}\right)=d_{G}\left(v_{2}\right)=d_{G}\left(v_{3}\right)=d_{G}\left(v_{5}\right)=d_{G}\left(v_{6}\right)=2$ and $v_{4}$ is a support vertex. Hence $G$ may be obtained from a tree $T$ and the cycle $C_{6}$ by identifying one vertex of $C_{6}$ with a support vertex of $T$. Clearly, $D-\left\{v_{1}\right\}$ is a dominating set of $T$, so

$$
\begin{equation*}
\gamma^{2}(T) \leq \gamma(T) \leq \gamma(G)-1=\gamma^{2}(G)-1 \tag{3}
\end{equation*}
$$

On the other hand, any 2-distance dominating set of $T$ may be extended to a dominating set of $G$ by adding to it $v_{1}$. Thus $\gamma^{2}(G) \leq \gamma^{2}(T)+1$ and we have equalities throught the inequality chain (3). In particular, $\gamma^{2}(T)=\gamma(T)$. By Theorem 7, $T$ belongs to the family $\mathcal{T}$. Hence $G$ may be obtained from $T \in \mathcal{T}$ and the cycle $C_{6}$ by identifying one vertex of $C_{6}$ with a support vertex of $T$. Thus $G \in \mathcal{C}_{6}^{+}$.
3. If $s_{2} \notin S(G)$ and $s_{3} \notin S(G)$, then $d_{G}\left(v_{2}\right)=2$ and $d_{G}\left(v_{3}\right)=2$. Moreover, $v_{1}=s_{2}$ and $v_{4}=s_{3}$. Since $v_{1}$ is not a support vertex, each element of $V(G)-\left\{v_{1}\right\}$ is within distance 2 from an element of $D-\left\{v_{1}\right\}$. Thus, $D-\left\{v_{1}\right\}$ 2-distance dominates $V(G)-\left\{v_{1}\right\}$. By the same reasoning, $D-\left\{v_{4}\right\}$ 2-distance dominates $V(G)-\left\{v_{4}\right\}$. Similarly as in previous case, we deduce that $d_{G}\left(v_{1}\right)=d_{G}\left(v_{4}\right)=2$. Since $v_{1} \neq v_{4}$, the unique cycle contains at least 6 vertices, $v_{5}, v_{6} \notin D$ and $v_{5}, v_{6} \notin S(G)$.

If $d_{G}\left(v_{5}\right) \geq 3$, then since $v_{5}$ is not a support vertex, $D \cap V\left(T\left(v_{5}\right)\right) \neq \emptyset$. Denote by $x$ an element of $D \cap V\left(T\left(v_{5}\right)\right)$, which is at distance 3 from $v_{4}$ and let $\left(x, y, v_{5}, v_{4}\right)$ be the shortest path from $x$ to $v_{4}$. Define $D^{\prime}=(D-$ $\left.\left\{x, v_{4}\right\}\right) \cup\{y\}$. Now $D^{\prime}$ is a 2 -distance dominating set of $G$, which contradicts that $\gamma(G)=\gamma^{2}(G)$. Therefore $d_{G}\left(v_{5}\right)=2$.

Since $D$ is dominating, $v_{6}$ has a neighbour in $D$. If there exists $x \in$ $N_{G}\left(v_{6}\right) \cap D$ such that $x \neq v_{1}$, then $\left(D-\left\{v_{1}, v_{4}\right\}\right) \cup\left\{v_{3}\right\}$ is a 2-distance dominating set of $G$, which contradicts that $\gamma(G)=\gamma^{2}(G)$. Thus we conclude that $\left\{v_{1}\right\}=N_{G}\left(v_{6}\right) \cap D$. Therefore the unique cycle of $G$ contains exactly 6 vertices. By similar reasoning as for $v_{5}$, we obtain that $d_{G}\left(v_{6}\right)=2$. Hence
each vertex of the unique cycle is of degree 2 and $G=C_{2}$. Therefore $G$ belongs to the family $\mathcal{C}$.

The following results are consequences of Theorem 7 and Lemmas 9 and 11.
Theorem 12. Let $G$ be a connected unicyclic graph. Then $\gamma(G)=\gamma^{2}(G)$ if and only if $G$ belongs to the family $\mathcal{C}$.

Theorem 13. Let $G$ be a unicyclic graph. Then $\gamma(G)=\gamma^{2}(G)$ if and only if exactly one connected component of $G$ is a unicyclic graph belonging to the family $\mathcal{C}$ and each other connected compoment of $G$ is a tree belonging to the family $\mathcal{T}$.

## References

[1] M. Borowiecki and M. Kuzak, On the $k$-stable and $k$-dominating sets of graphs, in: Graphs, Hypergraphs and Block Systems. Proc. Symp. Zielona Góra 1976, ed. by M. Borowiecki, Z. Skupień, L. Szamkołowicz, (Zielona Góra, 1976).
[2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker Inc., 1998).

