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# GENERALIZED CIRCULAR COLOURING OF GRAPHS

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**Abstract**

Let $\mathcal{P}$ be a graph property and $r, s \in \mathbb{N}$, $r \geq s$. A strong circular $(\mathcal{P}, r, s)$-colouring of a graph $G$ is an assignment $f : V(G) \to \{0, 1, \ldots, r-1\}$, such that the edges $uv \in E(G)$ satisfying $|f(u) - f(v)| < s$ or $|f(u) - f(v)| > r - s$, induce a subgraph of $G$ with the propery $\mathcal{P}$. In this paper we present some basic results on strong circular $(\mathcal{P}, r, s)$-colourings. We introduce the strong circular $\mathcal{P}$-chromatic number of a graph and we determine the strong circular $\mathcal{P}$-chromatic number of complete graphs for additive and hereditary graph properties.

**Keywords:** graph property, $\mathcal{P}$-colouring, circular colouring, strong circular $\mathcal{P}$-chromatic number.

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## 1. Introduction

Throughout this paper, by a graph property $\mathcal{P}$, we mean a nonempty isomorphism closed subclass of the class $\mathcal{I}$ of all finite simple graphs. We say that a graph $G$ has a property $\mathcal{P}$ if $G \in \mathcal{P}$. The empty set is called the *empty* property and it is denoted by $\mathcal{E}$. The class of graphs without edges is denoted by $\mathcal{O}$.

A graph property $\mathcal{P}$ is called *hereditary* whenever it is closed under taking subgraphs, that is, if $H$ is a subgraph of a graph $G$ and $G \in \mathcal{P}$, then $H \in \mathcal{P}$, too.

A graph property $\mathcal{P}$ is called *additive* if it is closed under disjoint union, so that every graph $G$ whose components have property $\mathcal{P}$ satisfies $G \in \mathcal{P}$, too.

For each hereditary graph property $\mathcal{P}$, there exists nonnegative integer $c(\mathcal{P})$ (called the *completeness* of $\mathcal{P}$) such that $c(\mathcal{P}) = \sup\{k : K_{k+1} \in \mathcal{P}\}$.

The following list shows several well-known hereditary and additive graph properties $\mathcal{P}$ with $c(\mathcal{P}) = k$ (we use in this paper the notations of [3, 4]):

$$\begin{aligned}
\mathcal{O}_k &= \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\},\\
\mathcal{S}_k &= \{G \in \mathcal{I} : \Delta(G) \le k\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : \delta(H) \le k \text{ for each } H \subseteq G\},\\
\mathcal{O}^{k+1} &= \{G \in \mathcal{I} : G \text{ is } k+1 \text{ colourable}\},\\
\mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ contains no } K_{k+2}\}.
\end{aligned}$$

In this paper we consider vertex colourings of graphs. The proper graph colouring requires that for each colour $i$ the subgraph induced by vertices coloured by the colour $i$ is independent, so that it belongs to the property $\mathcal{O}$. One of generalizations of proper vertex graph colouring is the vertex $\mathcal{P}$-colouring. For a graph property $\mathcal{P}$, by a $\mathcal{P}$-colouring of a graph $G$ we mean a partition $(V_1, V_2, \ldots, V_k)$ of vertices of $G$ such that, for each $i = 1, 2, \ldots, k$, the subgraph $G[V_i]$ induced by $V_i$ has the property $\mathcal{P}$.

If we restrict ourselves to additive hereditary graph properties, the definition of $\mathcal{P}$-colouring may be reformulated as follows: for a graph $G$ and a $k$-colouring $f : V(G) \to \{0, 1, \ldots, k-1\}$, $k \in \mathbb{N}$, let us define the graph $G_f$ with the vertex set $V(G_f) = V(G)$ and the edge set $E(G_f) = \{uv \in E(G) : f(u) = f(v)\}$. We say that $G$ has a $(\mathcal{P}, k)$-colouring (or $G$ is $(\mathcal{P}, k)$-colourable), if there exists a colouring $f : V(G) \to \{0, 1, \ldots, k-1\}$ such that $G_f \in \mathcal{P}$. Then the $\mathcal{P}$-chromatic number of $G$ is defined as

$$\chi_{\mathcal{P}}(G) = \min\{k : G \text{ is } (\mathcal{P}, k)\text{-colourable}\}.$$

In order to simplify the notation, the set of $n$ consecutive integers $\{a, a+1, \ldots, a+n-1\}$ will be denoted by $[a, a+n-1]$.

As a refinement of proper vertex colouring of graphs, one may consider $(k, q)$-colouring, called also the circular graph colouring, as follows: a graph $G$ has a $(k, q)$-colouring with $k \geq q > 1$, if there exists a mapping $f : V(G) \to [0, k-1]$ such that, for each pair of adjacent vertices $u$ and $v$, $q \leq |f(u) - f(v)| \leq k - q$ holds.

The circular chromatic number of $G$ (defined and called originally by Vince [8] "'the star chromatic number"') is the infimum of rational numbers $k/q$ such that there is a $(k, q)$-colouring of $G$. Note, a $(k, 1)$-colouring of a graph $G$ is an ordinary $k$-colouring of $G$, for any $k \in \mathbb{N}$.

As a generalization of proper graph colouring, we define *the strong circular $\mathcal{P}$-colouring* of graphs: let $r, s \in \mathbb{N}$, $r \geq s$ and $\mathcal{P}$ be a hereditary and additive graph property. Let $f : V(G) \to [0, r-1]$ be an $r$-colouring of a graph $G$. Then, for $G$ and $f$, define the graph $G_{f,s}$ with the vertex set $V(G_{f,s}) = V(G)$, where the edge $uv \in E(G)$ belongs to the set $E(G_{f,s})$ if and only if $|f(u) - f(v)| < s$ or $|f(u) - f(v)| > r - s$. We say that the graph $G$ has a strong circular $(\mathcal{P}, r, s)$-colouring (or $G$ is $(\mathcal{P}, r, s)$-colourable), if there exists a colouring $f : V(G) \to [0, r-1]$ such that $G_{f,s} \in \mathcal{P}$ (such colouring will be called also "'strong circular $\mathcal{P}$-colouring"'. The strong circular $\mathcal{P}$-chromatic number of the graph $G$ is defined as follows:

$$\chi_{c,\mathcal{P}}(G) = \inf\left\{\frac{r}{s} : G \text{ is } (\mathcal{P}, r, s)\text{-colourable}\right\}.$$

The introduced colouring is called "'strong"' because there is also a weaker version of the natural generalisation of the fractional and circular colouring (see [7]), however we shall not deal with this parameter here.

For $s = 1$ in a $(\mathcal{P}, r, s)$-colouring $f$ of a graph $G$ $uv \in E(G)$ is an edge of $G_{f,s}$ if and only if $|f(u) - f(v)| = 0$ and in this case the colouring $f$ is a $(\mathcal{P}, r)$-colouring of $G$, so that $\chi_{c,\mathcal{P}}(G) \leq \chi_{\mathcal{P}}(G)$. The strong circular $\mathcal{P}$-chromatic number $\chi_{c,\mathcal{P}}$ is a generalization of the circular chromatic number

$\chi_c$ (for which $\mathcal{P} = \mathcal{O}$). In fact, e.g. the strong circular $\mathcal{S}_k$-colouring, $k \in \mathbb{N}$, is the defective circular colouring introduced by Klostermeyer in [5]. He investigated the defective circular vertex colouring of planar, outerplanar and series-parallel graphs. Let us remark here, that the famous Borodin's Five Colour Theorem (see [2]) implies that each planar graph $G$ has a strong circular $(\mathcal{D}_1, 5, 2)$-colouring.

In Chapter 2 we introduce the basic properties of the strong circular $\mathcal{P}$-chromatic number of graphs. Borowiecki a Mihók showed in [3] that the set of all additive hereditary properties partially ordered by set inclusion is a complete distributive lattice $(\mathbb{L}^a, \subseteq)$ with the smallest element $\mathcal{E}$ and the greatest element $\mathcal{I}$. Moreover, the set of properties $\mathcal{P} \in \mathbb{L}^a$ with $c(\mathcal{P}) = k$, $k \in \mathbb{N}$, with partial order $\subseteq$ is a complete distributive lattice $(\mathbb{L}^a_k, \subseteq)$ with the smallest element $\mathcal{O}_k$ and the greatest element $\mathcal{I}_k$. Remark $\mathcal{O}_k \subseteq \mathcal{S}_k \subset \mathcal{D}_k \subset \mathcal{O}^{k+1} \subset \mathcal{I}_k$. More details on the lattices of hereditary properties may be found in [6]. Therefore it is interesting to study strong circular $\mathcal{P}$-chromatic number for $\mathcal{P} = \mathcal{O}_k$ or $\mathcal{P} = \mathcal{I}_k$. It will be our intention in Chapter 3, where the strong circular $\mathcal{P}$-chromatic numbers of complete graphs are determined.

## 2. Basic Properties

First we show that for determining the strong circular $\mathcal{P}$-chromatic number of graphs it is sufficient to consider only those rational numbers $\frac{r}{s}$ for which $r$ and $s$ are coprime.

**Lemma 1.** *Let $r, s \in \mathbb{N}$, $r \geq s$. Then, for any $n \in \mathbb{N}$, the graph $G$ is $(\mathcal{P}, r, s)$-colourable if and only if it is $(\mathcal{P}, nr, ns)$-colourable.*

***Proof.*** Suppose that a graph $G$ has $(\mathcal{P}, r, s)$-colouring $f : V(G) \to [0, r-1]$. Define a new colouring $g : V(G) \to [0, nr-1]$ of $G$ in the following way: $g(v) = nf(v)$ for each $v \in V(G)$. Then, for each edge $uv \in E(G)$, $s \leq |f(u) - f(v)| \leq r - s$ if and only if $ns \leq |g(u) - g(v)| \leq nr - ns$; thus $G_{g,ns} \cong G_{f,s}$ and so $G_{g,ns} \in \mathcal{P}$. Hence, $g$ is a $(\mathcal{P}, nr, ns)$-colouring of $G$.

Conversely, suppose that $G$ has $(\mathcal{P}, nr, ns)$-colouring $g' : V(G) \to [0, nr-1]$ and define new vertex colouring $f'$ of $G$ in the following way: $f'(v) = \left\lfloor \frac{g'(v)}{n} \right\rfloor$. Then for each vertex $v \in V(G)$, $f'(v) \in [0, r-1]$. Without loss of generality, let us consider the edge $uv \in E(G)$ satisfying $g'(v) \leq g'(u)$.

If $ns \le g'(u) - g'(v) \le n(r-s)$, then

$$f'(v) + s = \left\lfloor \frac{g'(v)}{n} \right\rfloor + s = \left\lfloor \frac{g'(v)+ns}{n} \right\rfloor \le \left\lfloor \frac{g'(u)}{n} \right\rfloor = f'(u)$$

and, also

$$f'(u) = \left\lfloor \frac{g'(u)}{n} \right\rfloor \le \left\lfloor \frac{g'(v)+n(r-s)}{n} \right\rfloor = \left\lfloor \frac{g'(v)}{n} \right\rfloor + (r-s) = f'(v) + (r-s).$$

Thus the graph $G_{f',s}$ is isomorphic with a subgraph of the graph $G_{g',ns}$ which implies that $f'$ is $(\mathcal{P}, r, s)$-colouring of $G$. ■

**Corollary 2.** *If a graph is $(\mathcal{P}, r, s)$-colourable, then it is also $(\mathcal{P}, a, b)$-colourable with $a/b = r/s$ and $a, b$ are coprime.*

**Lemma 3.** *Let $r, s, a, b \in \mathbb{N}$. If a graph $G$ is $(\mathcal{P}, r, s)$-colourable, then it is $(\mathcal{P}, a, b)$-colourable for each $a/b \ge r/s$.*

***Proof.*** Suppose that a graph $G$ is $(\mathcal{P}, r, s)$-colourable and $a/b \ge r/s$. Let $t = nsn(s, b)$. Adjust the fractions $r/s$ and $a/b$ such that

$$\frac{r}{s} = \frac{rr'}{t} \quad , \quad \frac{a}{b} = \frac{aa'}{t}.$$

By Lemma 1, $G$ is $(\mathcal{P}, rr', t)$-colourable. Since $a/b \ge r/s$, we have $aa' \ge rr'$, thus $(\mathcal{P}, rr', t)$-colouring of $G$ is also its $(\mathcal{P}, aa', t)$-colouring. Then, by Lemma 1, the graph $G$ is $(\mathcal{P}, a, b)$-colourable. ■

The strong circular chromatic number is a refinement of the classical chromatic number, that is, for each finite graph $G$, $\chi(G) - 1 < \chi_c(G) \le \chi(G)$. We prove here an analogical statement for the strong circular $\mathcal{P}$-chromatic number.

**Theorem 4.** *Let $\mathcal{P}$ be graph property. Then, for each finite graph $G$,*

$$\chi_{\mathcal{P}}(G) - 1 < \chi_{c,\mathcal{P}}(G) \le \chi_{\mathcal{P}}(G).$$

***Proof.*** Since each $(\mathcal{P}, r, 1)$-colouring of a graph $G$ is also its $(\mathcal{P}, r)$-colouring, we have $\chi_{c,\mathcal{P}}(G) \le \chi_{\mathcal{P}}(G)$.

If $\chi_{\mathcal{P}}(G) - 1 \ge \chi_{c,\mathcal{P}}(G)$, then there exists a $(\mathcal{P}, r, s)$-colouring of $G$, for which $r/s \le \chi_{\mathcal{P}}(G) - 1$. Then, by Lemma 3, there exists $(\mathcal{P}, \chi_{\mathcal{P}}(G) - 1, 1)$-colouring of $G$ which is also its $(\mathcal{P}, \chi_{\mathcal{P}}(G) - 1)$-colouring — *a contradiction.* ■

Before showing that the strong circular $\mathcal{P}$-chromatic number exists and is rational for each finite graph, we prove that, in every $(\mathcal{P}, r, s)$-colouring of a graph $G$ with $\chi(G) = \frac{r}{s}$, each of $r$ colours is assigned to a vertex of $G$. The proof of the following lemma is inspired by the proof of Bondy and Hell in [1].

**Lemma 5.** *Let $G$ have $(\mathcal{P}, r, s)$-colouring $f$ with $gcd(r, s) = 1$ and $r > |\{f(v) : v \in V(G)\}|$. Then $G$ is $(\mathcal{P}, a, b)$-colourable with $a < r$ and $a/b < r/s$.*

***Proof.*** Suppose that a graph $G$ has a $(\mathcal{P}, r, s)$-colouring $f : V(G) \to [0, r-1]$ such that, in this colouring, at least one colour is not used; denote this colour by $s$. Recolour each vertex having the colour $2s$ with the colour $2s - 1$. By this recolouring, we obtain a colouring $f_2$ which satisfies $G_{f_2,s} \subseteq G_{f,s}$, hence, $f_2$ is also a $(\mathcal{P}, r, s)$-colouring of $G$. In the colouring $f_2$, the colour $2s$ is not assigned to a vertex of $G$, hence, each vertex coloured with $3s$ may be assigned with the colour $3s - 1$. The colouring $f_3$ obtained in this way is also a $(\mathcal{P}, r, s)$-colouring of $G$. Now, perform described recolouring for colours $2s, 3s, \ldots, \sigma s$, where $\sigma s \equiv 1 \pmod r$ (such a $\sigma$ exists because $gcd(r, s) = 1$). Note that the values $2s, 3s, \ldots, \sigma s$ are considered modulo $r$ and are pairwise different. The colouring $f_\sigma$ uses $r - \sigma$ colours. Let $F = \{s, 2s, \ldots, \sigma s\}$. Define the colouring $g : V(G) \to [0, r - \sigma - 1]$ in the following way: $g(v) = f_\sigma(v) - |\{x \in F : x < f_\sigma(v)\}|$.

Let $t := \frac{\sigma s - 1}{r}$. We show that the colouring $g$ is $(\mathcal{P}, r-\sigma, s-t)$-colouring of the graph $G$.

For each $i = 0, 1, \ldots, r-1$, consider the set $M_i = \{i, i+1, \ldots, i+s-1\} \subseteq [0, r-1]$ (where the value $r - 1$ is followed by 0). Each of the sets $M_i$, $i \neq 1$ contains exactly $t$ values which are not used in the colouring $f_\sigma$; the set $M_1$ contains $t+1$ such values. From this follows that, if $s \le |f(u)-f(v)| \le r-s$ for an edge $uv \in E(G)$ in the colouring $f$, then, in the colouring $g$, for the edge $uv$, we have $s-t \le |g(u)-g(v)| \le r-\sigma-(s-t)$. Hence, $G_{g,s-t} \subseteq G_{f,s}$. Moreover,

$$\frac{r-\sigma}{s-t} = \frac{r(r-\sigma)}{rs-(\sigma s-1)} = \frac{r(r-\sigma)}{s(r-\sigma)+1} < \frac{r}{s}.$$

■

Lemma 5 and Corollary 2 imply that the strong circular $\mathcal{P}$-chromatic number can be defined as the minimum of a finite set of rational numbers.

**Theorem 6.** *For strong circular $\mathcal{P}$-chromatic number of a simple graph $G$,*

$$\chi_{c,\mathcal{P}}(G) = \min\left\{\frac{r}{s} : \textit{ the graph } G \textit{ has a } (\mathcal{P}, r, s)\textit{-colouring and } r \le |V(G)|\right\}.$$

***Proof.*** By Corollary 2, when determining the strong circular $\mathcal{P}$-chromatic number of a graph, it is enough to consider those rational numbers $\frac{r}{s}$, for which $gcd(r, s) = 1$. Also, by Lemma 5, if the graph $G$ has a $(\mathcal{P}, r', s')$-colouring with $r' > |V(G)|$, then $G$ has also a $(\mathcal{P}, r, s)$-colouring with $r \le |V(G)|$ and $\frac{r}{s} < \frac{r'}{s'}$. This implies that

$$\chi_{c,\mathcal{P}}(G) = \inf\left\{\frac{r}{s} : \text{ the graph } G \text{ has a } (\mathcal{P}, r, s)\text{-colouring and } r \le |V(G)|\right\}.$$

Since this set is finite, we can change infimum by minimum. ■

Now let us remark that the strong circular $\mathcal{P}$-chromatic number is an monotone graph invariant.

**Lemma 7.** *Let $H$ be a subgraph of a graph $G$. Then for each hereditary additive graph property $\mathcal{P}$, $\chi_{c,\mathcal{P}}(H) \le \chi_{c,\mathcal{P}}(G)$.*

***Proof.*** By restricting the $(\mathcal{P}, r, s)$-colouring $f : V(G) \to [0, r-1]$ on the set $V(H)$, we obtain the $(\mathcal{P}, r, s)$-colouring of the graph $H$. ■

**Lemma 8.** *Let $\mathcal{P} \subseteq \mathcal{Q}$. Then $\chi_{c,\mathcal{P}}(G) \ge \chi_{c,\mathcal{Q}}(G)$.*

***Proof.*** Let a colouring $f : V(G) \to [0, r-1]$ of a graph $G$ be a $(\mathcal{P}, r, s)$-colouring. Then $G_{f,s} \in \mathcal{P}$. Since $\mathcal{P} \subseteq \mathcal{Q}$, we have that $G_{f,s} \in \mathcal{Q}$; thus, the colouring $f$ is also a $(\mathcal{Q}, r, s)$-colouring of $G$, and so $\chi_{c,\mathcal{Q}}(G) \le \chi_{c,\mathcal{P}}(G)$. ■

Let us denote by $\mathcal{P} \circ \mathcal{P}$ the class of all $(\mathcal{P}, 2)$-colourable graphs.

**Theorem 9.** *For a graph $G$ and an additive hereditary property $\mathcal{P}$ it holds:*

(1) $\chi_{c,\mathcal{P}}(G) = 1$ *if and only if* $G \in \mathcal{P}$.

(2) $\chi_{c,\mathcal{P}}(G) = 2$ *if and only if* $G \in (\mathcal{P} \circ \mathcal{P}) - \mathcal{P}$.

(3) $\chi_{c,\mathcal{P}}(G) > 2$ *if and only if* $G \notin \mathcal{P} \circ \mathcal{P}$.

***Proof.*** (1) If $\chi_{c,\mathcal{P}}(G) = 1$ then there is $(\mathcal{P}, 1, 1)$-colouring $f : V(G) \to \{0\}$ of $G$ such that $G_{f,1} \in \mathcal{P}$. Whereas $G_{f,1} \cong G$, that $G \in \mathcal{P}$. On the other hand if $G \in \mathcal{P}$, then if we colour all vertices of $G$ with the same colour, we obtain a colouring $f$, for which $G_{f,1} \cong G$, so $G_{f,1} \in \mathcal{P}$. Then $f$ is a $(\mathcal{P}, 1, 1)$-colouring of $G$ and $\chi_{c,\mathcal{P}}(G) = 1$.

(2) Suppose $1 < r/s < 2$ and $\chi_{c,\mathcal{P}}(G) = \frac{r}{s}$. Consider $(\mathcal{P}, r, s)$-colouring $f$ of a graph $G$ and arbitrary two adjacent vertices $u, v \in V(G)$. Then either $|f(u) - f(v)| < s$ or $|f(u) - f(v)| \geq s > r - s$. Therefore $G_{f,s} \cong G$. Then by (1.) $\chi_{c,\mathcal{P}}(G) = 1$ — a contradiction. This implies that if $\chi_{c,\mathcal{P}}(G) > 1$, then $\chi_{c,\mathcal{P}}(G) \geq 2$.

Let us assume that $\chi_{c,\mathcal{P}}(G) = 2$. Then from (1) it follows that $G \notin \mathcal{P}$. Consider some $(\mathcal{P}, 2, 1)$-colouring $f$ of $G$. Since the property $\mathcal{P}$ is hereditary, a subgraph of $G_{f,s}$ induced by vertices of colour 0 (or colour 1), has the property $\mathcal{P}$. Whereas $V(G_{f,1}) = V(G)$, so $G \in (\mathcal{P} \circ \mathcal{P}) \backslash \mathcal{P}$.

On the other hand if $G \in (\mathcal{P} \circ \mathcal{P}) \backslash \mathcal{P}$, then from (1) and previous considerations it follows that $\chi_{c,\mathcal{P}}(G) \geq 2$. Simultaneously vertices of $G$ can be divided into two classes $V_1, V_2$ such that $G[V_1] \in \mathcal{P}$ and $G[V_2] \in \mathcal{P}$. Then by colouring of vertices from $V_1$ with colour 0 and vertices from $V_2$ with colour 1 we obtain a $(\mathcal{P}, 2, 1)$-colouring of $G$. Therefore $\chi_{c,\mathcal{P}}(G) \leq 2$.

(3) Let $G \notin \mathcal{P} \circ \mathcal{P}$, then (by (2)) $G$ has no $(\mathcal{P}, 2, 1)$-colouring. ■

## 3. Strong Circular Chromatic Number of Complete Graphs

By Theorem 9 for the graph $K_n$, $n \in \mathbb{N}$ and the property $\mathcal{P}$ with $c(\mathcal{P}) = k$, $k \in \mathbb{N}$, we obtain:

- $\chi_{c,\mathcal{P}}(K_n) = 1$ if and only if $n \leq k + 1$.
- $\chi_{c,\mathcal{P}}(K_n) = 2$ if and only if $k + 2 \leq n \leq 2k + 2$.
- $\chi_{c,\mathcal{P}}(K_n) > 2$ if and only if $n \geq 2k + 3$.

As we have mentioned in the first chapter, for any additive and hereditary property $\mathcal{P}$ with completeness $c(\mathcal{P}) = k$ it holds: $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{I}_k$ and thus $\chi_{c,\mathcal{I}_k}(G) \leq \chi_{c,\mathcal{P}}(G) \leq \chi_{c,\mathcal{O}_k}(G)$ for every $G$. Therefore we will investigate strong circular $\mathcal{P}$-chromatic number of graphs for $\mathcal{P} = \mathcal{O}_k$ or $\mathcal{P} = \mathcal{I}_k$.

For every property $\mathcal{P}$ and graph $G$ it holds: $\chi_{\mathcal{P}}(G) \geq \frac{\omega(G)}{c(\mathcal{P})+1}$. We show, that $\frac{\omega(G)}{c(\mathcal{P})+1}$ is the lower bound for strong circular $\mathcal{P}$-chromatic number of graphs and simultaneously we prove, there is a graph property, for which this value is attained.

**Theorem 10.** $\chi_{c,\mathcal{O}_k}(K_n) = \lceil \frac{n}{k+1} \rceil$.

***Proof.*** The complete graph $K_n$ is $(\mathcal{O}_k, \lceil \frac{n}{k+1} \rceil, 1)$-colourable, because $\chi_{c,\mathcal{O}_k}(K_n) \leq \chi_{\mathcal{O}_k}(K_n) = \lceil \frac{n}{k+1} \rceil$.

Suppose that $K_n$ has a $(\mathcal{O}_k, r, s)$-colouring $f$, where $\frac{r}{s} \leq \lceil \frac{n}{k+1} \rceil$. Then each component of the graph $G_{f,s}$ has at most $k+1$ vertices, thus, the graph $G_{f,s}$ has at least $\lceil \frac{n}{k+1} \rceil$ components.

Consequently, any colouring of $K_n$ requires at least $s \cdot \lceil \frac{n}{k+1} \rceil$ colours. Hence, $r \geq s\lceil \frac{n}{k+1} \rceil$, which implies that $\frac{r}{s} \geq \lceil \frac{n}{k+1} \rceil$. ■

In the second chapter we have shown that if $K_n \subseteq G$, then $\chi_{c,\mathcal{P}}(G) \geq \chi_{c,\mathcal{P}}(K_n)$. Thus by evaluating $\chi_{c,\mathcal{I}_k}(K_n)$ we have also the lower bound for strong circular $\mathcal{P}$-chromatic number of graphs with clique number at least $n$ and properties $\mathcal{P}$ with $c(\mathcal{P}) = k$.

**Theorem 11.** *Let $n, k \in \mathbb{N}$, $n \geq 2k+3$. Then $\chi_{c,\mathcal{I}_k}(K_n) = \frac{n}{k+1}$.*

***Proof.*** For the graph $K_n$ with vertex set $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$, consider the colouring $f : V(G) \to [0, n-1]$ defined as follows: $f(v_i) = i$ for each $i = 0, 1, \ldots, n-1$. Then the graph $(K_n)_{f,k+1}$ (isomorphic to the circulant graph $C_n(1, 2, \ldots, k)$) has $\omega((K_n)_{f,k+1}) = k+1$. Therefore $(K_n)_{f,k+1}$ belongs to the property $\mathcal{I}_k$ and $f$ is a $(\mathcal{I}_k, n, k+1)$-colouring of $K_n$ (thus $\chi_{c,\mathcal{I}_k}(K_n) \leq \frac{n}{k+1}$).

Let $r, s \in \mathbb{N}$, $r \geq s$ and let a mapping $f : V(G) \to [0, r-1]$ be a $(\mathcal{I}_k, r, s)$-colouring of $K_n$. Consider the sets $V_j = \{v \in V(G) : f(v) \in [j, j+s-1]\}$ for $j = 0, 1, \ldots, r-1$, where the values $j, j+1, \ldots, j+s-1$ are taken modulo $r$. For each $j = 0, 1, \ldots, r-1$, the graph $G[V_j] \subseteq (K_n)_{f,s}$ is complete and, since $(K_n)_{f,s} \in \mathcal{I}_k$, we have $|V_j| \leq k+1$ for each $j = 0, \ldots, r-1$. Then there are at most $(k+1)r$ pairs $[v, V_j]$ such that $v \in V_j$. On the other hand, $K_n$ has $n$ vertices and each fixed vertex belongs to $s$ of the sets $V_j$. We conclude that there are $sn$ pairs $[v, V_j]$ such that $v$ belongs to $V_j$. Therefore, $sn \leq (k+1)r$, which implies that $\frac{n}{k+1} \leq \frac{r}{s}$. Since this argument holds for each $(\mathcal{I}_k, r, s)$-colouring of $K_n$, $\chi_{c,\mathcal{I}_k}(K_n) \geq \frac{n}{k+1}$. ■

The following statement is a direct consequence of Theorem 10 and Theorem 11 for the property $\mathcal{P}$, where $\mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{I}_k$.

**Corollary 12.** *For each property $\mathcal{P}$ with $c(\mathcal{P}) = k$,*

$$\frac{n}{k+1} \leq \chi_{c,\mathcal{P}}(K_n) \leq \left\lceil \frac{n}{k+1} \right\rceil.$$

**Theorem 13.** $\chi_{c,\mathcal{O}^{k+1}}(K_n) = \lceil \frac{n}{k+1} \rceil$.

***Proof.*** Let us suppose to the contrary that $\chi_{c,\mathcal{O}^{k+1}}(K_n) = \frac{r}{s} = t + \frac{r_0}{s}$, where $t \in \mathbb{Z}$ and $0 < r_0 < s$. From Corollary 12 we obtain that $\frac{n}{k+1} \leq \chi_{c,\mathcal{O}^{k+1}}(K_n) \leq \lceil \frac{n}{k+1} \rceil$, thus, in this case, $t < \frac{n}{k+1}$. Consider now the corresponding $(\mathcal{O}^{k+1}, r, s)$-colouring $f$ of $K_n$. This colouring is such that $(K_n)_{f,s} \in \mathcal{O}^{k+1}$. Hence, consider the proper circular vertex colouring $g : V((K_n)_{f,s}) \to [1, k+1]$. Let $i \in [1, k+1]$ be a colour and a vertex $v$ be coloured with $i$, so $g(v) = i$. Let us put $f(v) = \alpha$ and let $V_j = \{u \in V((K_n)_{f,s}) : f(u) \in [j, j+s-1]\}$ for $j = 0, \dots, r-1$ (where the values $j, j+1, \dots, j+s-1$ are taken modulo $r$). But now the sequence $V_\alpha \cup V_{\alpha+ts}, V_{\alpha+s}, V_{\alpha+2s}, \dots, V_{\alpha+(t-1)s}$ contains all vertices of $K_n$ (because $r_0 < s$). Moreover, in each of these sets, there is at most one vertex coloured with $i$ in the colouring $g$, because, except the set $V_\alpha \cup V_{\alpha+ts}$, all other sets induce a complete subgraph of the graph $(K_n)_{f,s}$. However, the considered vertex $v$ belongs to the set $V_\alpha \cup V_{\alpha+ts}$ and it is adjacent to all other vertices from this set; therefore, this set cannot contain any other vertex coloured with $i$ in the colouring $g$.

Hence it follows that $|g^{-1}(i)| \leq t$, and this argument can be used for each colour $i \in [1, k+1]$. Thus $n = |V((K_n)_{f,s})| = |g^{-1}([1, k+1])| \leq t(k+1)$, which implies $\frac{n}{k+1} \leq t$ — a contradiction. ∎

For the complete graph $K_n$, Theorems 11 and 13 imply that

- For each $\mathcal{P} : \mathcal{O}_k \subseteq \mathcal{P} \subseteq \mathcal{O}^{k+1}$ we have: $\chi_{c,\mathcal{P}}(K_n) = \lceil \frac{n}{k+1} \rceil$ and
- For each $\mathcal{P} : \mathcal{O}^{k+1} \subseteq \mathcal{P} \subseteq \mathcal{I}_k$ we have: $\chi_{c,\mathcal{P}}(K_n) \in \langle \frac{n}{k+1}, \lceil \frac{n}{k+1} \rceil \rangle$.

**Corollary 14.** *For each property $\mathcal{P}$ and each finite graph $G$,*

$$\chi_{c,\mathcal{P}}(G) \geq \frac{\omega(G)}{c(\mathcal{P})+1}.$$

***Proof.*** If $\omega(G) = d$, then $K_d \subseteq G$. Then $\chi_{c,\mathcal{P}}(G) \geq \chi_{c,\mathcal{P}}(K_d) \geq \frac{d}{c(\mathcal{P})+1}$ by Theorem 7 and Corollary 12. ∎

We shall denote by $G_a^b, a \geq b$ the graph with the set of vertices $\{0, \dots, a-1\}$ and edges $\{ij : b \leq |i-j| \leq a-b\}$. In [1, 8] it was shown that for any pair of integers $a, b$ with $a \geq 2b$ and $gcd(a,b) = 1$, the graph $G_a^b$ is vertex critical and circulal chromatic number $\chi_c(G_a^b) = \frac{a}{b}$. We shall use this fact in the proof of Theorem 15.

This statement is an answer to the question if any rational number from $\langle \frac{n}{k+1}, \lceil \frac{n}{k+1} \rceil \rangle$ is the strong circular $\mathcal{P}$-chromatic number for some property $\mathcal{P}$ and finite graph $G$.

**Theorem 15.** *Let $k \in \mathbb{N}$ and $n \geq 2(k+1)$. For any $\frac{r}{s} \in \langle \frac{n}{k+1}, \lceil \frac{n}{k+1} \rceil \rangle$, $r \leq n$ there is a graph property $\mathcal{P}$ such that $\chi_{c,\mathcal{P}}(K_n) = \frac{r}{s}$ and $c(\mathcal{P}) = k$.*

***Proof.*** Let $n$ be positive integer and $n \geq 2(k+1)$. Consider finite set of rational numbers $M = \{\frac{r}{s} \mid \frac{n}{k+1} \leq \frac{r}{s} \leq \lceil \frac{n}{k+1} \rceil \wedge\ r \leq n\}$. Put $|M| = t+1$ and sort its elements in increasing order: $\frac{n}{k+1} = \frac{r_0}{s_0} < \frac{r_1}{s_1} < \cdots < \frac{r_i}{s_i} < \cdots < \frac{r_t}{s_t} = \lceil \frac{n}{k+1} \rceil$.

We shall provide a property $\mathcal{P}_i$ for each $\frac{r_i}{s_i} \in M$ such that $\chi_{c,\mathcal{P}}(K_n) = \frac{r_i}{s_i}$ and $c(\mathcal{P}_i) = k$.

Put $\mathcal{P}_0 = \mathcal{I}_k$ by Theorem 11. Also by Theorem 10, put $\mathcal{P}_t = O_k$ (or $O^{k+1}$ by Theorem 13).

If $t \geq 2$, then for $i = 1, \ldots, t-1$ we define

$$\mathcal{P}_i := \mathcal{I}_k - \{\ G \mid (\exists\ j < i)(\exists\ f : V(K_n) \to [0, r_j - 1]) : \chi_c(\overline{(K_n)_{f,s_j}}) = \tfrac{r_j}{s_j} \text{ and there is a component } H \subseteq (K_n)_{f,s_j} : H \subseteq G\ \}.$$

Note that each property $\mathcal{P}_i$ is hereditary and additive.

First we show that if $(K_n)_{f,s_j} \in \mathcal{I}_k$, with $s_j \geq 2$, then graph $(K_n)_{f,s_j}$ is connected. We consider $(\mathcal{P}, r_j, s_j)$-colouring $f$ of $K_n$ and denote $V_i = \{v \in V(K_n) \mid f(v) \in [i, i+s_j-1]\}$ for $i = 0, \ldots, r-1$ (values $i, \ldots, i+s_j-1$ are reduced modulo $r_j$). Graph $G[V_i]$ is complete, therefore $|V_i| \leq k+1$ for any $i = 0, \ldots, r_j - 1$.

If $(K_n)_{f,s_j}$ is disconnected, then there are $a, b \in [0, r_j - 1]$ and $s_j + 1 \leq |a-b| \leq r_j - (s_j+1)$ such that $V_a = \emptyset$ and $V_b = \emptyset$. We shall show, that there is no set $V_a$ such that $V_a = \emptyset$. For the proof by contradiction we suppose, there exists an empty set $V_a$ for some $a \in [0, r_j - 1]$. Then $n \leq \lfloor r_j/s_j \rfloor (k+1)$ and so $n/(k+1) \leq \lfloor r_j/s_j \rfloor = \lfloor n/(k+1) \rfloor$ — a contradiction.

Therefore we can write

$$\mathcal{P}_i := \mathcal{I}_k - \{\ G \mid (\exists\ j < i)(\exists\ f : V(K_n) \to [0, r_j - 1]) : (K_n)_{f,s_j} \in \mathcal{I}_k\ \wedge\ \chi_c(\overline{(K_n)_{f,s_j}}) = \tfrac{r_j}{s_j}\ \wedge\ G \supseteq (K_n)_{f,s_j}\ \}.$$

As $\mathcal{P}_i \subseteq \mathcal{I}_k$, we have $c(\mathcal{P}_i) \leq k$. Next we shall show $c(\mathcal{P}_i) = k$ and thus $K_{k+1} \in \mathcal{P}_i$. If $K_{k+1} \notin \mathcal{P}_i$, then there exists a colouring $f$ such that $K_{k+1} \supseteq (K_n)_{f,s_j}$. It follows that $k+1 \geq n \geq 2(k+1)$ — a contradiction.

Finally we shall prove, for each $\frac{r_i}{s_i} \in M$ there is a colouring $f_i$ such that $\chi_c(\overline{(K_n)_{f,s_i}}) = \frac{r_i}{s_i}$. We shall denote by $U_a$ the set of vertices of $K_n$ coloured by $a$. We shall construct a colouring $f_i : V(K_n) \to [0, r_i - 1]$ as follows: we colour vertices of $K_n$ such that $|U_a| = \lfloor \frac{(a+1)n}{r} \rfloor - \lfloor \frac{an}{r} \rfloor$, for

$a = 0, \ldots, r_i - 1$. Then $|U_a| \in \{\lfloor \frac{n}{r_i} \rfloor, \lfloor \frac{n}{r_i} \rfloor + 1\}$ and $|V_a| = \sum_{p=0}^{s_i-1} |U_{a+p}| = \sum_{p=0}^{s_i-1} (\lfloor \frac{(a+p+1)n}{r_i} \rfloor - \lfloor \frac{(a+p)n}{r_i} \rfloor) = \lfloor \frac{(a+s_i)n}{r_i} \rfloor - \lfloor \frac{an}{r_i} \rfloor \leq \lfloor \frac{s_i n}{r_i} \rfloor + 1 \leq k+1$. Because $G_{r_i}^{s_i} \subseteq \overline{(K_n)_{f,s_j}}$ that $\chi_c((\overline{(K_n)_{f,s_j}}) = \frac{r_i}{s_i}$. ■


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