

## ON DOUBLY LIGHT VERTICES IN PLANE GRAPHS

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### Abstract

A vertex is said to be doubly light in a family of plane graphs if its degree and sizes of neighbouring faces are bounded above by a finite constant. We provide several results on the existence of doubly light vertices in various families of plane graph.

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### 1. INTRODUCTION

Throughout this paper we consider connected plane graphs without loops or multiple edges. For a plane graph  $G$ ,  $V = V(G)$ ,  $E = E(G)$  and  $F = F(G)$  denotes the set of its vertices, edges and faces, respectively. A  $k$ -vertex ( $k$ -face) will stand for a vertex (a face) of degree  $k$ , a  $\geq k$ -vertex/ $\leq k$ -vertex ( $\geq k$ -face/ $\leq k$ -face) for those of degree at least  $k$ /at most  $k$ . The *minimum edge weight* of a graph  $G$  is the number  $w(G) = \min_{uv \in E(G)} (\deg_G(u) + \deg_G(v))$ ; the *minimum dual edge weight*  $w^*(G)$  of  $G$  is the minimum edge weight of the dual graph of  $G$  (that is, the minimum sum of sizes of adjacent faces of  $G$ ).

The classical consequence of Euler's theorem states that every plane graph contains a vertex of degree at most 5. This result provides no information on sizes of faces surrounding that vertex, but, Lebesgue [3] proved

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that, in a 3-connected plane graph, there exist an  $\leq 5$ -vertex incident with at least two  $\leq 6$ -faces. In general, it is not true that every 3-connected plane graph contains a vertex of degree  $\leq 5$  incident *only* with faces of size bounded above by a fixed constant: in the graph of  $n$ -prism or  $n$ -antiprism, each vertex is incident with an  $n$ -face. However, by [3], each 3-connected plane graph of minimum degree 5 contains a 5-vertex incident with four 3-faces and the remaining fifth face is of size at most 5 (this bound is best possible).

These examples show that if the minimum vertex degree is large enough, plane graphs contain small degree vertices incident with small faces (in the sequel, they are referred as doubly light vertices). The similar effect is obtained also when restricting the minimum face size: from [1], it follows that each plane graph of minimum face size 5 contains a 3-vertex such that the sum of sizes of incident faces is at most 17. On the other hand, the graph of  $n$ -prism shows that the minimum face size at least 4 is not enough to ensure the existence of doubly light vertices. Nevertheless, plane graphs contain doubly light vertices if their minimum dual edge weight is large enough: from [2], it follows that each 3-connected plane graph of minimum dual edge weight at least 11 (or 12 or 13, respectively) contains an  $\leq 5$ -vertex with all incident faces of sizes at most 22 (or 15 or 13, respectively), and if the dual edge weight is at least 10, the size of neighbouring faces of such a vertex is at most 41 (see [3]).

The aim of this paper is to explore in detail other conditions that enforce doubly light vertices in plane graphs. We will show that these vertices are also found in 3-connected plane graphs with sufficiently high minimum edge weight, or under certain requirements on all four constraints on vertex degree, face size, edge and dual edge weights.

For proving the structural results in this paper, we use the approach called the Discharging Method; its earliest application can be found in [4] (see also [5] for review of applications in graph theory and combinatorial geometry). Each theorem is proved by contradiction: we assume the existence of a hypothetical counterexample  $G$ . Now, each vertex and each face of  $G$  is assigned with the *initial charge*  $c$ ; in this paper, we use the following assignments:

$$\begin{aligned} c(v) &= \deg_G(v) - 4 \text{ for each vertex } v \in V, \\ c(\alpha) &= \deg_G(\alpha) - 4 \text{ for each face } \alpha \in F. \end{aligned} \tag{1}$$

$$\begin{aligned} c(v) &= \deg_G(v) - 6 \text{ for each vertex } v \in V, \\ c(\alpha) &= 2 \deg_G(\alpha) - 6 \text{ for each face } \alpha \in F. \end{aligned} \tag{2}$$

By Euler's formula, we obtain that  $\sum_{x \in V \cup F} c(x) = -8$  or  $-12$  for the first and the second assignments of initial charges, respectively.

Next, the initial charges of elements of  $G$  are locally redistributed in such a way that the total sum of all charges remains the same (hence, negative). This redistribution is performed by a set of discharging rules which specify the way of transfer of a charge from one element to another one in specific configurations of vertices and faces. Finally, by a case analysis, it is shown that, after discharging, each element of  $G$  has a nonnegative final charge  $c^*$ ; thus, the total sum of final charges is also nonnegative, which gives a contradiction.

## 2. RESULTS

Firstly, we explore the family of plane graphs which is "between" (according to inclusion ordering) families of plane graphs of minimum degree  $\geq 4$  and 5:

**Theorem 2.1.** *Each connected plane graph of minimum degree  $\geq 4$  and minimum edge weight  $\geq 9$  contains a vertex of degree  $\leq 5$  incident only with  $\leq 10$ -faces. The bound 10 is best possible.*

**Proof.** We use the Discharging Method with the initial charge assignment (2) and the following discharging rules:

**Rule.** Each  $\geq 11$ -face sends 2 to each incident 4-vertex and 1 to each incident 5-vertex.

We check the nonnegativity of final charges; since  $\geq 6$ -vertices and  $\leq 10$ -faces are not influenced by the discharging rule, it is enough to check 4- and 5-vertices, and  $\geq 11$ -faces. Moreover, in a counterexample, each 4- and 5-vertex is incident with an  $\geq 11$ -face from which it receives, by Rule, the charge that exactly neutralizes its initial negative charge  $-2$  or  $-1$ .

Let  $\alpha$  be an  $r$ -face,  $r \geq 11$  and let  $p, q$  be the number of 4- and 5-vertices incident with  $\alpha$ , respectively. Since the minimum edge weight of  $G$  is at least 9, we have  $p \leq \lfloor \frac{r}{2} \rfloor$  and  $p + q \leq r$ . Thus  $c^*(\alpha) \geq 2r - 6 - 2p - q = 2r - 6 - (p + q) - p \geq 2r - 6 - r - \lfloor \frac{r}{2} \rfloor \geq 0$  as  $r \geq 11$ .

Consider the graph of the Archimedean polytope of type  $(4, 6, 10)$  (which is the plane cubic graph such that each its vertex is incident with a 4-, 6- and 10-face) and the bipartition  $(B, W)$  of its vertex set. Into each 4-face, insert a diagonal connecting two vertices from  $W$ , and, into each 6-face, insert a

new vertex and join it by new edges with all vertices on the face boundary. The resulting plane graph is of minimum degree 4, minimum edge weight 9 and each its 4- and 5- vertex is incident with a 10-face. Thus, the bound 10 is best possible. ■

Similar results hold (with smaller bounds on face sizes) for plane graphs of minimum degree  $\geq 4$  and higher minimum edge weight:

**Theorem 2.2.** *Each connected plane graph of minimum degree  $\geq 4$  and minimum edge weight  $\geq 10$  contains a vertex of degree  $\leq 5$  incident only with  $\leq 5$ -faces.*

**Proof.** The proof goes essentially in the same way as for the previous theorem, the only difference is in the analysis of the final charge of big faces: if  $p, q$  are numbers of 4- and 5-vertices incident with an  $r$ -face  $\alpha$ , then  $p \leq \lfloor \frac{r-q}{2} \rfloor$  (because, in the counterexample, each 4-vertex is incident only with  $\geq 6$ -vertices) and  $p + q \leq r$ . Thus  $c^*(\alpha) \geq 2r - 6 - 2p - q \geq 2r - 6 - 2 \lfloor \frac{r-q}{2} \rfloor - q \geq 2r - 6 - 2 \frac{r-q}{2} - q = r - 6 \geq 0$  as  $r \geq 6$ . ■

**Theorem 2.3** *Each connected plane graph of minimum degree  $\geq 4$  and minimum edge weight  $\geq 11$  contains a vertex of degree  $\leq 5$  incident only with 3-faces.*

**Proof.** We use the Discharging Method with the initial charge assignment (2) and the following discharging rules:

**Rule 1.** Each  $\geq 4$ -face distributes its charge equally among incident 4- and 5-vertices.

**Rule 2.** Each  $\geq 7$ -vertex sends  $\frac{1}{7}$  to each adjacent 4-vertex.

**Rule 3.** Let  $[xyz]$  be a 3-face,  $y$  be an  $\geq 7$ -vertex,  $x$  be a 4-vertex and  $z$  be an  $\geq 5$ -vertex. Then  $y$  sends additional  $\frac{1}{14}$  to  $x$ .

We check the nonnegativity of final charges; by the formulation of discharging rules, it is enough to check only 4-, 5- and  $\geq 7$ -vertices. Each 5-vertex of the counterexample graph must be incident with an  $\geq 4$ -face and receives from this face  $\geq 1$  (note that, since the edge weight is  $\geq 11$ , there is at most  $\lfloor \frac{r}{2} \rfloor \leq 5$ -vertices incident with an  $r$ -face; hence, the contribution by Rule 1 is  $\geq \frac{2r-6}{\lfloor \frac{r}{2} \rfloor} \geq \frac{2 \cdot 4 - 6}{\lfloor \frac{4}{2} \rfloor} \geq 1$ ), so its final charge is nonnegative.

Let  $x$  be a 4-vertex. Then again,  $x$  must be incident with an  $\geq 4$ -face and adjacent to four  $\geq 7$ -vertices. If  $x$  is incident with at least two  $\geq 4$ -faces, then  $c^*(x) \geq -2 + 2 \cdot 1 = 0$ ; otherwise,  $x$  is incident with three 3-faces, and by Rules 1, 2 and 3,  $c^*(x) \geq -2 + 1 + 4 \cdot \frac{1}{7} + 6 \cdot \frac{1}{14} = 0$ . ■

Next theorem shows that, for the existence of doubly light vertices in plane graphs, one can consider also plane graphs of minimum degree at least 3 provided the minimum edge weight is large enough:

**Theorem 2.4.** *Each connected plane graph of minimum degree  $\geq 3$  and minimum edge weight  $\geq 9$  contains a vertex of degree  $\leq 5$  incident only with  $\leq 11$ -faces.*

**Proof.** We use the Discharging Method with the initial charge assignment (2) and the following discharging rules:

**Rule.** Each  $\geq 12$ -face sends  $6 - i$  to each incident  $i$ -vertex,  $i \in \{3, 4, 5\}$ .

We check the nonnegativity of final charges; since  $\geq 6$ -vertices and  $\leq 11$ -faces are not influenced by the discharging rule, it is enough to check  $\leq 5$ -vertices, and  $\geq 12$ -faces. In the counterexample graph, each  $i$ -vertex,  $i \in \{3, 4, 5\}$  is incident with an  $\geq 12$ -face from which it receives, by Rule, the charge that exactly neutralizes its initial negative charge  $i - 6$ .

Let  $\alpha$  be an  $r$ -face,  $r \geq 12$  and let  $p, q, s$  be the number of 3-, 4- and 5-vertices incident with  $\alpha$ , respectively. Since the minimum edge weight of  $G$  is at least 9, we have  $p \leq \lfloor \frac{r-q-s}{2} \rfloor$  and  $q \leq \lfloor \frac{r-2p}{2} \rfloor$ . Thus  $c^*(\alpha) \geq 2r - 6 - 3p - 2q - s = (2r - 6) - (2p + q + s) - (p + q) \geq (2r - 6) - \left( 2 \lfloor \frac{r-q-s}{2} \rfloor + \lfloor \frac{r-2p}{2} \rfloor + s \right) - (p + q) \geq (2r - 6) - \left( r - q - s + \frac{r}{2} - \frac{2p}{2} + s \right) - (p + q) = (2r - 6) - \left( \frac{3}{2}r - (p + q) \right) - (p + q) = \frac{r}{2} - 6 \geq 0$  as  $r \geq 12$ . ■

Note that even if the minimum edge weight is high, connected plane graphs need not contain doubly light vertices (an example is the star graph  $K_{1,n}$ ); but, for 2-connected plane graphs, we obtain the following

**Theorem 2.5.** *Each 2-connected plane graph of minimum edge weight  $\geq 11$  ( $12, 13, 14, 17$  or  $26$ , respectively) contains a vertex of degree  $\leq 5$  incident only with  $\leq 16$ - ( $\leq 14, \leq 12, \leq 10, \leq 8, \leq 6$ -) faces.*

**Proof.** We will handle all particular instances of the theorem simultaneously, using the Discharging Method with the initial charge assignment (2) and the following common discharging rules:

**Rule 1.** Each  $\geq 5$ -face distributes its charge equally among all incident  $\leq 5$ -vertices.

**Rule 2.** Each  $\geq 7$ -vertex  $x$  distributes its charge equally among all adjacent  $\leq 5$ -vertices.

From the formulation of discharging rules, it is enough to check only final charges of  $\leq 5$ -vertices. Given a counterexample  $G$  of minimum edge weight  $w$  ( $w \geq 11$ ), each  $d$ -vertex  $x$ ,  $d \in \{2, 3, 4, 5\}$  is adjacent to  $d$  vertices of degree  $\geq w - d$  and incident to at least one big  $r$ -face ( $r$  being greater than values described by theorem). Then, by Rules 1 and 2,  $c^*(x) \geq d - 6 + d \frac{w-d-6}{w-d} + \frac{2r-6}{\lfloor \frac{r}{2} \rfloor}$  (note that, due to the fact that  $w \geq 11$ , no two  $\leq 5$ -vertices are adjacent and an  $r$ -face is incident to at most  $\lfloor \frac{r}{2} \rfloor$  such vertices).

Let  $f(d, r, w) = d - 6 + d \frac{w-d-6}{w-d} + \frac{2r-6}{\lfloor \frac{r}{2} \rfloor}$  and  $f^*(d, r, w) = d - 6 + d \frac{w-d-6}{w-d} + \frac{2r-6}{\frac{r}{2}} = \frac{2(rw(d-1) - rd(d+2) - 6(w-d))}{(w-d)r}$ . It is not hard to check that, for  $w \geq 11$  and  $d \in \{2, 3, 4\}$ ,  $f(d+1, r, w) > f(d, r, w)$  holds. Thus, for ensuring that  $c^*(x) \geq 0$ , it is enough to check the nonnegativity of  $f(2, r, w)$ . Although  $f(2, r, w)$  is not increasing in  $r$ , we can see that, for fixed  $w$ , there exists  $r_w$  such that  $f(2, r, w) \geq 0$  for all  $r > r_w$ : using the estimation  $f(2, r, w) \geq f^*(2, r, w)$  and the fact that  $f^*$  is increasing in  $w$  (note that  $\frac{\partial f^*}{\partial w} = \frac{6d}{(d-w)^2} > 0$  for  $d \in \{2, 3, 4, 5\}$  and  $w \geq 11$ ), we obtain that

- if  $w \geq 11$  then  $f^*(2, r, w) \geq f^*(2, r, 11) = \frac{2(r-18)}{3r} \geq 0$  for  $r \geq 18$ ; furthermore,  $f(2, 17, 11) = \frac{1}{6} > 0$ ,  $f(2, 16, 11) = -\frac{1}{12}$ , thus,  $r_{11} = 16$ ,
- if  $w \geq 12$  then  $f^*(2, r, w) \geq f^*(2, r, 12) = \frac{4(r-15)}{5r} \geq 0$  for  $r \geq 15$ ; furthermore,  $f(2, 14, 12) = -\frac{2}{35}$ , thus,  $r_{12} = 14$ ,
- if  $w \geq 13$  then  $f^*(2, r, w) \geq f^*(2, r, 13) = \frac{2(5r-66)}{11r} \geq 0$  for  $r \geq 13$ ; furthermore,  $f(2, 12, 13) = -\frac{1}{11}$ , thus,  $r_{13} = 12$ ,
- if  $14 \leq w \leq 16$  then  $f^*(2, r, w) \geq f^*(2, r, 14) = \frac{r-12}{r} \geq 0$  for  $r \geq 12$ ; furthermore,  $f(2, 11, 14) = \frac{1}{5}$ ,  $f(2, 10, 14) = -\frac{1}{5}$ ,  $f(2, 11, 15) = \frac{18}{65}$ ,  $f(2, 10, 15) = -\frac{8}{65}$ ,  $f(2, 11, 16) = \frac{12}{35}$ ,  $f(2, 10, 16) = -\frac{2}{35}$ , thus,  $r_{14} = r_{15} = r_{16} = 10$ ,

- if  $17 \leq w \leq 25$  then  $f^*(2, r, w) \geq f^*(2, r, 17) = \frac{6(r-10)}{5r} \geq 0$  for  $r \geq 10$ ; furthermore,  $f(2, 9, w) > 0$  and  $f(2, 8, w) < 0$  for mentioned values of  $w$ , thus,  $r_{17} = \dots = r_{25} = 8$ ,
- if  $w \geq 26$  then  $f^*(2, r, w) \geq f^*(2, r, 26) = \frac{3(r-8)}{2r} \geq 0$  for  $r \geq 8$ ; furthermore,  $f(2, 7, 26) = \frac{1}{6}$  and  $f(2, 6, 26) = -\frac{1}{2}$ , thus,  $r_w = 6$  for  $w \geq 26$ . ■

As noted before, the condition of minimum degree at least 4 is not sufficient for existence of doubly light vertices in plane graphs (as seen from the graph of an  $n$ -antiprism); but, under additional requirement of slightly higher dual edge weight, we obtain

**Theorem 2.6.** *Each connected plane graph of minimum degree  $\geq 4$  and minimum dual edge weight  $\geq 7$  contains a 4-vertex incident only with  $\leq 7$ -faces; the bound 7 is best possible.*

**Proof.** We use the Discharging Method with the initial charge assignment (1) and the following discharging rules:

**Rule 1.** Each  $\geq 5$ -face distributes its charge equally among all incident vertices.

Let  $\bar{c}(x)$  denote the charge of a vertex  $x$  after application of Rule 1.

**Rule 2.** Each vertex  $x$  distributes its charge  $\bar{c}(x)$  equally among all incident 3-faces.

Observe that, due to the condition of minimum dual edge weight  $\geq 7$ , each  $d$ -vertex is incident with at most  $\lfloor \frac{d}{2} \rfloor$  3-faces; consequently, its contribution to a 3-face by Rule 2 is at least  $\frac{d-4}{\lfloor \frac{d}{2} \rfloor} \geq \frac{1}{2}$  if  $d \geq 5$ .

We check the nonnegativity of final charges; according to the formulation of discharging rules, it is enough to consider only 3-faces.

*Case 1.* Let  $\alpha$  be a 3-face incident with at least two  $\geq 5$ -vertices. Then, by Rule 2,  $c^*(\alpha) \geq -1 + 2 \cdot \frac{1}{2} = 0$ .

*Case 2.* Let  $\alpha$  be a 3-face incident with exactly one  $\geq 5$ -vertex  $z$ ; let  $x, y$  be 4-vertices incident with  $\alpha$ . Note that each of  $x, y$  is incident with at most two 3-faces, and with at least one  $\geq 8$ -face. Thus,  $\bar{c}(x) \geq \frac{8-4}{8} = \frac{1}{2}$  (similarly for  $\bar{c}(y)$ ) and, by Rules 1 and 2,  $c^*(\alpha) \geq -1 + \frac{1}{2} + 2 \cdot \frac{1}{2} = 0$ .

*Case 3.* Let  $\alpha$  be a 3-face incident only with 4-vertices; then each of them is incident with an  $\geq 8$ -face. Now, if at least two of them are incident with exactly one 3-face, then  $c^*(\alpha) \geq -1 + 2 \cdot \frac{1}{1} = 0$ . Hence, assume that at least two of vertices incident with  $\alpha$  are incident also with 3-faces different from  $\alpha$ . If there exists a vertex of  $\alpha$  which is incident with at least two  $\geq 8$ -faces, then  $c^*(\alpha) \geq -1 + \frac{2 \cdot \frac{1}{2}}{2} + 2 \cdot \frac{1}{2} = 0$ . Hence, assume that each vertex of  $\alpha$  is incident with exactly one  $\geq 8$ -face. Then there must be a vertex on  $\alpha$  that is incident with no other 3-face, and we obtain  $c^*(\alpha) \geq -1 + \frac{1}{1} + 2 \cdot \frac{1}{2} = 0$ .

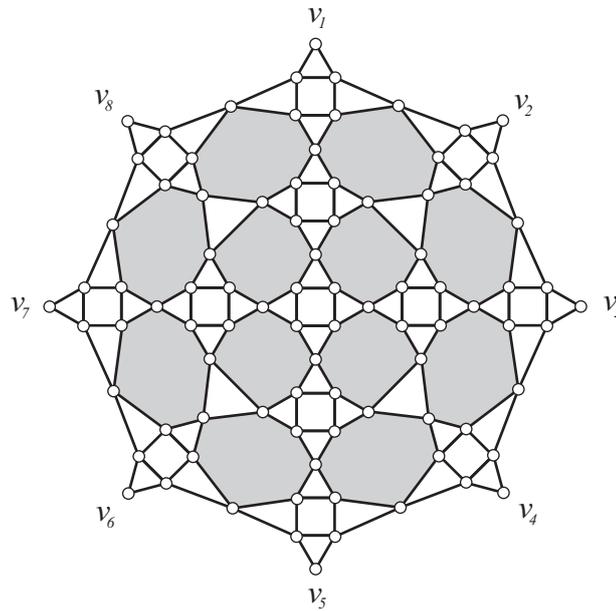


Figure 1

Consider two mirror copies  $G, G'$  of the graph on the Figure 1 drawn on two hemispheres  $\mathcal{H}, \mathcal{H}'$  such that vertices  $v_1, \dots, v_8$  and  $v'_1, \dots, v'_8$  lie on equators of  $\mathcal{H}$  and  $\mathcal{H}'$ , respectively. Now, glue  $\mathcal{H}$  and  $\mathcal{H}'$  into the sphere  $\mathcal{S}$  and identify the corresponding vertices  $v_i, v'_i$  for  $i = 1, \dots, 8$ . The resulting graph is drawn on  $\mathcal{S}$  without crossings, hence, it can be transformed to a plane graph which is 4-regular, has minimum dual edge weight 7 and each its 4-vertex is incident with a 7- or 8-face; thus, the bound 7 of this theorem is best possible. ■

Similarly, with the condition on minimum face size being at least 4, one cannot enforce the existence of doubly light vertices in plane graphs (see the graph of  $n$ -sided prism), but, they appear under additional condition on higher edge weight:

**Theorem 2.7.** *Each connected plane graph of minimum face size  $\geq 4$  and edge weight  $\geq 7$  contains a 3-vertex incident only with  $\leq 6$ -faces; the bound 6 is best possible.*

**Proof.** We use the Discharging Method with the initial charge assignment (1) and the following discharging rule:

**Rule.** Each  $\geq 7$ -face distributes its charge equally among incident 3-vertices.

We check the nonnegativity of final charges; due to formulation of discharging rule, it is enough to check only 3-vertices. Each 3-vertex must be incident with a  $d$ -face,  $d \geq 7$ ; as edge weight of the counterexample graph is at least 7, each  $d$ -face is incident with at most  $\lfloor \frac{d}{2} \rfloor$  3-vertices. Consequently, each  $d$ -face sends at least  $\frac{d-4}{\lfloor \frac{d}{2} \rfloor}$  to an incident 3-vertex. Now, for  $d \geq 7$ ,  $\frac{d-4}{\lfloor \frac{d}{2} \rfloor} \geq \frac{7-4}{\lfloor \frac{7}{2} \rfloor} = 1$ , so the contribution of an  $\geq 7$ -face to an incident 3-vertex is sufficient to make its final charge nonnegative.

Consider two mirror copies  $G, G'$  of the graph on the Figure 2 and, in the similar way as in the previous proof, glue them (by identifying the corresponding vertices  $v_i, v'_i$  for  $i = 1, \dots, 8$ ) to a graph embedded in the sphere. This graph can be transformed to the graph which is plane, has minimum face size 4, minimum edge weight 7 and each its 3-vertex is incident with a 6-face; thus, the bound 6 of this theorem is best possible.

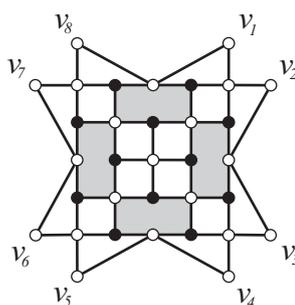


Figure 2



Finally, we show that if the minimum edge weight and dual edge weight is at least 8, the condition on minimum vertex degree or face size being at least 4 can be dropped:

**Theorem 2.8.** *Each connected plane graph of minimum degree  $\geq 3$ , minimum edge weight  $\geq 8$  and minimum dual edge weight  $\geq 8$  contains a  $\leq 5$ -vertex incident only with  $\leq 11$ -faces.*

**Proof.** We use the Discharging Method with the initial charge assignment (2) and the following discharging rules (here, a face of size at least 12 is called a big face):

**Rule 1.** Each  $\geq 4$ -face distributes its charge equally among all incident  $\leq 5$ -vertices.

Let  $\bar{c}(x)$  denote the charge of a vertex  $x$  after application of Rule 1.

**Rule 2.** Each vertex  $x$  distributes its charge  $\bar{c}(x)$  equally among all adjacent 3-vertices.

We check the nonnegativity of final charges; according to the formulation of discharging rules, it is enough to consider only  $\leq 5$ -vertices.

*Case 1.* Let  $x$  be a 5-vertex. Then  $x$  is incident with at least one big face; moreover, of the remaining four neighbouring faces, either at least two are  $\geq 5$ -faces, or at least three of them are  $\geq 4$ -faces (otherwise, in the neighbourhood of  $x$ , there exist two adjacent faces with size sum at most 7). Thus, by Rule 1,  $c^*(x) \geq -1 + \frac{2 \cdot 12 - 6}{12} + 3 \cdot \frac{2 \cdot 4 - 6}{4} = 2 > 0$  (as a consequence, we obtain that a 5-vertex always sends, by Rule 2, at least  $\frac{2}{5}$ ).

*Case 2.* Let  $x$  be a 4-vertex. Again,  $x$  is incident with at least one big face and, except this face, either with at least one  $\geq 5$ -face or with at least two  $\geq 4$ -faces; we obtain  $c^*(x) \geq -2 + \frac{2 \cdot 12 - 6}{12} + \frac{2 \cdot 5 - 6}{5} = \frac{3}{10} > 0$ .

*Case 3.* Let  $x$  be a 3-vertex. Note that all neighbours of  $x$  are  $\geq 5$ -vertices. As before,  $x$  is incident with a big face  $\alpha$ ; denote the remaining incident faces by  $\beta, \gamma$ .

*Case 3.1.* Let one of  $\beta, \gamma$  – say,  $\beta$  – be a 3-face; then  $\gamma$  is an  $\geq 5$ -face.

*Case 3.1.1.* If all neighbours of  $x$  are  $\geq 6$ -vertices, then, by Rule 1,  $c^*(x) \geq -3 + \frac{2 \cdot 5 - 6}{5 - 2} + \frac{2 \cdot 12 - 6}{12 - 2} = \frac{2}{15} > 0$ .

*Case 3.1.2.* If  $x$  is adjacent to exactly one 5-vertex, then  $\gamma$  and  $\alpha$  are incident with at least one  $\geq 6$ -vertex; hence, by Rules 1 and 2,  $c^*(x) \geq -3 + \frac{2 \cdot 5 - 6}{5 - 1} + \frac{2 \cdot 12 - 6}{12 - 1} + \frac{2}{5} = \frac{2}{55} > 0$ .

*Case 3.1.3.* If  $x$  is adjacent to precisely two 5-vertices, then, by Rules 1 and 2,  $c^*(x) \geq -3 + \frac{2 \cdot 5 - 6}{5} + \frac{2 \cdot 12 - 6}{12} + 2 \cdot \frac{2}{5} = \frac{1}{10} > 0$ .

*Case 3.1.4.* Let  $x$  be adjacent only to 5-vertices. Then, by Rules 1 and 2,  $c^*(x) \geq -3 + \frac{2 \cdot 5 - 6}{5} + \frac{2 \cdot 12 - 6}{12} + 3 \cdot \frac{2}{5} = \frac{1}{2} > 0$ .

*Case 3.2.* Let one of  $\beta, \gamma$  – say,  $\beta$  – be a 4-face; then  $\gamma$  is an  $\geq 4$ -face.

*Case 3.2.1.* If  $x$  is incident only with  $\geq 6$ -vertices, then, by Rule 1 we obtain  $c^*(x) \geq -3 + \frac{2 \cdot 12 - 6}{12 - 2} + 2 \cdot \frac{2 \cdot 4 - 6}{4 - 2} = \frac{4}{5} > 0$ .

*Case 3.2.2.* If exactly one of neighbours of  $x$  is a 5-vertex, then each neighbouring face of  $x$  is incident with an  $\geq 6$ -vertex, and by Rules 1 and 2, we obtain  $c^*(x) \geq -3 + \frac{2 \cdot 12 - 6}{12 - 1} + 2 \cdot \frac{2 \cdot 4 - 6}{4 - 1} + \frac{2}{5} = \frac{61}{165} > 0$ .

*Case 3.2.3.* If at least two neighbours of  $x$  are 5-vertices, then  $c^*(x) \geq -3 + \frac{2 \cdot 12 - 6}{12} + 2 \cdot \frac{2 \cdot 4 - 6}{4} + 2 \cdot \frac{2}{5} = \frac{3}{10} > 0$ .

*Case 3.3.* Let both  $\beta, \gamma$  be  $\geq 5$ -faces. Then  $c^*(x) \geq -3 + 2 \cdot \frac{2 \cdot 5 - 6}{5} + \frac{2 \cdot 12 - 6}{12} = \frac{1}{10} > 0$ . ■

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