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THE CROSSING NUMBERS OF JOIN PRODUCTS OF PATHS WITH GRAPHS OF ORDER FOUR

Marián Klešč[1,2] and Štefan Schrötter[1]

Department of Mathematics
Faculty of Electrical Engineering and Informatics
Technical University, 042 00 Košice, Slovak Republic

e-mail: Marian.Klesc@tuke.sk
Stefan.Schrotter@tuke.sk

Abstract

Kulli and Muddebihal [V.R. Kulli, M.H. Muddebihal, Characterization of join graphs with crossing number zero, Far East J. Appl. Math. 5 (2001) 87–97] gave the characterization of all pairs of graphs which join product is planar graph. The crossing number $cr(G)$ of a graph G is the minimal number of crossings over all drawings of G in the plane. There are only few results concerning crossing numbers of graphs obtained as join product of two graphs. In the paper, the exact values of crossing numbers for join of paths with all graphs of order four, as well as for join of all graphs of order four with n isolated vertices are given.

Keywords: graph, drawing, path, crossing number, join product.

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1. Introduction

Let G be a graph, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A drawing of G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. For simplicity, we

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assume that in a drawing (a) no edge passes through any vertex other than its end-points, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point. The *crossing number* $cr(G)$ is the smallest number of edge crossings in any drawing of G. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross.

Crossing number problems were introduced by Turán, who first inquired about the crossing number of the complete bipartite graph $K_{m,n}$. Turán devised a natural drawing of $K_{m,n}$ with $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ crossings, but the conjecture of Zarankiewicz that such a drawing is the best possible, is still open. Crossing number problems are, in general, very difficult. It was proved by Garey and Johnson [4] that computing the crossing number of a graph is an NP-hard problem. The exact values of crossing numbers are known only for few specific families of graphs. The Cartesian product is one of few graph classes, for which exact results concerning crossing numbers are known. Harary *at al.* [6] conjectured that the crossing number of $C_m \times C_n$ is $(m-2)n$, for all m, n satisfying $3 \leq m \leq n$. This has been proved only for m, n satisfying $n \geq m$, $m \leq 7$. It was recently proved by Glebsky and Salazar [5] that the crossing number of $C_m \times C_n$ equals its long-conjectured value at least for $n \geq m(m+1)$. Besides of Cartesian product of two cycles, there are several other exact results. In [2] and [7], the crossing numbers of $G \times C_n$ for all graphs G of order four are given. Bokal in [3] confirmed the general conjecture for the crossing number of Cartesian product of path and star formulated in [7]. The table in [8] shows the summary of known crossing numbers for Cartesian products of path, cycle and star with connected graphs of order five.

Kulli and Muddebihal [11] gave the characterization of all pairs of graphs which join is planar graph. It thus seems natural to inquire about crossing numbers of join product of graphs. In [9], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle are given. In addition, the exact values of crossing numbers for join products $G + P_n$ and $G + C_n$ for all graphs of order at most three and for some graphs of order four are given. In the paper, we give the crossing numbers for join products of all graphs on four vertices with discrete graphs nK_1 and with paths P_n.

Let D be a good drawing of the graph G. We denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G.

We denote by $cr_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $cr_D(G_i)$ the number of crossings among edges of G_i in D. It is easy to see that for three edge-disjoint graphs G_i, G_j, and G_k, the following equations hold:

$$cr_D(G_i \cup G_j) = cr_D(G_i) + cr_D(G_j) + cr_D(G_i, G_j),$$

$$cr_D(G_i \cup G_j, G_k) = cr_D(G_i, G_k) + cr_D(G_j, G_k).$$

2. The Join Product of Two Graphs

The join product of two graphs G and H, denoted by $G + H$, is obtained from vertex-disjoint copies of G and H by adding all edges between $V(G)$ and $V(H)$. For $|V(G)| = m$ and $|V(H)| = n$, the edge set of $G + H$ is the union of disjoint edge sets of the graphs G, H, and the complete bipartite graph $K_{m,n}$. It has been long-conjectured in [12] that the crossing number $cr(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ equals the Zarankiewicz's Number $Z(m,n) := \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. This conjecture has been verified by Kleitman for $\min\{m,n\} \leq 6$. More precisely, in [10] he proved that

$$cr(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6.$$

Let nK_1 denote the graph on n isolated vertices and let P_n be the path on n vertices. For a graph G with $|V(G)| = m$, the join product $G + nK_1$ consists of one copy of the graph G and n vertices $t_1, t_2, \ldots, t_n$, where every vertex t_i, $i = 1, 2, \ldots, n$, is adjacent to every vertex of G. For $i = 1, 2, \ldots, n$, let T^i denote the subgraph induced by m edges incident with the vertex t_i. Hence,

$$G + nK_1 = G \cup K_{m,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right).$$

The similar union of edge disjoint graphs forms the join product $G + P_n$:

$$G + P_n = G \cup K_{m,n} \cup P_n = G \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup P_n.$$

The decomposition into edge disjoint subgraphs K_4, P_n, and $K_{4,n}$ it is possible to see in the drawing of the graph $K_4 + P_n$ in Figure 1. The graph K_4

is induced on the vertices a, b, c, and d, and the path P_n is induced on the vertices $t_1, t_2, \ldots, t_n$.

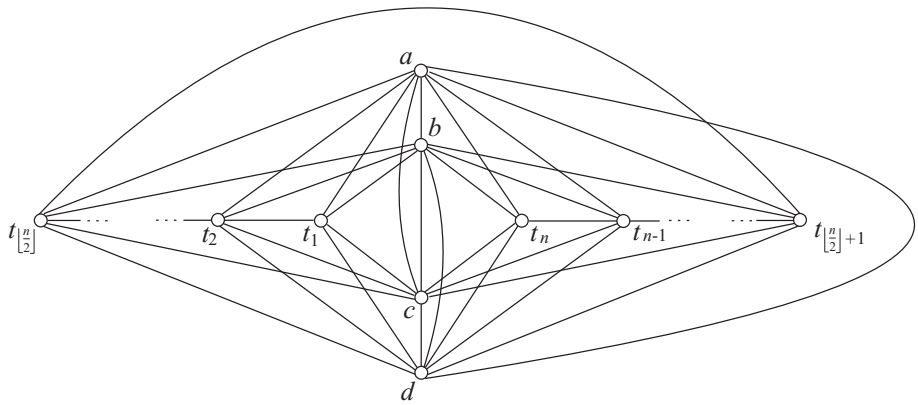

Figure 1. The drawing of the graph graph $K_4 + P_n$.

In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map". Several parts of proofs are based on the next Lemma 2.1 which was proved in [9], and on its Corollary 2.1. Let C_n be the cycle on n vertices. For the join product of the cycle C_n with m isolated vertices, the following holds.

Lemma 2.1. *Let D be a good drawing of $mK_1 + C_n$, $m \geq 2$, $n \geq 3$, in which no edge of C_n is crossed, and C_n does not separate the other vertices of the graph. Then, for all $i, j = 1, 2, \ldots, m$, two different subgraphs T^i and T^j cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times.*

Assume now that the edges of C_n can cross each other. Then, in the view of the subdrawing of the cycle C_n, there is only one region with all n vertices of C_n on its boundary. If, in this case, some subgraph T^i does not cross C_n, then it is placed in the considered region and the next Corollary 2.1 is obvious.

Corollary 2.1. *Let D be a good drawing of $mK_1 + C_n$, $m \geq 2$, $n \geq 3$, in which the edges of C_n cross each other and none of r subgraphs $T^{i_1}, T^{i_2}, \ldots, T^{i_r}$, $2 \leq r \leq m$, crosses the edges of C_n. Then, for all $j, k = 1, 2, \ldots, r$, two different subgraphs T^{i_j} and T^{i_k} cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times.*

3. THE CROSSING NUMBERS OF $G_j + nK_1$

There are eleven graphs $G_1 = 4K_1$, $G_2 = P_2 \cup 2K_1$, $G_3 = 2P_2$, $G_4 = P_3 \cup K_1$, $G_5 = P_4$, $G_6 = K_3 \cup K_1$, $G_7 = K_{1,3}$, $G_8 = C_4$, $G_9 = K_{1,3} \cup \{e\}$, $G_{10} = K_{1,1,2}$, and $G_{11} = K_4$ of order four. All graphs G_j, $j = 1, 2, \ldots, 11$, one can find in the first column of the Table 5.1 in Section 5. Figure 1 shows the drawing of the graph $K_4 + P_n$ in which there are $2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor = Z(4, n)$ crossings among the edges of $T^1 \cup T^2 \cup \cdots \cup T^n = K_{4,n}$, n crossings between the edges of $G_{11} = K_4$ and the edges of $K_{4,n}$, and one crossing between the edges of K_4 and the edges of P_n.

Let us consider the drawing of the graph $K_4 + nK_1$ obtained by deleting the edges of P_n from the drawing in Figure 1. This gives the upper bound $Z(4, n) + n$ for the crossing number of the graph $G_{11} + nK_1$. The deleting of all edges of the subgraph $G_{11} = K_4$ from the drawing of $G_{11} + nK_1$ results in the drawing of the graph $G_1 + nK_1 = K_{4,n}$ with $Z(4, n)$ crossings. This drawing is optimal, because $cr(K_{4,n}) = Z(4, n)$. By deleting the edges $\{a, c\}$, and $\{b, d\}$ from the considered drawing of $G_{11} + nK_1$, the drawing of the graph $G_8 + nK_1 = C_4 + nK_1$ with $Z(4, n)$ crossings is obtained. As every of the graphs $G_2 + nK_1$, $G_3 + nK_1$, $G_4 + nK_1$, and $G_5 + nK_1$ is a subgraph of the graph $C_4 + nK_1$ and all these graphs contain $K_{4,n}$ as a subgraph, for $n \geq 1$ we have that the crossing number of the graphs $G_1 + nK_1$, $G_2 + nK_1$, $G_3 + nK_1$, $G_4 + nK_1$, $G_5 + nK_1$, and $G_8 + nK_1$ is $Z(4, n)$.

The deleting of the edge $\{b, d\}$ from the considered drawing of the graph $K_4 + nK_1$ with $Z(4, n) + n$ crossings results in the drawing of the graph $G_{10} + nK_1$ with $Z(4, n) + \lfloor \frac{n}{2} \rfloor$ crossings. Asano [1] in 1986 proved that the crossing number of the complete tripartite graph $K_{1,3,n}$ is $Z(4, n) + \lfloor \frac{n}{2} \rfloor$. The graph $G_7 + nK_1$ is isomorphic to the graph $K_{1,3,n}$, hence the crossing number of the graph $G_7 + nK_1$ is $Z(4, n) + \lfloor \frac{n}{2} \rfloor$. As the graph $G_{10} + nK_1$ contains $G_7 + nK_1$ as a subgraph, $cr(G_{10} + nK_1) \geq Z(4, n) + \lfloor \frac{n}{2} \rfloor$. Moreover, as the graph $G_9 + nK_1$ is a subgraph of $G_{10} + nK_1$ and $G_7 + nK_1$ is a subgraph of $G_9 + nK_1$, for $n \geq 1$ we have that $cr(G_7 + nK_1) = cr(G_9 + nK_1) = cr(G_{10} + nK_1) = Z(4, n) + \lfloor \frac{n}{2} \rfloor$. The crossing numbers of the remaining two graphs $G_6 + nK_1$ and $G_{11} + nK_1$ are given below.

Theorem 3.1. *$cr(K_4 + nK_1) = 2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n$ for $n \geq 1$.*

Proof. The drawing in Figure 1 without the edges of P_n shows that $cr(K_4 + nK_1) \leq Z(4, n) + n$. We prove the reverse inequality by induction on n. The graph $K_4 + K_1$ is isomorphic to the complete graph K_5 and, as $cr(K_5) = 1$,

the case $n = 1$ is proved. Assume that for $n \geq 2$ there is a drawing D of the graph $K_4 + nK_1$ with fewer than $Z(4,n) + n$ crossings and that $cr(K_4 + rK_1) \geq Z(4,r) + r$ for every integer $r < n$. As $K_4 + nK_1 = K_4 \cup K_{4,n}$, the assumption $cr_D(K_4 + nK_1) < Z(4,n) + n$ implies that the subgraph K_4 has at most $n-1$ crossings in D and that there is a subgraph T^i with $cr_D(K_4, T^i) = 0$. Without loss of generality, let $cr_D(K_4, T^n) = 0$. Hence, the subdrawing of the subgraph K_4 induced from D has all four vertices on the boundary of one, say unbounded, region and all edges of T^n are placed in this region without crossings. Such unique subdrawing of the graph $K_4 \cup T^n = K_5$ with one crossing on the edges of K_4 is shown in Figure 2. There are exactly two vertices of the subgraph K_4 on the boundary of every region. This implies that, in D, every subgraph T^i, $i \neq n$, crosses the edges of $K_4 \cup T^n$ at least two times. Thus, using the fact that $K_4 + nK_1 = K_{4,n-1} \cup (K_4 \cup T^n)$ we have

$$\begin{aligned} cr_D(K_4 + nK_1) &= cr_D(K_{4,n-1}) + cr_D(K_4 \cup T^n) + cr_D(K_{4,n-1}, K_4 \cup T^n) \\ &\geq Z(4,n-1) + 1 + 2(n-1) > Z(4,n) + n - 1. \end{aligned}$$

This contradiction completes the proof. ■

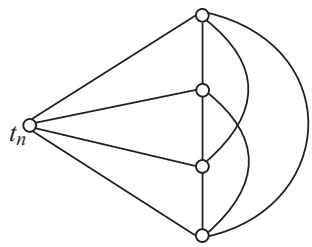

Figure 2. The drawing of $K_4 \cup T^n$ with $cr_D(K_4, T^n) = 0$.

Theorem 3.2. *$cr((K_3 \cup K_1) + nK_1) = 2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ for $n \geq 1$.*

Proof. By deleting all edges of P_n and the edges $\{a,d\}$, $\{b,d\}$, $\{c,d\}$ of K_4 from the drawing of the graph $K_4 + P_n$ in Figure 1, the drawing of the graph $(K_3 \cup K_1) + nK_1 = G_6 + nK_1$ with $2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = \frac{1}{2}n(n-1)$ crossings is obtained. Hence, $cr((K_3 \cup K_1) + nK_1) \leq \frac{1}{2}n(n-1)$. We prove the reverse inequality by induction on n. Let us denote by H_n the graph $(K_3 \cup K_1) + nK_1$ in this proof. The graph H_1 is planar, so the case $n = 1$ is

trivial. As, the graph H_2 is a subdivision of K_5, $cr(H_2) = 1$ and the theorem is true for $n = 2$. Suppose now that for $n \geq 3$

$$cr(H_{n-2}) \geq \frac{1}{2}(n-2)(n-3)$$

and consider such a drawing D of H_n that

$$cr_D(H_n) < \frac{1}{2}n(n-1).$$

Assume that there are two different subgraphs T^i and T^j that do not cross each other in D. Without loss of generality, let $cr_D(T^{n-1}, T^n) = 0$. In the good drawing D, $cr_D(T^{n-1} \cup T^n) = 0$ and, as $cr(K_{4,3}) = 2$, $cr_D(T^i, T^{n-1} \cup T^n) \geq 2$ for all $i = 1, 2, \ldots, n-2$. Let us denote the 3-cycle of the graph G_6 by C_3. As $H_n = (C_3 \cup K_1) \cup (\cup_{i=1}^{n} T^i)$, using the fact that $cr(H_2) = 1$ we have that $cr_D(C_3, T^{n-1} \cup T^n) \geq 1$. This implies that

$$\begin{aligned}
&cr_D(H_n) = cr_D(H_{n-2}) + cr_D(T^{n-1} \cup T^n) + cr_D(H_{n-2}, T^{n-1} \cup T^n) \\
&= cr_D(H_{n-2}) + cr_D(T^{n-1} \cup T^n) + cr_D(C_3, T^{n-1} \cup T^n) \\
&\quad + \sum_{i=1}^{n-2} cr_D(T^i, T^{n-1} \cup T^n) \\
&\geq \frac{1}{2}(n-2)(n-3) + 0 + 1 + 2(n-2) = \frac{1}{2}n(n-1),
\end{aligned}$$

a contradiction with the assumption of D.

Hence, $cr_D(T^i, T^j) \neq 0$ for all $i, j = 1, 2, \ldots, n$, $i \neq j$, and in D there are at least $\binom{n}{2} = \frac{1}{2}n(n-1)$ crossings. This contradiction with $cr_D(H_n) < \frac{1}{2}n(n-1)$ completes the proof. ■

4. THE CROSSING NUMBERS OF $G_j + P_n$

Every join product $G_j + P_n$, $j = 1, 2, \ldots, 11$, contains $G_j + nK_1$ as a subgraph and therefore, $cr(G_j + P_n) \geq cr(G_j + nK_1)$. For $j = 1, 2, \ldots, 5$, one can easy obtain a drawing of $G_j + P_n$ with $Z(4, n)$ crossings by removing suitable edges from the drawing in Figure 1. For $j = 6, 7$, and 9, the deleting of suitable edges from the drawing in Figure 1 results in the drawings of the graphs $G_j + P_n$ with $Z(4, n) + \lfloor \frac{n}{2} \rfloor$ crossings. Hence, $cr(G_j + P_n) = cr(G_j + nK_1)$

for $j = 1, 2, \ldots, 7$ and also for $j = 9$. It was proved in [9] that $cr(G_8 + P_n) = cr(C_4 + P_n) = Z(4, n) + 1$ for $n \geq 2$. The graph $G_8 + P_1$ is planar. It is the aim of this section to establish the crossing numbers for the graphs $G_{10} + P_n$ and $G_{11} + P_n$.

Theorem 4.1. *$cr(K_{1,1,2}+P_1) = 0$ and $cr(K_{1,1,2}+P_n) = 2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ for $n \geq 2$.*

Proof. The graph $K_{1,1,2} + P_1 = G_{10} + P_1$ is planar. For $n \geq 2$, the deleting of the edge $\{b, d\}$ from the drawing in Figure 1 results in the drawing of $G_{10}+P_n$ with $Z(4, n)+\lfloor \frac{n}{2} \rfloor+1 = \frac{1}{2}n(n-1)+1$ crossings. As $G_{10}+P_n$ contains the graph $G_{10} + nK_1$ as a subgraph, we have $\frac{1}{2}n(n-1) \leq cr(G_{10} + P_n) \leq \frac{1}{2}n(n-1) + 1$.

Assume that there is a drawing D of the graph $G_{10} + P_n$ with $\frac{1}{2}n(n-1)$ crossings. In such a drawing, no edge of the path P_n is crossed, because $G_{10}+nK_1$ is a subgraph of $G_{10}+P_n$ and $cr(G_{10}+nK_1) = \frac{1}{2}n(n-1)$. Moreover, no edge of the 4-cycle of the subgraph G_{10} is crossed in D, otherwise in D there is a crossing on an edge of G_{10} not belonging to the subgraph $G_9 + nK_1$, a contradiction with $cr(G_9 + nK_1) = \frac{1}{2}n(n-1)$. The subdrawing of the considered 4-cycle induced from D divides the plane into two quadrangular regions in such a way that, in D, all vertices of the path P_n lie in one of these two regions and none of $T^1, T^2, \ldots, T^n$ crosses the considered 4-cycle. Hence, using Lemma 2.1, we have $cr_D(G_{10} + P_n) \geq 2\binom{n}{2} > \frac{1}{2}n(n-1)$. This contradiction completes the proof. ■

Theorem 4.2. *$cr(K_4 + P_1) = 1$ and $cr(K_4 + P_n) = 2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 1$ for $n \geq 2$.*

Proof. The graph $K_4 + P_1$ is isomorphic to the complete graph K_5 and $cr(K_5) = 1$. For $n \geq 2$, the drawing in Figure 1 shows that $cr(K_4 + P_n) \leq Z(4, n) + n + 1$. We prove the reverse inequality by induction on n. The graph $K_4 + P_2$ is isomorphic to the graph K_6 and, as $cr(K_6) = 3$, the theorem is true for $n = 2$. Suppose now that for $n \geq 3$ there is a drawing D of the graph $G_{11} + P_n$ with fewer than $Z(4, n) + n + 1$ crossings and that $cr(G_{11} + P_r) \geq Z(4, r) + r + 1$ for every integer $2 \leq r < n$. As $cr(K_{4,n}) = Z(4, n)$, the edges of G_{11} are crossed in D at most n times. Moreover, since $G_{11}+P_n$ contains $G_{11}+nK_1$ as a subgraph and $cr(G_{11}+nK_1) = Z(4, n)+n$, none of the edges of P_n is crossed in D.

Assume first that the edges of G_{11} do not cross each other in D. Then $cr_D(G_{11}, T^i) = 1$ for all $i = 1, 2, \ldots, n$, because the graph $G_{11} = K_4$ is not outerplanar and $cr_D(K_{4,n}, G_{11}) \leq n$. In D there are at least two different subgraphs T^i and T^j with $cr_D(T^i, T^j) = 0$, otherwise $cr_D(G_{11} + P_n) \geq \binom{n}{2} + n > Z(4, n) + n$, and this contradicts our assumption. The subgraph consisting of G_{11}, T^i, T^j, and the path joining t_i with t_j is homeomorphic to the complete graph K_6. Now we have $cr_D(G_{11}) = 0$, $cr_D(T^i, T^j) = 0$, $cr_D(G_{11}, T^i) = cr_D(G_{11}, T^j) = 1$ and no edge of the path joining t_i with t_j is crossed. This contradiction with $cr(K_6) = 3$ confirms that the edges of G_{11} cross each other in D.

In a good drawing, the edges of $G_{11} = K_4$ do not cross each other more than once. As $cr(K_{4,n}) = Z(4, n)$ and $cr_D(G_{11}) = 1$, in D there is at least one subgraph T^i with $cr_D(G_{11}, T^i) = 0$. The edges of T^i do not cross each other and, as $G_{11} \cup T^i = K_5$, the only possible subdrawing of $G_{11} \cup T^i$ induced from D divides the plane in the same way as shown in Figure 2. Hence, in this subdrawing there are exactly two vertices of G_{11} on the boundary of every region and, in D, every subgraph T^j, $j = 1, 2, \ldots, n$, $j \neq i$, crosses the edges of $G_{11} \cup T^i$ at least two times. This implies that

$$cr_D(G_{11} + P_n) = cr(K_{4,n-1}) + cr_D(G_{11} \cup T^i) + cr_D(K_{4,n-1}, G_{11} \cup T^i)$$

$$\geq Z(4, n-1) + 1 + 2(n-1) \geq Z(4, n) + n.$$

For $cr_D(G_{11} + P_n) > Z(4, n) + n$ we have a contradiction. If $cr_D(G_{11} + P_n) = Z(4, n) + n$, then $cr_D(G_{11} \cup T^i, T^j) = 2$ for all $j \neq i$. Since $cr_D(G_{11}, P_n) = 0$, all vertices $t_1, t_2, \ldots, t_n$ lie in D in the same region in the view of the subdrawing of G_{11}. All vertices of G_{11} are placed on the boundary of this considered region. One can easy to see that at least two edges of T^j, $j \neq i$, cross in D the edges of $G_{11} \cup T^i$ and that if some edge $\{t_j, x\}$, $x \in \{a, b, c, d\}$, crosses the edges of G_{11}, then it crosses G_{11} at least twice and therefore, T^j crosses $G_{11} \cup T^i$ at least three times. This implies that $cr_D(G_{11}, T^i) = 0$ for all $i = 1, 2, \ldots, n$ and all vertices $t_1, t_2, \ldots, t_n$ lie in D in one region of a 4-cycle of G_{11}. Since the graph $G_{11} + P_n$ contains the subgraph $C_4 + nK_1$, using Corollary 2.1 and the fact that $cr_D(G_{11}) = 1$ we have $cr_D(G_{11} + P_n) \geq 2\binom{n}{2} + 1 > Z(4, n) + n$. This contradiction with the assumption completes the proof. ■

5. Conclusion

In the Table 5.1 below there are collected all values of crossing numbers of the join products for all graphs on four vertices with discrete graphs nK_1 and with paths P_n. It is easily seen that for all nonconnected subgraphs H of the path P_n with some restricted number of edges in every component, the crossing number of the graph $G_j + H$ is the same as for the graph $G_j + nK_1$ for all $j = 1, 2, \ldots, 11$.

Table 5.1. Summary of crossing numbers for $G_i + nK_1$ and $G_i + P_n$.

G_i		$cr(G_i + nK_1)$		$cr(G_i + P_n)$	
G_1		$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$
G_2		$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$
G_3		$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$
G_4		$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$
G_5		$Z(4, n)$	$n \geq 1$	$Z(4, n)$	$n \geq 1$
G_6		$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$
G_7		$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$
G_8		$Z(4, n)$	$n \geq 1$	$Z(4, n) + 1$	$n \geq 2$
G_9		$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$
G_{10}		$Z(4, n) + \lfloor \frac{n}{2} \rfloor$	$n \geq 1$	$Z(4, n) + \lfloor \frac{n}{2} \rfloor + 1$	$n \geq 2$
G_{11}		$Z(4, n) + n$	$n \geq 1$	$Z(4, n) + n + 1$	$n \geq 2$

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