# GRAPHS WITH RAINBOW CONNECTION NUMBER TWO 

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#### Abstract

An edge-coloured graph $G$ is rainbow connected if any two vertices are connected by a path whose edges have distinct colours. The rainbow connection number of a connected graph $G$, denoted $r c(G)$, is the smallest number of colours that are needed in order to make $G$ rainbow connected. In this paper we prove that $r c(G)=2$ for every connected graph $G$ of order $n$ and size $m$, where $\binom{n-1}{2}+1 \leq m \leq\binom{ n}{2}-1$. We also characterize graphs with rainbow connection number two and large clique number.


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## 1. Introduction

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-coloured graph $G$ is called rainbow-connected if any two vertices are connected by a path whose edges have different colours. This concept of rainbow connection in graphs was recently introduced by Chartrand et al. in [4]. The rainbow connection number of a connected graph $G$, denoted $r c(G)$, is the smallest number of colours that are needed in order to make $G$ rainbow connected. An easy observation is that if $G$ has $n$ vertices then $r c(G) \leq n-1$, since one may colour the edges of a given spanning tree of $G$ with different colours, and colour the remaining edges with one of the already used colours. Chartrand et al. computed the precise rainbow connection number of several graph classes including complete multipartite graphs [4]. The rainbow connection number has been studied for further graph classes in $[3]$ and for graphs with fixed minimum degree in $[3,6,8]$.

Rainbow connection has an interesting application for the secure transfer of classified information between agencies (cf. [5]). While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. This twofold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct?

The computational complexity of rainbow connectivity has been studied in $[2,7]$. It is proved that the computation of $r c(G)$ is NP-hard ([2],[7]). In fact it is already NP-complete to decide if $r c(G)=2$, and in fact it is already NP-complete to decide whether a given edge-coloured (with an unbounded number of colours) graph is rainbow connected [2]. More generally it has been shown in [7], that for any fixed $k \geq 2$, deciding if $r c(G)=k$ is NPcomplete.

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition 1. Let $G$ be a connected graph of order $n$. Then

1. $1 \leq r c(G) \leq n-1$,
2. $r c(G) \geq \operatorname{diam}(G)$,
3. $r c(G)=1 \Leftrightarrow G$ is complete,
4. $r c(G)=n-1 \Leftrightarrow G$ is a tree.

## 2. Rainbow Connection and Size of Graphs

In this section we consider the following
Problem 1. For every $k, 1 \leq k \leq n-1$, compute and minimize the function $f(n, k)$ with the following property: If $|E(G)| \geq f(n, k)$, then $r c(G) \leq k$.

We first show a lower bound for $f(n, k)$.
Proposition 2. $f(n, k) \geq\binom{ n-k+1}{2}+(k-1)$.
Proof. We construct a graph $G_{k}$ as follows: Take a $K_{n-k+1}-e$ and denote the two vertices of degree $n-k-1$ with $u_{1}$ and $u_{2}$. Now take a path $P_{k}$ with vertices labeled $w_{1}, w_{2}, \ldots, w_{k}$ and identify the vertices $u_{2}$ and $w_{1}$. The resulting graph $G_{k}$ has order $n$ and size $|E(G)|=\binom{n-k+1}{2}+(k-2)$. For its diameter we obtain $d\left(u_{1}, w_{k}\right)=\operatorname{diam}(G)=k+1$. Hence $f(n, k) \geq$ $(\underset{2}{n-k+1})+(k-1)$.
Using Propositions 1 and 2 we can compute $f(n, k)$ for $k \in\{1, n-2, n-1\}$.

## Proposition 3.

$f(n, 1)=\binom{n}{2}$,
$f(n, n-1)=n-1$,
$f(n, n-2)=n$.
We will now show that $f(n, 2)=\binom{n-1}{2}+1$. In fact we will prove a stronger result.

Theorem 1. Let $G$ be a connected graph of order $n$ and size m. If $\binom{n-1}{2}+$ $1 \leq m \leq\binom{ n}{2}-1$, then $r c(G)=2$.

Proof. Since $m \leq\binom{ n}{2}-1$, we obtain $r c(G) \geq \operatorname{diam}(G) \geq 2$ by Proposition 1.

Now we want to colour the edges of $G$ blue and red in such a way that $G$ is rainbow connected. Equivalently we can colour the edges of the complete graph $K_{n}$ blue, red and black, where the edges of $\bar{G}$ are coloured black. Then for every black edge we need a blue-red path of length two between the endvertices. Let $H$ be the subgraph spanned by the edges of $\bar{G}$. Then $1 \leq$ $|E(H)| \leq n-2$. Let $H=\cup_{i=1}^{s} H_{i}$, where $H_{i}$ are the connected components of $H$, and let $F$ be a maximal bipartite spanning subgraph of $H$ with $F=$ $\cup_{i=1}^{s} F_{i}$. For $1 \leq i \leq s$ let $\left|V\left(H_{i}\right)\right|=\left|V\left(F_{i}\right)\right|=n_{i}, q_{i}=\left|E\left(F_{i}\right)\right| \leq\left|E\left(H_{i}\right)\right|=$ $p_{i}$, and let $q=|E(F)| \leq|E(H)|=p$. For each $F_{i}, 1 \leq i \leq s$, let $U_{i}, W_{i}$ with $V\left(F_{i}\right)=U_{i} \cup W_{i}$ be the partite sets of $F_{i}$.

Let $E[F, H]$ be the set of edges of $G$ between vertices of $V(F)$ and vertices of $V(H)$ and $E[v, H]$ be the set of edges of $G$ between $v \in F$ and the vertices of $V(H)$. Finally let $R=V(G) \backslash V(H)$ and $r=|R|$.

We now distinguish several cases. In each of these cases we will colour some edges blue or red. All remaining edges can be coloured arbitrarily blue or red.

Case 1. $q=p(F=H)$.
Subcase 1.1. $s=1$.
Then $n_{1} \leq n-1$. Choose a vertex $v_{1} \in R$ and colour all edges of $E\left[v_{1}, U_{1}\right]$ blue and all edges of $E\left[v_{1}, W_{1}\right]$ red.

Subcase 1.2. $s \geq 2$.
In this subcase the blue-red stars will form a circular structure within the components $H_{i}$. For each $H_{i}, 1 \leq i \leq s$, choose a vertex $u_{i} \in U_{i}$ and colour all edges of $E\left[u_{i}, U_{i+1}\right]$ blue and all edges of $E\left[u_{i}, W_{i+1}\right]$ red (indices reduced modulo s).

Case 2. $q<p$.
Then $p-q \leq n-\sum_{i=1}^{s} n_{i}+(s-2)=r+s-2$.
Suppose $p-q>n-\sum_{i=1}^{s} n_{i}+(s-2)$. Then $p>n-\sum_{i=1}^{s} n_{i}+(s-2)+q \geq$ $n-\sum_{i=1}^{s} n_{i}+(s-2)+\sum_{i=1}^{s}\left(n_{i}-1\right)=n-2$, since $q_{i} \geq n_{i}-1$, a contradiction.

For each of the $q$ black edges we can construct a blue-red path of length two as in the previous case. For the remaining $p-q \leq r+s-2$ black edges we choose a vertex $w_{i} \in W_{i}$ for $3 \leq i \leq s$ and the $r$ vertices $v_{1}, \ldots, v_{r}$ of $R$. We may assume that the components $H_{i}$ are labeled in such a way that $p_{1}-q_{1} \geq p_{2}-q_{2} \geq \ldots \geq p_{s}-q_{s}$. Now picking up the vertices in the order $w_{3}, w_{4}, \ldots, w_{s}, v_{1}, v_{2}, \ldots, v_{r}$ and the black edges in the order $E\left(H_{1}\right) \backslash$
$E\left(F_{1}\right), E\left(H_{2}\right) \backslash E\left(F_{2}\right), \ldots, E\left(H_{s}\right) \backslash E\left(F_{s}\right)$, we can construct $p-q$ blue-red paths of length two between the endvertices of the black edges.

## 3. Rainbow Connection and Clique Number

In this section we characterize graphs with rainbow connection two with respect to their clique number.

Proposition 4. Let $G$ be a connected graph of order $n$ and clique number $\omega(G)$. If $\omega(G)=n+1-i$ for $i=1$ or $i=2$, then $r c(G)=i$.

Proof. If $i=1$ then $\omega(G)=n$ and thus $G$ is complete which implies $r c(G)=1$ by Proposition 1. If $i=2$ then $\omega(G)=n-1$. Hence $|E(G)| \geq$ $\binom{n-1}{2}+1$ since $G$ is connected. The result follows now by Theorem 1 .

Suppose now that $G$ is connected and that $2 \leq \omega(G) \leq n-2$. Let $H$ be a subgraph of $G$ which induces a maximum clique, i.e., a clique of size $\omega=\omega(G)$. Let $F=G[V(G) \backslash V(H)]$ be the subgraph of $G$ induced by the vertices of $V(G) \backslash V(H)$. Let $V(H)=\left\{w_{1}, w_{2}, \ldots, w_{\omega}\right\}$ and $V(F)=\left\{v_{1}, v_{2}, \ldots, v_{n-\omega}\right\}$. If $F$ is not connected then let $F_{1}, F_{2}, \ldots, F_{p}$ be the components of $F$. Let $N_{H}(v)$ be the set of neighbors of $v$ in $H$ and $d_{H}(v)=\left|N_{H}(v)\right|$.

Proposition 5. Let $G$ be a connected graph of order $n$, clique number $\omega(G)$ with $2 \leq \omega(G) \leq n-2$ and rainbow connection number $r c(G)=2$. Then
(N1) $1 \leq d_{H}(v) \leq \omega(G)-1$ for every vertex $v \in V(F)$,
(N2) $N_{H}\left(v_{i}\right) \cap N_{H}\left(v_{j}\right) \neq \emptyset$ and $\max \left\{d_{H}\left(v_{i}\right), d_{H}\left(v_{j}\right)\right\} \geq 2$ for every pair of nonadjacent vertices $v_{i} \in V\left(F_{i}\right), v_{j} \in V\left(F_{j}\right)$,
(N3) $\left|\left(N_{H}\left(v_{i}\right) \cap N_{H}\left(v_{j}\right)\right) \cup\left(N_{H}\left(v_{i}\right) \cap N_{H}\left(v_{k}\right)\right) \cup\left(N_{H}\left(v_{j}\right) \cap N_{H}\left(v_{k}\right)\right)\right| \geq 2$ for every triple of independent vertices $v_{i} \in V\left(F_{i}\right), v_{j} \in V\left(F_{j}\right), v_{k} \in$ $V\left(F_{k}\right)$.

Proof. By Proposition 1 we have that $\operatorname{diam}(G)=2$. Since $H$ induces a maximum clique in $G$ we obtain (N1). Suppose $w \in N_{H}\left(v_{1}\right) \cap N_{H}\left(v_{2}\right)$ for two nonadjacent vertices $v_{1}, v_{2} \in V(F)$ and $d_{H}\left(v_{1}\right)=d_{H}\left(v_{2}\right)=1$. Since $G$ is rainbow connected we may assume that $c\left(v_{1} w\right)=1$ and $c\left(v_{2} w\right)=2$. Then $c(w u)=2$ for all vertices $u \in(V(H) \backslash\{w\})$ with respect to $v_{1}$ and $c(w u)=$ 1 for all vertices $u \in(V(H) \backslash\{w\})$ with respect to $v_{2}$, a contradiction. This shows (N2). If $\mid\left(N_{H}\left(v_{1}\right) \cap N_{H}\left(v_{2}\right)\right) \cup\left(N_{H}\left(v_{1}\right) \cap N_{H}\left(v_{3}\right)\right) \cup\left(N_{H}\left(v_{2}\right) \cap\right.$
$\left.N_{H}\left(v_{3}\right)\right) \mid=1$, then not all three pairs of vertices of $F$ are rainbow connected. This shows (N3).

Theorem 2. Let $G$ be a connected graph of order n, diameter 2 and clique number $n-2$. Then $\operatorname{rc}(G)=2$ with the exception that $G$ is isomorphic to $K_{n-2}$ with two pendant edges at the same vertex.

Proof. If $F \cong K_{2}$ then colour all edges of $E(H)$ blue, all edges of $E[F, H]$ red and the edge of $F$ arbitrarily to obtain an edge colouring of $G$ with $r c(G)=2$.

If $F \cong 2 K_{1}$ then $v_{1}$ and $v_{2}$ have a common neighbor in $H$ by ( N 2 ), say $w_{1}$. If $N_{H}\left(v_{1}\right)=N_{H}\left(v_{2}\right)=\left\{w_{1}\right\}$, then $G$ is isomorphic to $K_{n-2}$ with two pendant edges at $w_{1}$. Now (N2) is violated and thus $r c(G) \geq 3$. Hence we may assume that $\max \left\{\left|E\left[v_{1}, H\right]\right|,\left|E\left[v_{2}, H\right]\right|\right\} \geq 2$, say $\left|E\left[v_{1}, H\right]\right| \geq 2$ and $w_{1}, w_{2} \in N\left(v_{1}, H\right)$. Colour the edges of $E(H)$ as well as edge $v_{1} w_{1}$ blue and the edges $v_{2} w_{1}$ and $v_{1} w_{2}$ red to obtain an edge colouring of $G$ with $r c(G)=2$.

Theorem 3. Let $G$ be a connected graph of order n, diameter 2 and clique number $n-3$. Then $r c(G)=2$ with the exception of the following three cases:
(1) $F=G[V(G) \backslash V(H)] \cong K_{2} \cup K_{1}$ where $H$ is a clique of size $n-3$, $V\left(K_{2}\right)=\left\{v_{1}, v_{2}\right\}, V\left(K_{1}\right)=\left\{v_{3}\right\}$ and $\min \left\{\left|E\left[v_{1}, H\right]\right|,\left|E\left[v_{2}, H\right]\right|\right\}=$ $\left|E\left[v_{3}, H\right]\right|=1$.
(2) $F=G[V(G) \backslash V(H)] \cong K_{2} \cup K_{1}, V\left(K_{2}\right)=\left\{v_{1}, v_{2}\right\}, V\left(K_{1}\right)=\left\{v_{3}\right\}$, $\left|E\left[v_{1}, H\right]\right|+\left|E\left[v_{2}, H\right]\right|=\left|E\left[v_{3}, H\right]\right|=2$ and $N_{H}\left(v_{1}\right) \neq N_{H}\left(v_{2}\right)$.
(3) $F=G[V(G) \backslash V(H)] \cong 3 K_{1}, V(F)=\left\{v_{1}, v_{2}, v_{3}\right\}$ and $\left|E\left[v_{1}, H\right]\right|=$ $\left|E\left[v_{2}, H\right]\right|=\left|E\left[v_{3}, H\right]\right|=1$.

Proof. 1. If $F \cong K_{3}$ or $F \cong P_{3}$ then colour all edges of $E(H)$ blue, all edges of $E[F, H]$ red and the edges of $F$ blue and red such that $F$ is rainbow connected. This is an edge colouring of $G$ with $r c(G)=2$.
2. If $F \cong K_{2} \cup K_{1}$ then let $V\left(K_{2}\right)=\left\{v_{1}, v_{2}\right\}, V\left(K_{1}\right)=\left\{v_{3}\right\}$. We distinguish three cases.

Case 1. $\left|E\left[v_{3}, H\right]\right|=1$.
Let $N_{H}\left(v_{3}\right)=\left\{w_{1}\right\}$. If $\min \left\{\left|E\left[v_{1}, H\right]\right|,\left|E\left[v_{2}, H\right]\right|\right\}=1$, then (N2) is violated and thus $r c(G) \geq 3$. Hence we may assume that $\min \left\{\left|E\left[v_{1}, H\right]\right|\right.$,
$\left.\left|E\left[v_{2}, H\right]\right|\right\} \geq 2$, say $\left\{w_{1}, w_{2}\right\} \subseteq N_{H}\left(v_{1}\right)$ and $\left\{w_{1}, w_{3}\right\} \subseteq N_{H}\left(v_{2}\right)\left(w_{2}=w_{3}\right.$ is possible; $w_{1} \in N_{H}\left(v_{1}\right) \cap N_{H}\left(v_{2}\right)$ since $\left.\operatorname{diam}(G)=2\right)$. The following colouring $c$ with colours 1 (blue) and 2 (red) induces an edge colouring of $G$ with rainbow connection $r c(G)=2: c\left(w_{i} w_{j}\right)=1$ for all $w_{i}, w_{j} \in V(H)$, $c\left(v_{1} w_{1}\right)=c\left(v_{2} w_{1}\right)=1, c\left(v_{1} w_{2}\right)=c\left(v_{2} w_{3}\right)=c\left(v_{3} w_{1}\right)=2$ and an arbitrary colour for the remaining edges.

Case 2. $\left|E\left[v_{3}, H\right]\right|=2$.
Let $N_{H}\left(v_{3}\right)=\left\{w_{1}, w_{2}\right\}$. If $\left|E\left[v_{1}, H\right]\right|=\left|E\left[v_{2}, H\right]\right|=1$ and $N_{H}\left(v_{1}\right) \neq$ $N_{H}\left(v_{2}\right)$, then we may assume $v_{1} w_{1}, v_{2} w_{2} \in E(G)$ and $c\left(v_{1} w_{1}\right)=c\left(v_{2} w_{2}\right)=$ 2. Assume that $r c(G)=2$. This implies $v_{3} w_{1}, v_{3} w_{2} \in E(G)$ and $c\left(v_{3} w_{1}\right)=$ $c\left(v_{3} w_{2}\right)=1$, which is not possible. Therefore, $r c(G) \geq 3$.

Hence we may assume that $N_{H}\left(v_{1}\right)=N_{H}\left(v_{2}\right)=\{w\}$ or $\left|E\left[v_{1}, H\right]\right|+$ $\left|E\left[v_{2}, H\right]\right| \geq 3$. If $N_{H}\left(v_{1}\right)=N_{H}\left(v_{2}\right)=\{w\}$ then $w \in N_{H}\left(v_{3}\right)$, say $w=$ $w_{1}$, since $\operatorname{diam}(G)=2$. Choose $c\left(w_{i} w_{j}\right)=1$ for all $w_{i}, w_{j} \in V(H)$, $c\left(v_{1} w_{1}\right)=c\left(v_{2} w_{1}\right)=c\left(v_{3} w_{2}\right)=2, c\left(v_{3} w_{1}\right)=1$ and an arbitrary colour for the remaining edges to obtain an edge colouring of $G$ with $\operatorname{rc}(G)=2$. If $\left|E\left[v_{1}, H\right]\right|+\left|E\left[v_{2}, H\right]\right| \geq 3$ and $\left|E\left[v_{3}, H\right]\right|=2$ then, without loss of generality, $N_{H}\left(v_{3}\right)=\left\{w_{1}, w_{2}\right\},\left\{w_{3}, w_{4}\right\} \subseteq N_{H}\left(v_{1}\right), w_{5} \in N_{H}\left(v_{2}\right)$ with $w_{3}=w_{1}$ and $w_{5}=w_{1}$ or $w_{5}=w_{2}$. Choose $c\left(w_{i} w_{j}\right)=1$ for all $w_{i}, w_{j} \in V(H)$, $c\left(v_{1} w_{4}\right)=2$, and $c\left(v_{1} w_{1}\right)=c\left(v_{2} w_{1}\right)=c\left(v_{3} w_{2}\right)=2, c\left(v_{3} w_{1}\right)=1$ in case $w_{5}=w_{1}$ or $c\left(v_{1} w_{1}\right)=c\left(v_{3} w_{2}\right)=1, c\left(v_{2} w_{2}\right)=c\left(v_{3} w_{1}\right)=2$ in case $w_{5}=w_{2}$, respectively, and an arbitrary colour for the remaining edges in both cases.

Case 3. $\left|E\left[v_{3}, H\right]\right| \geq 3$.
Obviously an analogous coloring like the previous one induces an edge colouring of $G$ with $r c(G)=2$.
3. If $F \cong 3 K_{1}$ then let $V(F)=\left\{v_{1}, v_{2}, v_{3}\right\}$. If $\left|E\left[v_{1}, H\right]\right|=\left|E\left[v_{2}, H\right]\right|=$ $\left|E\left[v_{3}, H\right]\right|=1$, then $N_{H}\left(v_{1}\right)=N_{H}\left(v_{2}\right)=N_{H}\left(v_{3}\right)=\{w\}$ for a vertex $w \in$ $V(H)$ by (N2). However, (N3) is violated and thus $r c(G) \geq 3$. Hence we may assume $\left|\left(N_{H}\left(v_{1}\right) \cap N_{H}\left(v_{2}\right)\right) \cup\left(N_{H}\left(v_{1}\right) \cap N_{H}\left(v_{3}\right)\right) \cup\left(N_{H}\left(v_{2}\right) \cap N_{H}\left(v_{3}\right)\right)\right| \geq$ 2. If there are three pairwise different vertices $w_{1} \in N_{H}\left(v_{1}\right) \cap N_{H}\left(v_{2}\right)$, $w_{2} \in N_{H}\left(v_{1}\right) \cap N_{H}\left(v_{3}\right)$, and $w_{3} \in N_{H}\left(v_{2}\right) \cap N_{H}\left(v_{3}\right)$, then choose $c\left(v_{1} w_{1}\right)=$ $c\left(v_{2} w_{3}\right)=c\left(v_{3} w_{2}\right)=1$ and $c\left(v_{1} w_{2}\right)=c\left(v_{2} w_{1}\right)=c\left(v_{3} w_{3}\right)=2$. If two of the vertices $w_{1}, w_{2}, w_{3}$ coincide, say $w_{1}=w_{2}$, then choose $c\left(v_{2} w_{3}\right)=c\left(v_{3} w_{1}\right)=$ $1, c\left(v_{1} w_{1}\right)=c\left(v_{2} w_{1}\right)=c\left(v_{3} w_{3}\right)=2$. Choose in both cases $c\left(w_{i} w_{j}\right)=1$ for all $w_{i}, w_{j} \in V(H)$ and an arbitrary colour for the remaining edges to obtain an edge colouring of $G$ with rainbow connection $\operatorname{rc}(G)=2$.

It would be possible to characterize all connected graphs of order $n$, diameter 2 and rainbow connection number 2 with clique number $n-s, s \geq 4$. However, the case analysis will enlarge extensively since the number of exceptional graph classes with $|V(G)|=n, \operatorname{diam}(G)=2, \omega(G)=n-s$, but rainbow connection number $r c(G)>2$ increases.

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