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GRAPHS WITH RAINBOW CONNECTION NUMBER TWO

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Abstract

An edge-coloured graph G is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colours. The *rainbow connection number* of a connected graph G, denoted $rc(G)$, is the smallest number of colours that are needed in order to make G rainbow connected. In this paper we prove that $rc(G) = 2$ for every connected graph G of order n and size m, where $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2} - 1$. We also characterize graphs with rainbow connection number two and large clique number.

Keywords: edge colouring, rainbow colouring, rainbow connection.

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1. Introduction

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-coloured graph G is called *rainbow-connected* if any two vertices are connected by a path whose edges have different colours. This concept of rainbow connection in graphs was recently introduced by Chartrand *et al.* in [4]. The rainbow connection number of a connected graph G, denoted $rc(G)$, is the smallest number of colours that are needed in order to make G rainbow connected. An easy observation is that if G has n vertices then $rc(G) \leq n-1$, since one may colour the edges of a given spanning tree of G with different colours, and colour the remaining edges with one of the already used colours. Chartrand *et al.* computed the precise rainbow connection number of several graph classes including complete multipartite graphs [4]. The rainbow connection number has been studied for further graph classes in [3] and for graphs with fixed minimum degree in [3, 6, 8].

Rainbow connection has an interesting application for the secure transfer of classified information between agencies (cf. [5]). While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct?

The computational complexity of rainbow connectivity has been studied in [2, 7]. It is proved that the computation of $rc(G)$ is NP-hard ([2],[7]). In fact it is already NP-complete to decide if $rc(G) = 2$, and in fact it is already NP-complete to decide whether a given edge-coloured (with an unbounded number of colours) graph is rainbow connected [2]. More generally it has been shown in [7], that for any fixed $k \geq 2$, deciding if $rc(G) = k$ is NP-complete.

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition 1. *Let G be a connected graph of order n. Then*

1. $1 \leq rc(G) \leq n-1$,
2. $rc(G) \geq diam(G)$,
3. $rc(G) = 1 \Leftrightarrow G$ *is complete*,
4. $rc(G) = n-1 \Leftrightarrow G$ *is a tree.*

2. Rainbow Connection and Size of Graphs

In this section we consider the following

Problem 1. For every $k, 1 \leq k \leq n-1$, compute and minimize the function $f(n,k)$ with the following property: If $|E(G)| \geq f(n,k)$, then $rc(G) \leq k$.

We first show a lower bound for $f(n,k)$.

Proposition 2. $f(n,k) \geq \binom{n-k+1}{2} + (k-1)$.

Proof. We construct a graph G_k as follows: Take a $K_{n-k+1} - e$ and denote the two vertices of degree $n-k-1$ with u_1 and u_2. Now take a path P_k with vertices labeled $w_1, w_2, \ldots, w_k$ and identify the vertices u_2 and w_1. The resulting graph G_k has order n and size $|E(G)| = \binom{n-k+1}{2} + (k-2)$. For its diameter we obtain $d(u_1, w_k) = diam(G) = k+1$. Hence $f(n,k) \geq \binom{n-k+1}{2} + (k-1)$. ■

Using Propositions 1 and 2 we can compute $f(n,k)$ for $k \in \{1, n-2, n-1\}$.

Proposition 3.

$f(n,1) = \binom{n}{2}$,

$f(n,n-1) = n-1$,

$f(n,n-2) = n$.

We will now show that $f(n,2) = \binom{n-1}{2} + 1$. In fact we will prove a stronger result.

Theorem 1. *Let G be a connected graph of order n and size m. If* $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2} - 1$, *then* $rc(G) = 2$.

Proof. Since $m \leq \binom{n}{2} - 1$, we obtain $rc(G) \geq diam(G) \geq 2$ by Proposition 1.

Now we want to colour the edges of G blue and red in such a way that G is rainbow connected. Equivalently we can colour the edges of the complete graph K_n blue, red and black, where the edges of $\overline{G}$ are coloured black. Then for every black edge we need a blue-red path of length two between the endvertices. Let H be the subgraph spanned by the edges of $\overline{G}$. Then $1 \leq |E(H)| \leq n-2$. Let $H = \cup_{i=1}^{s} H_i$, where H_i are the connected components of H, and let F be a maximal bipartite spanning subgraph of H with $F = \cup_{i=1}^{s} F_i$. For $1 \leq i \leq s$ let $|V(H_i)| = |V(F_i)| = n_i, q_i = |E(F_i)| \leq |E(H_i)| = p_i$, and let $q = |E(F)| \leq |E(H)| = p$. For each $F_i, 1 \leq i \leq s$, let U_i, W_i with $V(F_i) = U_i \cup W_i$ be the partite sets of F_i.

Let $E[F,H]$ be the set of edges of G between vertices of $V(F)$ and vertices of $V(H)$ and $E[v,H]$ be the set of edges of G between $v \in F$ and the vertices of $V(H)$. Finally let $R = V(G) \setminus V(H)$ and $r = |R|$.

We now distinguish several cases. In each of these cases we will colour some edges blue or red. All remaining edges can be coloured arbitrarily blue or red.

Case 1. $q = p$ $(F = H)$.

Subcase 1.1. $s = 1$.
Then $n_1 \leq n-1$. Choose a vertex $v_1 \in R$ and colour all edges of $E[v_1, U_1]$ blue and all edges of $E[v_1, W_1]$ red.

Subcase 1.2. $s \geq 2$.
In this subcase the blue-red stars will form a circular structure within the components H_i. For each $H_i, 1 \leq i \leq s$, choose a vertex $u_i \in U_i$ and colour all edges of $E[u_i, U_{i+1}]$ blue and all edges of $E[u_i, W_{i+1}]$ red (indices reduced modulo s).

Case 2. $q < p$.
Then $p - q \leq n - \sum_{i=1}^{s} n_i + (s-2) = r + s - 2$.

Suppose $p - q > n - \sum_{i=1}^{s} n_i + (s-2)$. Then $p > n - \sum_{i=1}^{s} n_i + (s-2) + q \geq n - \sum_{i=1}^{s} n_i + (s-2) + \sum_{i=1}^{s} (n_i - 1) = n-2$, since $q_i \geq n_i - 1$, a contradiction.

For each of the q black edges we can construct a blue-red path of length two as in the previous case. For the remaining $p - q \leq r + s - 2$ black edges we choose a vertex $w_i \in W_i$ for $3 \leq i \leq s$ and the r vertices $v_1, \ldots, v_r$ of R. We may assume that the components H_i are labeled in such a way that $p_1 - q_1 \geq p_2 - q_2 \geq \ldots \geq p_s - q_s$. Now picking up the vertices in the order $w_3, w_4, \ldots, w_s, v_1, v_2, \ldots, v_r$ and the black edges in the order $E(H_1) \setminus$

$E(F_1), E(H_2) \setminus E(F_2), \ldots, E(H_s) \setminus E(F_s)$, we can construct $p - q$ blue-red paths of length two between the endvertices of the black edges. ■

3. Rainbow Connection and Clique Number

In this section we characterize graphs with rainbow connection two with respect to their clique number.

Proposition 4. *Let G be a connected graph of order n and clique number $\omega(G)$. If $\omega(G) = n + 1 - i$ for $i = 1$ or $i = 2$, then $rc(G) = i$.*

Proof. If $i = 1$ then $\omega(G) = n$ and thus G is complete which implies $rc(G) = 1$ by Proposition 1. If $i = 2$ then $\omega(G) = n - 1$. Hence $|E(G)| \geq \binom{n-1}{2} + 1$ since G is connected. The result follows now by Theorem 1. ■

Suppose now that G is connected and that $2 \leq \omega(G) \leq n-2$. Let H be a subgraph of G which induces a maximum clique, i.e., a clique of size $\omega = \omega(G)$. Let $F = G[V(G) \setminus V(H)]$ be the subgraph of G induced by the vertices of $V(G) \setminus V(H)$. Let $V(H) = \{w_1, w_2, \ldots, w_\omega\}$ and $V(F) = \{v_1, v_2, \ldots, v_{n-\omega}\}$. If F is not connected then let $F_1, F_2, \ldots, F_p$ be the components of F. Let $N_H(v)$ be the set of neighbors of v in H and $d_H(v) = |N_H(v)|$.

Proposition 5. *Let G be a connected graph of order n, clique number $\omega(G)$ with $2 \leq \omega(G) \leq n - 2$ and rainbow connection number $rc(G) = 2$. Then*

(N1) $1 \leq d_H(v) \leq \omega(G) - 1$ *for every vertex $v \in V(F)$,*

(N2) $N_H(v_i) \cap N_H(v_j) \neq \emptyset$ *and* $\max\{d_H(v_i), d_H(v_j)\} \geq 2$ *for every pair of nonadjacent vertices $v_i \in V(F_i), v_j \in V(F_j)$,*

(N3) $|(N_H(v_i) \cap N_H(v_j)) \cup (N_H(v_i) \cap N_H(v_k)) \cup (N_H(v_j) \cap N_H(v_k))| \geq 2$ *for every triple of independent vertices $v_i \in V(F_i), v_j \in V(F_j), v_k \in V(F_k)$.*

Proof. By Proposition 1 we have that $diam(G) = 2$. Since H induces a maximum clique in G we obtain (N1). Suppose $w \in N_H(v_1) \cap N_H(v_2)$ for two nonadjacent vertices $v_1, v_2 \in V(F)$ and $d_H(v_1) = d_H(v_2) = 1$. Since G is rainbow connected we may assume that $c(v_1 w) = 1$ and $c(v_2 w) = 2$. Then $c(wu) = 2$ for all vertices $u \in (V(H) \setminus \{w\})$ with respect to v_1 and $c(wu) = 1$ for all vertices $u \in (V(H) \setminus \{w\})$ with respect to v_2, a contradiction. This shows (N2). If $|(N_H(v_1) \cap N_H(v_2)) \cup (N_H(v_1) \cap N_H(v_3)) \cup (N_H(v_2) \cap$

$N_H(v_3))| = 1$, then not all three pairs of vertices of F are rainbow connected. This shows (N3). ■

Theorem 2. *Let G be a connected graph of order n, diameter 2 and clique number $n-2$. Then $rc(G) = 2$ with the exception that G is isomorphic to K_{n-2} with two pendant edges at the same vertex.*

Proof. If $F \cong K_2$ then colour all edges of $E(H)$ blue, all edges of $E[F, H]$ red and the edge of F arbitrarily to obtain an edge colouring of G with $rc(G) = 2$.

If $F \cong 2K_1$ then v_1 and v_2 have a common neighbor in H by (N2), say w_1. If $N_H(v_1) = N_H(v_2) = \{w_1\}$, then G is isomorphic to K_{n-2} with two pendant edges at w_1. Now (N2) is violated and thus $rc(G) \geq 3$. Hence we may assume that $\max\{|E[v_1, H]|, |E[v_2, H]|\} \geq 2$, say $|E[v_1, H]| \geq 2$ and $w_1, w_2 \in N(v_1, H)$. Colour the edges of $E(H)$ as well as edge v_1w_1 blue and the edges v_2w_1 and v_1w_2 red to obtain an edge colouring of G with $rc(G) = 2$. ■

Theorem 3. *Let G be a connected graph of order n, diameter 2 and clique number $n-3$. Then $rc(G) = 2$ with the exception of the following three cases:*

(1) *$F = G[V(G) \setminus V(H)] \cong K_2 \cup K_1$ where H is a clique of size $n-3$, $V(K_2) = \{v_1, v_2\}$, $V(K_1) = \{v_3\}$ and $\min\{|E[v_1, H]|, |E[v_2, H]|\} = |E[v_3, H]| = 1$.*

(2) *$F = G[V(G) \setminus V(H)] \cong K_2 \cup K_1$, $V(K_2) = \{v_1, v_2\}$, $V(K_1) = \{v_3\}$, $|E[v_1, H]| + |E[v_2, H]| = |E[v_3, H]| = 2$ and $N_H(v_1) \neq N_H(v_2)$.*

(3) *$F = G[V(G) \setminus V(H)] \cong 3K_1$, $V(F) = \{v_1, v_2, v_3\}$ and $|E[v_1, H]| = |E[v_2, H]| = |E[v_3, H]| = 1$.*

***Proof.* 1.** If $F \cong K_3$ or $F \cong P_3$ then colour all edges of $E(H)$ blue, all edges of $E[F, H]$ red and the edges of F blue and red such that F is rainbow connected. This is an edge colouring of G with $rc(G) = 2$.

2. If $F \cong K_2 \cup K_1$ then let $V(K_2) = \{v_1, v_2\}$, $V(K_1) = \{v_3\}$. We distinguish three cases.

Case 1. $|E[v_3, H]| = 1$.
Let $N_H(v_3) = \{w_1\}$. If $\min\{|E[v_1, H]|, |E[v_2, H]|\} = 1$, then (N2) is violated and thus $rc(G) \geq 3$. Hence we may assume that $\min\{|E[v_1, H]|,$

$|E[v_2, H]|\} \geq 2$, say $\{w_1, w_2\} \subseteq N_H(v_1)$ and $\{w_1, w_3\} \subseteq N_H(v_2)$ ($w_2 = w_3$ is possible; $w_1 \in N_H(v_1) \cap N_H(v_2)$ since $diam(G) = 2$). The following colouring c with colours 1 (blue) and 2 (red) induces an edge colouring of G with rainbow connection $rc(G) = 2$: $c(w_iw_j) = 1$ for all $w_i, w_j \in V(H)$, $c(v_1w_1) = c(v_2w_1) = 1$, $c(v_1w_2) = c(v_2w_3) = c(v_3w_1) = 2$ and an arbitrary colour for the remaining edges.

Case 2. $|E[v_3, H]| = 2$.
Let $N_H(v_3) = \{w_1, w_2\}$. If $|E[v_1, H]| = |E[v_2, H]| = 1$ and $N_H(v_1) \neq N_H(v_2)$, then we may assume $v_1w_1, v_2w_2 \in E(G)$ and $c(v_1w_1) = c(v_2w_2) = 2$. Assume that $rc(G) = 2$. This implies $v_3w_1, v_3w_2 \in E(G)$ and $c(v_3w_1) = c(v_3w_2) = 1$, which is not possible. Therefore, $rc(G) \geq 3$.

Hence we may assume that $N_H(v_1) = N_H(v_2) = \{w\}$ or $|E[v_1, H]| + |E[v_2, H]| \geq 3$. If $N_H(v_1) = N_H(v_2) = \{w\}$ then $w \in N_H(v_3)$, say $w = w_1$, since $diam(G) = 2$. Choose $c(w_iw_j) = 1$ for all $w_i, w_j \in V(H)$, $c(v_1w_1) = c(v_2w_1) = c(v_3w_2) = 2$, $c(v_3w_1) = 1$ and an arbitrary colour for the remaining edges to obtain an edge colouring of G with $rc(G) = 2$. If $|E[v_1, H]| + |E[v_2, H]| \geq 3$ and $|E[v_3, H]| = 2$ then, without loss of generality, $N_H(v_3) = \{w_1, w_2\}$, $\{w_3, w_4\} \subseteq N_H(v_1)$, $w_5 \in N_H(v_2)$ with $w_3 = w_1$ and $w_5 = w_1$ or $w_5 = w_2$. Choose $c(w_iw_j) = 1$ for all $w_i, w_j \in V(H)$, $c(v_1w_4) = 2$, and $c(v_1w_1) = c(v_2w_1) = c(v_3w_2) = 2$, $c(v_3w_1) = 1$ in case $w_5 = w_1$ or $c(v_1w_1) = c(v_3w_2) = 1$, $c(v_2w_2) = c(v_3w_1) = 2$ in case $w_5 = w_2$, respectively, and an arbitrary colour for the remaining edges in both cases.

Case 3. $|E[v_3, H]| \geq 3$.
Obviously an analogous coloring like the previous one induces an edge colouring of G with $rc(G) = 2$.

3. If $F \cong 3K_1$ then let $V(F) = \{v_1, v_2, v_3\}$. If $|E[v_1, H]| = |E[v_2, H]| = |E[v_3, H]| = 1$, then $N_H(v_1) = N_H(v_2) = N_H(v_3) = \{w\}$ for a vertex $w \in V(H)$ by (N2). However, (N3) is violated and thus $rc(G) \geq 3$. Hence we may assume $|(N_H(v_1) \cap N_H(v_2)) \cup (N_H(v_1) \cap N_H(v_3)) \cup (N_H(v_2) \cap N_H(v_3))| \geq 2$. If there are three pairwise different vertices $w_1 \in N_H(v_1) \cap N_H(v_2)$, $w_2 \in N_H(v_1) \cap N_H(v_3)$, and $w_3 \in N_H(v_2) \cap N_H(v_3)$, then choose $c(v_1w_1) = c(v_2w_3) = c(v_3w_2) = 1$ and $c(v_1w_2) = c(v_2w_1) = c(v_3w_3) = 2$. If two of the vertices w_1, w_2, w_3 coincide, say $w_1 = w_2$, then choose $c(v_2w_3) = c(v_3w_1) = 1$, $c(v_1w_1) = c(v_2w_1) = c(v_3w_3) = 2$. Choose in both cases $c(w_iw_j) = 1$ for all $w_i, w_j \in V(H)$ and an arbitrary colour for the remaining edges to obtain an edge colouring of G with rainbow connection $rc(G) = 2$. ■

It would be possible to characterize all connected graphs of order n, diameter 2 and rainbow connection number 2 with clique number $n-s$, $s \geq 4$. However, the case analysis will enlarge extensively since the number of exceptional graph classes with $|V(G)| = n$, $diam(G) = 2$, $\omega(G) = n-s$, but rainbow connection number $rc(G) > 2$ increases.

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