k-KERNELS IN GENERALIZATIONS
OF TRANSITIVE DIGRAPHS

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Abstract

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively.

A (k, l)-kernel N of D is a k-independent set of vertices (if $u, v \in N$, $u \neq v$, then $d(u, v), d(v, u) \geq k$) and l-absorbent (if $u \in V(D) - N$ then there exists $v \in N$ such that $d(u, v) \leq l$). A k-kernel is a $(k, k - 1)$-kernel. Quasi-transitive, right-pretransitive and left-pretransitive digraphs are generalizations of transitive digraphs. In this paper the following results are proved: Let D be a right-(left-) pretransitive strong digraph such that every directed triangle of D is symmetrical, then D has a k-kernel for every integer $k \geq 3$; the result is also valid for non-strong digraphs in the right-pretransitive case. We also give a proof of the fact that every quasi-transitive digraph has a (k, l)-kernel for every integers $k > l \geq 3$ or $k = 3$ and $l = 2$.

Keywords: digraph, kernel, (k, l)-kernel, k-kernel, transitive digraph, quasi-transitive digraph, right-pretransitive digraph, left-pretransitive digraph, pretransitive digraph.

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1. Introduction

In this work, $D = (V(D), A(D))$ will denote a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. For general concepts and notation we refer the reader to [1], [4] and [9], particularly we will use the notation of [9] for walks, if $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a walk and $i < j$ then $x_i \mathscr{C} x_j$ will denote the subwalk of $\mathscr{C}$ $(x_i, x_{i+1}, \ldots, x_{j-1}, x_j)$. Union of walks will be denoted by concatenation or with $\cup$. For a vertex $v \in V(D)$, we define the *out-neighbourhood* of v in D as the set $N_D^+(v) = \{u \in V(D) | (v, u) \in A(D)\}$; when there is no possibility of confusion we will omit the subscript D. The elements of $N^+(v)$ are called the *out-neighbours* of v, and the *out-degree* of v, $d_D^+(v)$, is the number of out-neighbours of v. Definitions of in-neighbourhood, in-neighbours and in-degree of v are analogously given.

A digraph is *strongly connected* (or strong) if for every $u, v \in V(D)$, there exists a uv-directed path, i.e., a directed path with initial vertex u and terminal vertex v. A *strong component* (or component) of D is a maximal strong subdigraph of D. The *condensation* of D is the digraph $D^\star$ with $V(D^\star)$ equal to the set of all strong components of D, and $(S, T) \in A(D^\star)$ if and only if there is an ST-arc in D. Clearly $D^\star$ is an acyclic digraph (a digraph without directed cycles), and thus, it has both vertices of out-degree equal to zero and vertices of in-degree equal to zero. A *terminal component* of D is a strong component T of D such that $d_{D^\star}^+(T) = 0$. An *initial component* of D is a strong component S of D such that $d_{D^\star}^-(S) = 0$.

An arc $(u, v) \in A(D)$ is called *asymmetrical* (resp. symmetrical) if $(v, u) \notin A(D)$ (resp. $(v, u) \in A(D)$). The asymmetrical part of D, $\mathrm{Asym}(D)$, is the subdigraph of D induced by the asymmetrical arcs. A subdigraph of D (e.g. a directed path or a directed triangle) is asymmetrical if all its arcs are asymmetrical.

A *biorientation* of the graph G is a digraph D obtained from G by replacing each edge $\{x, y\} \in E(G)$ by either the arc (x, y) or the arc (y, x) or the pair of arcs (x, y) and (y, x). A *semicomplete* digraph is a biorientation of a complete graph. An *orientation* of a graph G is an asymmetrical biorientation of G; thus, an *oriented* graph is an asymmetrical digraph. A tournament is an orientation of a complete graph. An orientation of a digraph D is a maximal asymmetrical subdigraph of D. A *complete* digraph is a biorientation of a complete graph obtained by replacing each edge $\{x, y\}$ by the arcs (x, y) and (y, x).

In [16], M. Kwaśnik introduces the concept of (k, l)-kernel in a digraph generalizing the concept of kernel of a digraph in Berge's sense which is a $(2, 1)$-kernel. As a special case of (k, l)-kernels we consider the k-kernels; we define a k-kernel to be a $(k, k-1)$-kernel. Under this definition a kernel is a 2-kernel. Another special case of (k, l)-kernels that have been studied by some authors are the $(2, 2)$-kernels, or *quasi-kernels*.

The solution of a digraph is the dual notion of a kernel, so Kwaśnik's concept can be adopted to generalize the concept of solution to (k, l)-solution of a digraph.

Definition. Let D be a digraph and $S \subseteq V(D)$.

- The set S is l-dominating if for every $v \in V(D) \setminus S$ there exists $u \in S$ such that $d(u, v) \leq l$.
- The set S is called a (k, l)-solution of D if it is both k-independent and l-dominating.
- A k-solution is a $(k, k-1)$-solution.

Another concept related to the kernel of a digraph is the one of semikernel. A subset $S \subseteq V(D)$ is a *semikernel* (or local kernel) of the digraph D if S is independent and S absorbs $N^+(S)$. This concept was generalized also by Kucharska and Kwaśnik in [15] as follows: A subset $S \subseteq V(D)$ is a k-*semikernel* of the digraph D if S is k-independent and for each $u \in V(D) \setminus S$ such that $d(S, u) \leq k-1$, it holds that $d(u, S) \leq k-1$. Clearly a semikernel is a 2-semikernel.

In the particular families of digraphs we will be studying in this work the existence of k-solutions will be very close to the existence of k-kernels, so we find it useful to recall the definition of the dual of a digraph.

Definition. If D is a digraph, the dual (or converse) of D, $\overleftarrow{D}$ is the digraph with vertex set $V(\overleftarrow{D}) = V(D)$ and such that $(u, v) \in A(\overleftarrow{D})$ if and only if $(v, u) \in A(D)$.

Remark 1. Clearly $\overleftarrow{\overleftarrow{D}} = D$. Also it is obvious that if N is a (k, l)-kernel of D then N is a (k, l)-solution of $\overleftarrow{D}$.

There are not many results concerning the existence of k-kernels nor (k, l)-kernels in large families of digraphs. Many of the existing results come from the study of products of graphs and digraphs and how the k-kernels are preserved (like the work of Włoch and Włoch, in particular with Szumny

in [20, 21]) or the superdigraphs of certain families of digraphs ([15]). We begin with some of the classical results in Kernel Theory that we will use as platform for the results we propose.

Since every (directed) cycle of odd length does not have a kernel, sufficient conditions for the existence of kernels in digraphs have been found imposing conditions on the cycles of a digraph, e.g., in [22] it is proved that

Theorem 2. *If D is a digraph without directed cycles, then D has a kernel.*

In [19], Richardson generalizes this result as follows[1]:

Theorem 3 (Richardson)**.** *If D is a digraph such that the length of every directed cycle is congruent to 0 (mod 2) then D has a kernel.*

In [17], M. Kwaśnik generalized this result for k-kernels in the following way.

Theorem 4 (Kwaśnik)**.** *Let D be a strongly connected digraph. If every directed cycle in D has length congruent to 0 (mod k) then D has a k-kernel.*

Kwaśnik's Theorem indeed proves that digraphs in a very large family have a k-kernel. Her result is equivalent to saying that every strongly connected cyclically k-partite digraph have a k-kernel. Recall that a digraph D is *transitive* if $(u, v), (v, w) \in A(D)$ implies $(u, w) \in A(D)$.

In [4], the following general result can be found:

Theorem 5. *If D is a transitive digraph, then D has a kernel. Moreover, every kernel consists in one vertex from every terminal strong component of D, so all kernels of D have the same cardinality.*

Theorem 5 has been a motivation for the results we present in this work. We focus on three families of digraphs which generalize transitive digraphs: quasi-transitive digraphs and right-/left-pretransitive digraphs.

Definition.
- A digraph D is *quasi-transitive* if $(u, v), (v, w) \in A(D)$ implies $(u, w) \in A(D)$ or $(w, u) \in A(D)$.
- A digraph D is *right-(left-)pretransitive* if $(u, v), (v, w) \in A(D)$ implies $(u, w) \in A(D)$ or $(w, v) \in A(D)$ $((v, u) \in A(D))$.

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[1]See [6] for a simpler proof of Theorem 3.

Every transitive digraph is quasi-transitive and right-/left-pretransitive (and thus the title of this article).

Related to right- and left-pretransitive digraphs is also Theorem 6 below (proved by Galeana-Sánchez and Rojas-Monroy in [12]), which generalizes a result of Duchet ([10]).

Definition. A digraph D is called *kernel-perfect* if every induced subdigraph of D has a kernel.

Theorem 6 (Galeana-Sánchez, Rojas-Monroy). *Let D be a* (*possibly infinite*) *digraph. Suppose that there exist two subdigraphs of D say D_1 and D_2 such that $D = D_1 \cup D_2$ (possibly $A(D_1) \cap A(D_2) \neq \emptyset$), where D_1 is a right-pretransitive digraph, D_2 is a left-pretransitive digraph, and D_i contains no infinite outward path for $i \in \{1, 2\}$. Then D is a kernel-perfect digraph.*

Corollary 7 (Duchet). *If D is a right-/left-pretransitive digraph, then D is kernel-perfect.*

So we have great motivation for studying these families of digraphs. Also, Bang-Jensen and Huang have studied quasi-transitive digraphs. Among other results related to our research they have relevant results concerning 3-kings in quasi-transitive digraphs [3] and a structural characterization of quasi-transitive digraphs [2].

To conclude this section, we present a result of Kernel Theory due to Neumann-Lara that inspired the following lemma, which has turned out to be very useful when dealing with k-kernels.

Theorem 8 (Neumann-Lara [18]). *Let D be a digraph. If every induced subdigraph of D has a nonempty semikernel, then D is a kernel-perfect digraph.*

Lemma 9. *Let $k \geq 2$ be an integer. If D is a digraph such that $\{v\}$ is a k-semikernel of D for every $v \in V(D)$, then D has k-kernel.*

Proof. Let D be a digraph such that $\{v\}$ is a k-semikernel of D for every $v \in V(D)$. Since D has at least one k-semikernel we can consider a ($\subseteq$-) maximal k-semikernel of D (because D is finite), say S. If S is a $(k-1)$-absorbent set, S is the k-kernel we have been looking for. Otherwise, there exists at least one vertex $v_0 \in V(D)$ such that it is not $(k-1)$-absorbed by S. In other words, there does not exist a $v_0 S$-directed path of length less than or equal to $k - 1$, and by the second k-semikernel condition neither

a Sv_0-directed path of length less than or equal to $k-1$ does (because, if it exists, also a v_0S-directed path of length less than or equal to $k-1$ would exist). Thence $S \cup \{v_0\}$ is a k-independent set. Besides, since S is a k-semikernel of D, for each Sv-directed path of length less than or equal to $k-1$ in D, there exists a vS-directed path of length less than or equal to $k-1$ in D. The same property can be claimed for every v_ov-directed path of length less than or equal to $k-1$ in D because $\{v_0\}$ is also a k-semikernel of D. Therefore $S \cup \{v_0\}$ fulfills the second k-semikernel condition, which contradicts the choice of S as a maximal k-semikernel. It follows that S is a k-kernel of D. ■

The rest of the paper is structured as follows. In Section 2 we study some properties of right-(left-)pretransitive digraphs as a setup to use Lemma 9 to prove that if D is a right-(left-)pretransitive strong digraph such that every directed triangle of D is symmetrical, then D has a k-kernel for every integer $k \geq 3$. This result will be used along with a brief structural analysis of non-strong right-pretransitive digraphs to prove that the result is also valid for non-strong digraphs in the right-pretransitive case. A conjecture and an open problem are proposed on the matter. In Section 3 a structural characterization of quasi-transitive digraphs is used along with a previous result about (k,l)-kernels in the composition of digraphs to prove that every quasi-transitive digraph has (k,l)-kernel for every integers $k > l \geq 3$ or $k = 3$ and $l = 2$. An analysis of the (2-)kernels in quasi-transitive digraphs is made from the point of view of the Strong Perfect Graph Theorem. At the end of both sections, results about (k,l)-solutions in digraphs are obtained by means of dualization.

2. Pretransitive Digraphs

Recall that a digraph D is called *right-pretransitive* (resp. *left-pretransitive*) if $(u,v) \in A(D)$ and $(v,w) \in A(D)$ implies $(u,w) \in A(D)$ or $(w,v) \in A(D)$ (resp. when $(u,v),(v,w) \in A(D)$ implies $(u,w) \in A(D)$ or $(v,u) \in A(D)$).

There is a notorious duality in the definitions of right and left-pretransitive digraphs and as there is also a duality in the definitions of k-kernels and k-solutions. In view of both definitions the next lemma will prove to be very useful once we have the appropriate tools.

Lemma 10. *Let D be a digraph. D is a right-pretransitive digraph if and only if $\overleftarrow{D}$ is a left-pretransitive digraph.*

Proof. Straightforward. ∎

We will prove two lemmas about the structure of right-pretransitive digraphs; the second one will be dualized using Lemma 10 to obtain an analogous result about left-pretransitive digraphs.

Lemma 11. *If D is a right-pretransitive digraph and $(x_0, x_1, \ldots, x_n)$ is an asymmetrical directed path in D then $(x_0, x_i) \in A(D)$ for every $i \in \{2, 3, \ldots, n\}$.*

Proof. Straightforward, by induction on n. ∎

Lemma 12. *If D is a right-pretransitive digraph then $\mathrm{Asym}(D)$ is acyclic. Moreover, every directed triangle in D has at least two symmetrical arcs.*

Proof. We will prove the second part first. Let $C_3 = (x, y, z, x)$ be a directed triangle in D. Since D is right-pretransitive and $(x, y), (y, z) \in A(D)$ we can conclude that $(x, z) \in A(D)$ or $(z, y) \in A(D)$. In either case the result is a directed triangle with a symmetrical arc, so let us assume without loss of generality that $(x, z) \in A(D)$. Then we can consider the arcs $(z, x) \in A(D)$ and $(x, y) \in A(D)$, for the right-pretransitivity of D we know that $(z, y) \in A(D)$ or $(y, x) \in A(D)$. In either case C_3 has at least two symmetrical arcs.

For the first part, suppose by contradiction that $\mathscr{C} = (x_0, x_1, \ldots, x_n = x_0)$ is a directed cycle in $\mathrm{Asym}(D)$, where D is a right-pretransitive digraph. If we consider the asymmetrical directed path $(x_0, x_1, \ldots, x_{n-1})$, it follows from Lemma 11 that $(x_0, x_{n-1}) \in A(D)$; we don't know whether (x_0, x_{n-1}) is an asymmetrical arc or not, but together with (x_{n-1}, x_n) and as a consequence of the right-pretransitivity of D we have that $(x_0, x_n) \in A(D)$ or $(x_n, x_{n-1}) \in A(D)$, but (x_n, x_0) and (x_{n-1}, x_n) are both asymmetrical arcs of D, thus we obtain the desired contradiction. Since the contradiction arises from the assumption that there is a directed cycle in $\mathrm{Asym}(D)$, then $\mathrm{Asym}(D)$ is acyclic. ∎

Lemma 13. *If D is a left-pretransitive digraph then $\mathrm{Asym}(D)$ is acyclic. Moreover, every directed triangle in D has at least two symmetrical arcs.*

Proof. The result follows straightforward from Lemmas 10 and 12. ∎

Our next result was part of our first attempt to implement a classical proof method in kernel theory to k-kernels. For kernels (2-kernels), once it is proved that digraphs in a certain family have nonempty semikernels it suffices to consider a ($\subseteq$-)maximal semikernel S for a digraph D. If the set of vertices not absorbed by S is not empty, then we can find a nonempty semikernel S' for $D \setminus (S \cup N^-(S))$. From here is easy to prove that $S \cup S'$ is a semikernel of D, contradicting the choice of S. When working with k-kernels we have a problem: suppose that we have proved that a certain family of digraphs have nonempty k-semikernel and consider a digraph D in such family. Then we can find a maximal k-semikernel S of D and, if S is $(k-1)$-absorbent, S is the desired k-kernel. But if not, we consider a k-semikernel S' for the subdigraph T of D induced by the vertices not $(k-1)$-absorbed by S; it remains clear that $S \cup S'$ is k-independent and that every vertex reached from S must reach $S \cup S'$ in D but, suppose that there is a vertex $v \in V(T)$ such that the only $S'v$-directed path of length less than or equal to $k-1$ in D is $(x_0, x_1, x_2, \ldots, x_n = v)$, where $x_0 \in S' \subseteq V(T)$, but $x_i \in V(D \setminus T)$ for some $1 \leq i \leq n-1$, then v is not reached by S' in T and then v may not reach S' in D, and as v is in T, v does not reach S in D, so $S \cup S'$ may not be a k-semikernel in D. It is in view of this problem that we proposed Lemma 9, were we prove that if **every** vertex $v \in V(D)$ is a k-semikernel **of D**, then D has a k-kernel. Nevertheless, this result is interesting by itself as a local property of the class of right-pretransitive digraphs is found.

Theorem 14. *If D is a right-pretransitive digraph, then D has a k-semikernel consisting of a single vertex for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. If D has no asymmetrical arcs, then D is a symmetrical digraph and each vertex is trivially a k-semikernel of D for every $k \geq 2$.

So, let us assume that $\mathrm{Asym}(D) \neq \emptyset$. In virtue of Lemma 12 $\mathrm{Asym}(D)$ is acyclic, so we can choose a vertex v with out-degree 0 in $\mathrm{Asym}(D)$. We claim that $\{v\}$ is a k-semikernel of D for every $k \geq 2$. As $\{v\}$ is k-independent for every $k \in \mathbb{N}$, it suffices to prove that for every $k \geq 2$ if a vw-directed path of length less than or equal to $k-1$ exists, then a wv-directed path of length less than or equal to $k-1$ exists.

Since v has out-degree 0 in $\mathrm{Asym}(D)$, if $(v, w) \in A(D)$ for some $w \in V(D)$, then such arc must be symmetrical, so $(w, v) \in A(D)$ and the second condition of k-semikernel is fulfilled for $k = 2$. Let k be greater than 2. We will prove by induction on n that if a vw-directed path of length $n \leq k-1$

exists, then there exists a wv-directed path of length less than or equal to $k-1$. The case $n = 1$ has been already proved, is the same as case $k = 2$. Let us assume the result valid for every vw-directed path of length $m < n$ and let $\mathscr{C} = (v = v_0, v_1, \ldots, v_n = w)$ be a vw-directed path of length $n \leq k - 1$. For the choice of v we know that (v_0, v_1) is a symmetrical arc of D. If every arc in $\mathscr{C}$ is symmetrical, then the directed path $\mathscr{C}^{-1}$ is the one we have been looking for. Otherwise, there must be a first asymmetrical arc in $A(\mathscr{C})$, let us say (v_i, v_{i+1}), $1 \leq i$. So we can consider the arcs $(v_{i-1}, v_i), (v_i, v_{i+1}) \in A(D)$ and, since D is right-pretransitive and $(v_{i+1}, v_i) \notin A(D)$, necessarily $(v_{i-1}, v_{i+1}) \in A(D)$, and hence $v_0 \mathscr{C} v_{i-1} \cup (v_{i-1}, v_{i+1}) \cup v_{i+1} \mathscr{C} w$ is a vw-directed path of length $n - 1$. Inductive hypothesis assures the existence of a wv-directed path of length less than or equal to $k - 1$, which concludes the proof. The desired result follows from the induction principle. ∎

We have already proved that right-/left-pretransitive digraphs have at least two symmetrical arcs in every directed triangle. In view of this property, it is not very restrictive to ask for a right-/left-pretransitive digraph to have only symmetrical directed triangles. As the next lemma shows (only after a little technical lemma), this is a sufficient condition along with strong connectedness to prove that every right-/left-pretransitive digraph have a k-kernel.

Lemma 15. *If D is a right-pretransitive digraph such that every directed triangle is symmetrical and $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a directed path such that (x_i, x_{i+1}) is a symmetrical arc for every $i \in \{0, 1, \ldots, n-2\}$ and (x_{n-1}, x_n) is an asymmetrical arc of D, then $(x_i, x_n) \in A(D)$ for every $i \in \{0, 1, \ldots, n - 1\}$. Moreover, every such arc is asymmetrical.*

Proof. By induction on n. For $n = 2$, $(x_0, x_1), (x_1, x_2) \in A(D)$ so, since D is right-pretransitive then $(x_2, x_1) \in A(D)$ or $(x_0, x_2) \in A(D)$, but (x_1, x_2) is an asymmetrical arc, hence $(x_0, x_2) \in A(D)$. Besides, if $(x_2, x_0) \in A(D)$, then (x_0, x_1, x_2, x_0) would be a directed triangle and it should be symmetrical by hypothesis, but (x_1, x_2) is an asymmetrical arc; it follows that (x_0, x_2) is also an asymmetrical arc. So, let us assume the result valid for every path with the required conditions and length less than n. If $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a directed path with the required conditions and length n, clearly $(x_1, x_2, \ldots, x_n)$ is a directed path with the required conditions and length $n - 1 < n$, and from the inductive hypothesis we have the existence of the asymmetrical arcs (x_i, x_n) for every $i \in \{1, 2, \ldots, n\}$.

To finish the inductive step we have to prove that $(x_0, x_n) \in A(D)$ and it is an asymmetrical arc. But $(x_0, x_1), (x_1, x_n)$ is a directed path of length 2 where the first arc is symmetrical and the second arc is asymmetrical, so it follows from the case $n = 2$ that $(x_0, x_n) \in A(D)$ is an asymmetrical arc. The desired result follows from the principle of mathematical induction. ∎

Lemma 16. *Let $k \geq 2$ be an integer. If D is a right-pretransitive strong digraph such that every directed triangle is symmetrical, then every vertex of D is a k-semikernel of D.*

Proof. Let $k \geq 2$ be an integer. Let $v \in V(D)$ be any vertex, consider $w \in V(D)$ such that there exists a vw-directed path of length less than or equal to $k-1$ and let $\mathscr{C} = (v = v_0, v_1, \ldots, v_n = w)$ be a vw-directed path of minimum length. Then $n \leq k-1$. For every pair of arcs $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}) \in A(\mathscr{C})$, the arc (v_i, v_{i+2}) can not exist in $A(D)$, because it would contradict the choice of $\mathscr{C}$ as a vw-directed path of minimum length, so, for the right pretransitve hypothesis, for every $0 \leq i \leq n - 2$ the arc $(v_{i+2}, v_{i+1}) \in A(D)$ must exist. If $(v_1, v_0) \in A(D)$, the directed path $(v_n, v_{n-1}, \ldots, v_0)$ would be a wv-directed path of length $n \leq k-1$. If $(v_1, v_0) \notin A(D)$, as D is strong, there exists a $v_1 v$-directed path in D, say $\mathscr{D} = (v_1 = z_0, z_1, \ldots, z_m = v)$. We can suppose without loss of generality that $\mathscr{D}$ is of minimum length and its length is greater than 1. So, we can consider the arcs $(z_i, z_{i+1}), (z_{i+1}, z_{i+2}) \in A(D)$, and since $\mathscr{D}$ has minimum length, once again we have the existence of the arcs $(z_{i+2}, z_{i+1}) \in A(D)$ for every $0 \leq i \leq m - 2$. Also, we have the arcs $(z_{m-1}, v), (v, z_0) \in A(D)$, and by hypothesis we know that $(v, z_0 = v_1)$ is not a symmetrical arc, thence it follows from right pretransitivity the existence of the arc $(z_{m-1}, z_0) \in A(D)$. But (z_{m-1}, z_0) must be an asymmetrical arc of D, in other case, (z_0, z_{m-1}, z_m, z_0) would be a directed triangle and all of its arcs would be symmetrical for hypothesis, in particular the arc $(z_m, z_0) = (v, v_1)$ would be symmetrical, contrary to our assumption. So the directed path $(z_1, z_2, \ldots, z_{m-1}, z_0)$ fulfills the hypothesis of Lemma 15 and as a consequence the arcs (z_i, z_0) are asymmetrical arcs of D for every $i \in \{1, 2, \ldots m - 1\}$; in particular $(z_1, z_0) \in A(D)$ and it should be an asymmetrical arc, but $(z_0, z_1) \in A(D)$, which turns out to be a contradiction. Since the contradiction arises from the assumption $(v_1, v_0) \notin A(D)$, we can conclude that $(v_1, v_0) \in A(D)$ and thence there exists a wv-directed path of length less than or equal to $k - 1$. ∎

Theorem 17. *If D is a right-pretransitive strong digraph such that every directed triangle is symmetrical, then D has k-kernel for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. It follows from Lemmas 9 and 16.   ∎

Lemma 18. *If D is a left-pretransitive strong digraph such that every directed triangle is symmetrical, then $\{v\}$ is a k-semikernel of D for every $v \in V(D)$.*

Proof. Let D be a left-pretransitive strong digraph such that every directed triangle is symmetrical. In virtue of Lemma 10 $\overleftarrow{D}$ is a right-pretransitive digraph such that every directed triangle is symmetrical, so it follows from Lemma 16 that $\{v\}$ is a k-semikernel of $\overleftarrow{D}$ for every $v \in V(D) = V(\overleftarrow{D})$. Let v be a vertex in $V(D)$ and $k \geq 2$ an integer. It is clear that $\{v\}$ is k-independent for every k, so let us consider a vw-directed path of length less than or equal to $k - 1$ $\mathscr{C}$. It is also obvious that $\mathscr{C}^{-1}$ is a wv-directed path of length less than or equal to $k - 1$ in $\overleftarrow{D}$, and since $\{w\}$ is a k-semikernel of $\overleftarrow{D}$, then there exists a vw-directed path of length less than or equal to $k - 1$ in $\overleftarrow{D}$, say $\mathscr{D}$. But $\mathscr{D}^{-1}$ is hence a wv-directed path of length $\leq k - 1$ in D, consequently $\{v\}$ fulfills both k-semikernel conditions and the result follows.   ∎

Theorem 19. *If D is a left-pretransitive strong digraph such that every directed triangle is symmetrical, then D has k-kernel for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. The result follows from Lemmas 9 and 18.   ∎

The following corollary is obtained directly by dualization.

Corollary 20. *If D is a right-(left-)pretransitive strong digraph such that every directed triangle is symmetrical, then D has a k-solution for every $k \in \mathbb{N}$, $k \geq 2$.*

For right-pretransitive digraphs we can improve our results. Let us state a lemma about the structure of non-strong right-pretransitive digraphs, but first we will need some notation. Let A and B be non-empty subsets of $V(D)$. If for every $a \in A$ and every $b \in B$ we have that $(a, b) \in A(D)$, we will write $A \mapsto B$. When $A = \{v\}$ for some $v \in V(D)$, we will simply write $v \mapsto B$, and analogously if $B = \{v\}$. If S and T are subdigraphs of D (*e.g.*, strong components) we will abuse notation to write $S \mapsto T$ instead of $V(S) \mapsto V(T)$.

Lemma 21. *Let be D a right-pretransitive digraph, S and T strong components of D. If there exist $s \in S$ and $t \in T$ such that $(s, t) \in A(D)$, then $S \mapsto t$.*

Proof. Let v be a vertex in $V(S) \setminus \{s\}$. Since S is a strong component of D, we have that $d(v, s) \in \mathbb{N}$. We will prove by induction on $n = d(v, s)$ that $(v, t) \in A(D)$. If $d(v, s) = 1$, then $(v, s), (s, t) \in A(D)$. By the right-pretransitivity of D we have that $(v, t) \in A(D)$ or $(t, s) \in A(D)$. But S and T are distinct strong components of D and $(s, t) \in A(D)$. Since $D^\star$ is an acyclic digraph, $(t, s) \notin A(D)$, thus $(v, t) \in A(D)$.

For the inductive step it suffices to observe that, if $d(v, s) = n$, then there exists a vs-directed path in S, $(v = x_0, x_1, \ldots, x_n = s)$, realizing the distance from v to s. It is clear that $d(x_1, s) = n - 1$, so by induction hypothesis, $(x_1, t) \in A(D)$. Together with $(v, x_1) \in A(D)$, we may use the same argument that we used in the base case. ∎

Theorem 22. *Let D be a right-pretransitive digraph such that every directed triangle is symmetrical. Then D has k-kernel for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. Let $k \geq 2$ be a fixed integer. We will proceed by induction on $n = |V(D^\star)|$. If $n = 1$, then D is a strong digraph and the result follows from Theorem 17. So let us assume that $n \geq 2$.

Let D be a digraph such that $|V(D^\star)| = n$ and S an initial strong component of D. Clearly $(D \setminus S)^\star = D^\star \setminus S$, so $|V((D \setminus S)^\star)| = n - 1$. By induction hypothesis, $D \setminus S$ has a k-kernel, say N'. Suppose first that $d_D(S, N') \geq k$. Then, by Theorem 17 we can choose a k-kernel N'' of S. We know that $d_D(N', S) = \infty$ because S is an initial component of D, so $N = N' \cup N''$ is k-independent. Also it follows from the fact that N' is $(k-1)$-absorbent in $D \setminus S$ and N'' is $(k-1)$-absorbent in S that N is $(k-1)$-absorbent in D. Thus, N is the desired k-kernel.

Suppose now that $d_D(S, N') \leq k - 1$. Then there is a vertex $s \in S$ and a vertex $t \in N'$ such that there exist a directed path $(s = x_0, x_1, \ldots, x_r = t)$ of length $r \leq k - 1$. We can choose s and t in such way that $x_1 \in V(D \setminus S)$. Since s and x_1 are in distinct strong components, in virtue of Lemma 21 we can conclude that $S \mapsto x_1$, which implies that $d(v, t) \leq k - 1$ for every $v \in V(S)$. Thus, N' is a $(k-1)$-absorbent set in D. Also, since S is an initial component, there are no $N'S$-directed paths, so N' is k-independent in D. Hence, N' is the desired k-kernel. ∎

Dualization does not work as good as we would like for Lemma 21 and Theorem 22. The next results have straightforward proofs by means of dualization.

Lemma 23. *Let be D a left-pretransitive digraph, S and T strong components of D. If there exist $s \in S$ and $t \in T$ such that $(s,t) \in A(D)$, then $s \mapsto T$.*

Theorem 24. *Let D be a left-pretransitive digraph such that every directed triangle is symmetrical, then D has k-solution for every $k \in \mathbb{N}$, $k \geq 2$.*

So, two obvious problems arise.

Problem 25. Is it true that every right-pretransitive digraph such that every directed triangle is symmetrical has a k-solution for every integer $k \geq 2$?

A positive answer for the question proposed in Problem 25 would imply that every left-pretransitive digraph such that every directed triangle is symmetrical has a k-kernel for every integer $k \geq 2$. The remaining question about existence of k-kernels in right-/left-pretransitive digraphs would be the following.

Problem 26. Are the hypotheses in Theorems 17 and 19 on the directed triangles sharp?

In virtue of Lemmas 12 and 13, Problem 26 is equivalent to asking if it is true that every right-/left-pretransitive strong digraph has a k-kernel for every integer $k \geq 3$ or if there is a right-/left-pretransitive strong digraph without a k-kernel for some integer $k \geq 3$.

3.   QUASI-TRANSITIVE DIGRAPHS

Recall that a digraph D is *quasi-transitive* if $(u,v), (v,w) \in A(D)$ implies $(u,w) \in A(D)$ or $(w,u) \in A(D)$.

As we have already mentioned, in [2] Bang-Jensen and Huang proved a structural characterization of quasi-transitive digraphs. This result uses the composition operation, defined next.

Definition. Let D be a digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $G_1, G_2, \ldots, G_n$ be digraphs which are pairwise vertex-disjoint. The *composition* $D[G_1, G_2, \ldots, G_n]$ is the digraph L with vertex set $\bigcup_{i=1}^{n} (V(G_i))$ and arc set $(\bigcup_{i=1}^{n} A(G_i)) \cup \{g_i g_j \big| g_i \in V(G_i), g_j \in V(G_j), v_i v_j \in A(D)\}$.

The characterization theorem is stated next.

Theorem 27 (Bang-Jensen and Huang [2]). *Let D be a digraph which is quasi-transitive.*

1. *If D is not strong, then there exists an acyclic, transitive oriented graph T with vertices $\{u_1, u_2, \ldots, u_t\}$ and quasi-transitive strong digraphs H_1, $H_2, \ldots, H_t$ such that $D = T[H_1, H_2, \ldots, H_t]$, where H_i is substituted u_i, $i = 1, 2, \ldots, t$.*

2. *If D is strong, then there exists a strong semicomplete digraph S with vertices $\{v_1, v_2, \ldots, v_s\}$ and quasi-transitive digraphs $Q_1, Q_2, \ldots, Q_s$ such that each Q_i is either a vertex or is non-strong and $D = S[Q_1, Q_2, \ldots, Q_s]$, where Q_i is substituted for v_i, $i = 1, 2, \ldots, s$.*

Using this characterization and a result due to Szumny, Włoch and Włoch about (k, l)-kernels in digraph compositions we are able to derive easily that every quasi-transitive digraph has a (k, l)-kernel for every $k \geq 4$, $k - 1 \geq l \geq 3$ or $k = 3$ and $l = 2$, in particular, every quasi-transitive digraph has a k-kernel for $k \geq 3$. Previous results include those of I. Goldfeder who has proven that every quasi-transitive digraph has a 3-kernel and has given a characterization of quasi-transitive digraphs with a (2-)kernel ([14]), and the work of Galeana-Sánchez and Rojas-Monroy ([12]) about sufficient conditions for a quasi-transitive digraph to have a kernel. We include this proof since is a very direct and easy consequence of Theorems 27 and 28, but also, the authors have developed another proof of this fact using local properties of the quasi-transitive digraphs rather than global arguments (like those from Theorems 27 and 28) that will not be included in this work. For the case $k = 2$ we simply mention the existing results about kernels in digraphs.

To state the next result we need some new notation. If D is a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$, we denote by $\mathcal{C}_D^\mu(v_i)$ the family of all circuits in D containing the vertex v_i and of length at most μ. If $D[G_1, G_2, \ldots, G_n]$ is a digraph composition, we denote by G_i^c the copy of G_i as an induced subdigraph of $D[G_1, G_2, \ldots, G_n]$.

Theorem 28 (Szumny, Włoch, Włoch [20]). *Let $k \geq 2$, $1 \leq l \leq k - 1$ be integers. A subset $J^* \subseteq V(D[G_1, G_2, \ldots, G_n])$ is a (k, l)-kernel of the composition $D[G_1, G_2, \ldots, G_n]$ if and only if there exists a (k, l)-kernel $J \subseteq V(D)$ such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i | v_i \in J\}$, $J_i \subseteq V(G_i^c)$ and for every $i \in \mathcal{I}$*

1. *J_i is a (k, l)-kernel of G_i^c if $\mathcal{C}_D^{k-1}(v_i) = \emptyset$ or*

2. J_i is 1-element set containing an arbitrary vertex of $V(G_i^c)$ if $\mathcal{C}_D^l(v_i) \neq \emptyset$ or

3. J_i is 1-element set containing an l-absorbent vertex of G_i^c, otherwise.

To make an adequate use of this theorem we need to prove the following lemma.

Lemma 29. *If D is a strong semicomplete digraph and $v \in V(D)$, then v is contained in a directed cycle of length 2 or 3.*

Proof. Let D be a semicomplete digraph and $v \in V(D)$ a vertex. If any arc incident to or from v is symmetrical, then v is contained in a directed cycle of length 2. If every arc indicent to and from v is asymmetrical, we can consider the in-neighbourhood and out-neighbourhood of v, $N^-(v)$ and $N^+(v)$. Since D is a strong semicomplete digraph, there must exist a $N^+(v)N^-(v)$-arc in D, say (u, w), and thus (v, u, w, v) is a directed cycle of length 3. ∎

To finish the setup to prove the main theorem of this section, we need to observe that Theorem 5 has a very nice generalization for (k, l)-kernels. Let D be a digraph and let $x_1, x_2, \ldots, x_n$ be an ordering of its vertices. We call this ordering an acyclic ordering if, for every arc $(x_i, x_j) \in A(D)$, we have $i < j$. In [1], the following characterization of transitive digraphs is left as an excersise.

Proposition 30. *Let D be a digraph with an acyclic ordering $D_1, D_2, \ldots, D_p$ of its strong components. The digraph D is transitive if and only if each D_i is complete, $D^\star$ is a transitive oriented graph, and $D = D^\star[D_1, D_2, \ldots, D_p]$, where $D^\star$ is the condensation of D.*

Using Proposition 30, Theorem 5 can be generalized as follows.

Theorem 31. *If D is a transitive digraph, then D has a (k, l)-kernel for every $k \geq 2$ and every $l \geq 1$. Moreover, every (k, l)-kernel consists in one vertex from every terminal strong component of D, so all (k, l)-kernels of D have the same cardinality.*

Proof. Let D be a transitive digraph with an acyclic ordering $D_1, D_2, \ldots, D_p$ of its strong components. From Proposition 30 we have that $D^\star$ is a transitive acyclic digraph and $D = D^\star[D_1, D_2, \ldots, D_p]$, so, if v is a vertex of D that does not belong to a terminal strong component of D, then there

exists a terminal strong component of D, say S, such that $(v, s) \in A(D)$ for every $s \in S$. Besides, D_i is a complete digraph for every $i \in \{1, 2, \ldots, p\}$, so every vertex in D_i is absorbed by every other vertex in D_i. From these observations we can conclude that if we choose one vertex in every terminal strong component, then we obtain an (1-)absorbent set, say N. Also, for every vertex $v \in N$, since v is in a terminal strong component of D, there are no directed paths from v to any other strong component of D, so N is k-independent for every $k \geq 2$. The set N is the desired (k, l)-kernel, we have already observed that it is k-independent, and every for every vertex $u \in V(D) \setminus N$, there exists a vertex $v \in N$ such that $d(u, v) = 1 \leq l$, for each $l \geq 1$. ∎

Theorem 32. *If D is a quasi-transitive digraph, then D has a (k, l)-kernel for every pair of integers k, l such that $k \geq 4$ and $3 \leq l \leq k - 1$ or $k = 3$ and $l = 2$.*

Proof. Let $k \geq 4$ and $3 \leq l \leq k - 1$ or $k = 3$ and $l = 2$ be a fixed pair of integers. The proof is by mathematical induction on the order of D. If $|V(D)| = 1$ the result follows trivially, so let us assume the result valid for every quasi-transitive digraph with fewer than m vertices and let D be a digraph with exactly m vertices. We have two cases, when D is strong and when D is non-strong.

Case 1. If D is non-strong, as a consequence of Theorem 27 there exists an acyclic, transitive oriented graph T with vertices $\{u_1, u_2, \ldots, u_t\}$ and quasi-transitive strong digraphs $H_1, H_2, \ldots, H_t$ such that $D = T[H_1, H_2, \ldots, H_t]$. Theorem 31 assures the existence of a (k, l)-kernel with $k \geq 3$ and $2 \leq l \leq k - 1$ for every transitive digraph, so we can consider a (k, l)-kernel of T, say J, and since $H_1, H_2, \ldots, H_t$ are quasi-transitive digraphs of order strictly smaller than m, it follows from the inductive hypothesis that every H_i has (k, l)-kernel J_i. Since T is acyclic, we just have to consider the first case of Theorem 28, which asks H_i to have a (k, l)-kernel for every $u_i \in J$ such that $\mathcal{C}_D^{k-1}(u_i) = \emptyset$. It follows from Theorem 28 that D has a (k, l)-kernel.

Case 2. If D is strong, as a consequence of Theorem 27 there exists a strong semicomplete digraph S with vertex set $\{v_1, v_2, \ldots, v_s\}$ and quasi-transitive digraphs $Q_1, Q_2, \ldots, Q_s$ such that Q_i is a single vertex or is non-strong, and $D = S[Q_1, Q_2, \ldots, Q_s]$. Since S is a semicomplete digraph, it

follows from a well known result[2] that S has a 1-vertex quasi-kernel, which without loss of generality can be chosen as $\{v_1\}$. So $\{v_1\}$ is k-independent for every k and 2-absorbent, which implies that is also l-absorbent for every $2 \leq l \leq k-1$. Being S a strong semicomplete digraph, and as a consequence of Lemma 29, for every vertex $v \in V(S)$ there exists a directed cycle of length 2 or 3 containing v. Therefore, if $l \geq 3$, $\mathcal{C}_S^l(v_1) \neq \emptyset$ and in such case, applying Theorem 28, it suffices to consider $J = \{v_1\}$ and $J_1 = \{u\}$, where $u \in V(Q_1^c)$ is an arbitrary vertex. If $k = 3$, $l = 2$ and $\mathcal{C}_S^l(v_1) \neq \emptyset$, it also suffices to consider $J = \{v_1\}$ and $J_1 = \{u\}$, where $u \in V(Q_1^c)$ is an arbitrary vertex. If $k = 3$, $l = 2$ and $\mathcal{C}_S^l(v_1) = \emptyset$, then, as $k - 1 = 2$, also $\mathcal{C}_S^{k-1}(x_1) = \emptyset$, and for the first case of Theorem 28, we only need to choose J_1 as a $(3, 2)$-kernel for Q_1 that exists for the inductive hypothesis. It follows from Theorem 28 that $J^* = J_1$ is a (k, l)-kernel of D.

The result now follows from the principle of mathematical induction. ∎

As mentioned above, case $k = 2$ is not covered by Theorem 32, but in this case a k-kernel is a kernel in the classical sense of Berge. For kernels in quasi-transitive digraphs we have a powerful sufficient condition given by the Strong Perfect Graph Theorem, conjectured by Berge and finally proven by Chudnovsky, Robertson, Seymour and Thomas in [8]. An odd hole in a graph is an induced odd cycle, and odd anti-hole in a graph is an induced subgraph isomorphic to the complement of an odd cycle.

Theorem 33 (Strong Perfect Graph Theorem). *A graph is perfect if and only if it has no odd holes nor odd anti-holes of length greater than or equal to 5.*

Together with an additional characterization and an observation we will reach the desired result.

Definition. Let G be a graph. We call an orientation D of G *clique-acyclic* if every clique in G has a kernel in D.

A graph G is called *kernel solvable* if every clique-acyclic orientation of G has a kernel.

Boros and Gurvich proved in [7] that a graph is kernel solvable if it is perfect. The converse of this result is a consequence of Theorem 33 so the next theorem can be stated.

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[2] Every tournament has a $(2, 2)$-kernel.

Theorem 34. *A graph G is perfect if and only if it is kernel solvable.*

Applying Theorem 34 and remembering that underlying graphs of asymmetrical quasi-transitive digraphs are comparability graphs[3] and that comparability graphs are perfect[4], we obtain a sufficient condition for an asymmetrical quasi-transitive digraph D to have a kernel (in fact, to be kernel perfect).

Theorem 35. *If D is an asymmetrical quasi-transitive digraph such that every maximal semicomplete subdigraph of D has a kernel, then D is kernel perfect.*

Also, Galeana-Sánchez and Rojas-Monroy proved in [12] the following sufficient condition for a quasi-transitive digraph to have a kernel.

Theorem 36. *Let D be a (possibly infinite) digraph such that $D = D_1 \cup D_2$ (possibly $A(D_1) \cap A(D_2) \neq \emptyset$), where D_i is a quasi-transitive subdigraph of D which contains no asymmetrical (in D) infinite outward path. If every triangle contained in D has at least two symmetrical arcs, then D is a kernel perfect digraph.*

Corollary 37. *If D is a quasi-transitive digraph such that every triangle contained in D has at least two symmetrical arcs, then D is kernel-perfect.*

Lemma 38. *Let D be a digraph. Then D is a quasi-transitive digraph if and only if $\overleftarrow{D}$ is a quasi-transitive digraph.*

Proof. Straightforward. ■

Corollary 39. *If D is a quasi-transitive digraph, then D has (k, l)-solution for every pair of integers k, l such that $k \geq 4$ and $3 \leq l \leq k - 1$ or $k = 3$ and $l = 2$.*

Acknowledgments

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[3] Proved by Ghouila-Houri in [13].
[4] Proved by Berge in [5].

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