Discussiones Mathematicae
Graph Theory 31 (2011) 283–292

MONOCHROMATIC CYCLES AND MONOCHROMATIC PATHS IN ARC-COLORED DIGRAPHS

Hortensia Galeana-Sánchez

Guadalupe Gaytán-Gómez

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Ciudad Universitaria, México, D.F. 04510, México

e-mail: hgaleana@matem.unam.mx
gaytan@matem.unam.mx

and

Rocío Rojas-Monroy

Facultad de Ciencias
Universidad Autónoma del Estado de México
Instituto Literario No. 100, Centro 50000, Toluca, Edo. de México, México

e-mail: mrrm@uaemex.mx

Abstract

We call the digraph D an m-colored digraph if the arcs of D are colored with m colors. A path (or a cycle) is called monochromatic if all of its arcs are colored alike. A cycle is called a quasi-monochromatic cycle if with at most one exception all of its arcs are colored alike. A subdigraph H in D is called rainbow if all its arcs have different colors. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions: (i) for every pair of different vertices $u, v \in N$ there is no monochromatic path between them and; (ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-monochromatic path. The closure of D, denoted by $\mathfrak{C}(D)$, is the m-colored multidigraph defined as follows: $V(\mathfrak{C}(D)) = V(D)$, $A(\mathfrak{C}(D)) = A(D) \cup \{(u, v)$ with color $i \mid$ there exists a uv-monochromatic path colored i contained in $D\}$. Notice that for any digraph D, $\mathfrak{C}(\mathfrak{C}(D)) \cong \mathfrak{C}(D)$ and D has a kernel by monochromatic paths if and only if $\mathfrak{C}(D)$ has a kernel.

Let D be a finite m-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of D such that every cycle in the subdigraph $D[C_i]$ spanned by the arcs with colors in C_i is monochromatic. We show that if $\mathfrak{C}(D)$ does not contain neither rainbow triangles nor rainbow P_3 involving colors of both C_1 and C_2, then D has a kernel by monochromatic paths.

This result is a wide extension of the original result by Sands, Sauer and Woodrow that asserts: Every 2-colored digraph has a kernel by monochromatic paths (since in this case there are no rainbow triangles in $\mathfrak{C}(D)$).

Keywords: kernel, kernel by monochromatic paths, monochromatic cycles.

2010 Mathematics Subject Classification: 05C20.

1. Introduction

For general concepts we may refer the reader to [1]. Let D be a digraph, and let $V(D)$ and $A(D)$ denote the sets of vertices and arcs of D, respectively. We recall that a subdigraph D_1 of D is a spanning subdigraph if $V(D_1) = V(D)$. If S is a nonempty subset of $V(D)$ then the subdigraph $D[S]$ induced by S is the digraph having vertex set S, and whose arcs are all those arcs of D joining vertices of S. An arc u_1u_2 of D will be called an S_1S_2-arc of D whenever $u_1 \in S_1$ and $u_2 \in S_2$.

A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D, that is an arc from z towards some vertex in N. A digraph D is called a kernel-perfect digraph when every induced subdigraph of D has a kernel. Sufficient conditions for the existence of kernels in digraphs have been investigated by several authors, Duchet and Meyniel [4]; Duchet [2, 3]; Galeana-Sánchez and Neumann-Lara [5, 6]. The concept of kernel is very useful in applications.

We call the digraph D an m-colored digraph if the arcs of D are colored with m colors. Along this paper, all the paths and cycles will be directed paths and directed cycles. A path is called monochromatic if all of its arcs are colored alike. A subdigraph H of D is called rainbow if all its arcs have distinct colors. A set $N \subseteq V(D)$ is called a kernel by monochromatic paths if for every pair of different vertices $u, v \in N$ there is no monochromatic path between them and for every vertex $v \in V(D) - N$ there is a monochromatic path from v to some vertex in N.

In [12] Sands, Sauer and Woodrow have proved that any 2-colored digraph D has an independent set S of vertices of D such that, for every vertex $x \notin S$, there is a monochromatic path from x to a vertex of S (i.e., D has a kernel by monochromatic paths, concept that was introduced later by Galeana-Sánchez [7].) In particular, they proved that any 2-colored tournament T has a kernel by monochromatic paths. They also raised the following problem: Let T be a 3-colored tournament such that every cycle of length 3 is a quasi-monochromatic cycle; must T have a kernel by monochromatic paths? (This question still remains open.) In [11] Shen Minggang proved that if T is an m-colored tournament such that every cycle of length 3 is a quasi-monochromatic cycle, and every transitive tournament of order 3 is quasi-monochromatic, then T has a kernel by monochromatic paths. He also proved that this result is the best possible for m-colored tournaments with $m \geq 5$. In fact, he proved that for each $m \geq 5$ there exists an m-colored tournament T such that every cycle of length 3 is quasi-monochromatic and T has no kernel by monochromatic paths. Also for every $m \geq 3$ there exists an m-colored tournament T' such that every transitive tournament of order 3 is quasi-monochromatic and T' has no kernel by monochromatic paths. In 2004 [10] H. Galeana-Sánchez and R. Rojas-Monroy presented a 4-colored tournament T such that every cycle of order 3 is quasi-monochromatic; but T has no kernel by monochromatic paths. The known sufficient conditions for the existence of kernel by monochromatic paths in m-colored ($m \geq 3$) tournaments (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of certain subdigraphs. More information on m-colored digraphs can be found in [7, 8, 9, 13, 14].

If $\mathcal{C} = (z_0, z_1, \ldots, z_n, z_0)$ is a cycle, we will denote by $\ell(\mathcal{C})$ its length and if $z_i, z_j \in V(\mathcal{C})$ with $i \leq j$ we denote by $(z_i, \mathcal{C}, z_j)$ the $z_i z_j$-path contained in $\mathcal{C}$, and $\ell(z_i, \mathcal{C}, z_j)$ will denote its length.

The following is our main result:

Theorem 1. *Let D be a finite m-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of D such that every cycle in the subdigraph $D[C_i]$ spanned by the arcs with colors in C_i is monochromatic. Suppose, moreover, that $\mathfrak{C}(D)$ does not contain neither rainbow triangles nor rainbow P_3 involving colors of both C_1 and C_2. Then D has a kernel by monochromatic paths.*

Notice that the Theorem 1 implies the Theorem of Sands, Sauer and Woodrow in the finite case by taking as a partition each of the two colors: all cycles in each color class are trivially monochromatic and $\mathfrak{C}(D)$ has no rainbow subdigraphs.

We will need the following basic elementary results.

Lemma 2. *Let D be a digraph; $u, v \in V(D)$. Every uv-monochromatic walk in D contains a uv-monochromatic path.*

Lemma 3. *Let D be a digraph. Every closed walk in D contains a cycle.*

Lemma 4. *Let D be a digraph. If for every $v \in V(D)$ fulfills that $\delta_D^-(v) \geq 1$ ($\delta_D^+(v) \geq 1$) then D contains a cycle.*

And the following Theorem.

Theorem 5 (Berge-Duchet [2])**.** *If D is a digraph such that every cycle of D has at least one symmetrical arc, then D is a kernel-perfect digraph.*

2. Monochromatic Cycles and Monochromatic Paths in Arc-colored Digraphs

The following lemmas are about m-colored digraphs such that each cycle is monochromatic, and they are useful to prove our main result.

Lemma 6. *Let D be a finite or infinite m-colored digraph such that every cycle in D is monochromatic. If $C = (u_0, u_1, \ldots, u_{n-1})$ is a sequence of $n \geq 2$ vertices, different by pairs, such that for every $i \in \{0, \ldots, n-1\}$ T_i is some $u_i u_{i+1}$-monochromatic path then the set of paths $\{T_i \mid i \in \{0, \ldots, n-1\}\}$ is monochromatic, that is, the paths T_i are of the same color by pairs (the indices of the vertices will be taken modulo n.)*

Proof. Assume, by contradiction, that there exists a sequence of vertices $(u_0, u_1, \ldots, u_{n-1})$ such that for every $i \in \{0, \ldots, n-1\}$ there exists a $T_i = u_i u_{i+1}$-monochromatic path in D and the set of paths $\{T_i \mid i \in \{0, \ldots, n-1\}\}$ is not monochromatic. Choose such a counterexample with a minimal number of arcs. Then from Lemma 3 the subdigraph induced by this walk contains a cycle which involves more than one path. Since all cycles in D are monochromatic, we can not consider the arcs of the cycle and obtain a counterexample with a smaller number of arcs, a contradiction. ∎

As a direct result from Lemma 6 we have:

Remark 7. If D is an m-colored digraph such that every cycle is monochromatic then in $\mathfrak{C}(D)$ every cycle is monochromatic.

Remark 8. If D is an m-colored digraph such that every cycle is monochromatic then in $\mathfrak{C}(D)$ every cycle is symmetrical.

Proof. It follows from Remark 7 and the fact that $\mathfrak{C}(\mathfrak{C}(D)) \cong \mathfrak{C}(D)$. ■

Lemma 9. *Let D be a finite m-colored digraph such that every cycle in D is monochromatic. Then there exists $x_0 \in V(D)$ such that for every $z \in V(D) - \{x_0\}$ if there exists a x_0z-monochromatic path contained in D then there exists a zx_0-monochromatic path contained in D.*

Proof. Assume, for a contradiction, that D is a digraph as in the hypothesis and that there is no vertex x_0 satisfying the affirmation from Lemma 9. It follows that $Asym\mathfrak{C}(D)$ has a cycle. On the other hand, from Remark 8 we have that every cycle in $\mathfrak{C}(D)$ is symmetric, a contradiction. ■

Let D be an m-colored digraph and let H be a subdigraph of D. We will say that $S \subseteq V(D)$ is a semikernel by monochromatic paths *modulo* H of D if S is independent by monochromatic paths in D and for every $z \in V(D) - S$, if there is a Sz-monochromatic path contained in $D - H$ then there is a zS-monochromatic path contained in D.

Lemma 10. *Let D be a finite m-colored digraph. Let H be a subdigraph of D such that every directed cycle in $D - H$ is monochromatic. Then there exists $x_0 \in V(D)$ which satisfies that $\{x_0\}$ is a semikernel by monochromatic paths mod H of D.*

Proof. It follows by applying Lemma 9 to $D - H$. ■

Let

$$\mathcal{S} = \{\emptyset \neq S \mid \ S \text{ is a semikernel by monochromatic paths } \textit{mod } D_2 \text{ of } D\}.$$

Whenever $\mathcal{S} \neq \emptyset$, we will denote by $D_{\mathcal{S}}$ the digraph defined as follows: $V(D_{\mathcal{S}}) = \mathcal{S}$ (i.e, for every element of $\mathcal{S}$ we put a vertex in $D_{\mathcal{S}}$) and $(S_1, S_2) \in A(D_{\mathcal{S}})$ if and only if for every $s_1 \in S_1$ there exists $s_2 \in S_2$ such that $s_1 = s_2$, or there exists a s_1s_2-monochromatic path contained in D_2 and there is no s_2S_1-monochromatic path contained in D.

Lemma 11. *Let D be a finite m-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of D such that every cycle in the subdigraph $D[C_i]$ spanned by the arcs with colors in C_i is monochromatic. Then $D_{\mathcal{S}}$ is an acyclic digraph.*

Proof. Observe that by Lemma 10, there exists a semikernel by monochromatic paths *mod* D_2 of D. Thus $\mathcal{S} \neq \emptyset$ and we can consider the digraph $D_{\mathcal{S}}$. Suppose for a contradiction, that $D_{\mathcal{S}}$ contains some cycle, say $\mathcal{C} = (S_0, S_1, \ldots, S_{n-1}, S_0)$ of length $n \geq 2$. Since $\mathcal{C}$ is a cycle in $D_{\mathcal{S}}$, we have that $S_i \neq S_j$ whenever $i \neq j$.

Claim 1. There exists $i_0 \in \{0, 1, 2, \ldots, n-1\}$ such that for some $z \in S_{i_0}$, $z \notin S_{i_0+1}$ $(mod\ n)$.

Otherwise, for every $i \in \{0, 1, \ldots, n-1\}$ and every $z \in S_i$ we have that $z \in S_{i+1}$ and then $S_i = S_j$ for all $i, j \in \{0, 1, \ldots, n-1\}$. So, $\mathcal{C} = (S_0)$, which is a contradiction since a cycle contains at least two vertices.

Claim 2. If there exists $i_0 \in \{0, 1, \ldots, n-1\}$ such that for some $z \in S_{i_0}$ and some $w \in S_{i_0+1}$ $(mod\ n)$ there exists a zw-monochromatic path; then there exists $j_0 \neq i_0$, $j_0 \in \{0, 1, \ldots, n-1\}$ such that $w \in S_{j_0}$ and $w \notin S_{j_0+1} (mod\ n)$.

Suppose without loss of generality that $i_0 = 0$. First, observe that $w \notin S_n = S_0$ since otherwise we have a zw-monochromatic path with $\{z, w\} \subseteq S_0$, contradicting that S_0 is independent by monochromatic paths. Since $w \in S_1$, let $j_0 = \max\{i \in \{0, 1, \ldots, n-1\} \mid w \in S_i\}$ (notice that for both previous observations j_0 is well defined.) So, $w \in S_{j_0}$ and $w \notin S_{j_0+1}$.

It follows from Claim 1 that there exists $i_0 \in \{0, \ldots, n-1\}$ and $t_0 \in S_{i_0}$ such that $t_0 \notin S_{i_0+1}$. It follows from the fact that $(S_{i_0}, S_{i_0+1}) \in F(D_{\mathcal{S}})$ that there exists $t_1 \in S_{i_0+1}$ such that there exists a t_0t_1-monochromatic path contained in D_2 and there is no $t_1S_{i_0}$-monochromatic path contained in D. From Claim 2, it follows that there exists an index $i_1 \in \{0, \ldots, n-1\}$ such that $t_1 \in S_{i_1}$ and $t_1 \notin S_{i_1+1}$. Since $(S_{i_1}, S_{i_1+1}) \in F(D_{\mathcal{S}})$ it follows that there exists $t_2 \in S_{i_1+1}$ such that there is a t_1t_2-monochromatic path contained in D_2 and there is no $t_2S_{i_1}$-monochromatic path contained in D. Since D is finite, we obtain a sequence of vertices $(t_0, t_1, t_2, \ldots, t_{m-1})$ such that there exists a t_it_{i+1}-monochromatic path contained in D_2 and there is no $t_{i+1}t_i$-monochromatic path contained in D for every $i \in \{0, 1, 2, \ldots, m-1\}$ $(mod\ m)$. But this contradicts Lemma 6. Therefore $D_{\mathcal{S}}$ is an acyclic digraph. ■

3. The Main Result

The following theorem is a particular case from our Main Result.

Theorem 12. *Let D be an m-colored digraph such that every cycle in D is monochromatic, then D has a kernel by monochromatic paths.*

Proof. It follows from Remark 8 and Theorem 5 that $\mathfrak{C}(D)$ has a kernel and so D has a kernel by monochromatic paths. ■

The main idea of the proof of our main theorem is to select $S \in V(D_{\mathcal{S}})$ such that $\delta^{+}_{D_{\mathcal{S}}}(S) = 0$ (such S exists since $D_{\mathcal{S}}$ is acyclic) and prove that S is a kernel by monochromatic paths of D.

We next proceed to prove our main result, Theorem 1.

***Proof of Theorem* 1.** Consider the digraph $D_{\mathcal{S}}$ of the digraph D. Since $D_{\mathcal{S}}$ is a finite digraph and from Lemma 11 it does not contain cycles, it follows that $D_{\mathcal{S}}$ contains at least a vertex of zero outdegree. Let $S \in V(D_{\mathcal{S}})$ be such that $\delta^{+}_{D_{\mathcal{S}}}(S) = 0$.

We will prove that S is a kernel by monochromatic paths of D.
Suppose for a contradiction, that S is not a kernel by monochromatic paths of D. Since $S \in V(D_{\mathcal{S}})$, we have that S is independent by monochromatic paths.

Let

$$X = \{z \in V(D) \mid \text{there is no } zS\text{-monochromatic path in } D\}.$$

It follows from our assumption that $X \neq \emptyset$. Since $D[X]$ is an induced subdigraph of D, we have that $D[X]$ satisfies the hypotheses from Lemma 11. So, it follows that there exists $x_0 \in X$ such that $\{x_0\}$ is a semikernel by monochromatic paths *mod* D_2 of D.

Let

$$T = \{z \in S \mid \text{there is no } zx_0\text{-monochromatic path in } D_2\}.$$

From the definition of T, we have that for every $z \in (S - T)$ there exists a zx_0-monochromatic path contained in D_2.

Claim 13. $T \cup \{x_0\}$ is independent by monochromatic paths.

It follows directly from the facts that $T \subseteq S$, $S \in \mathcal{S}$ and $x_0 \in X$.

Claim 14. For each $z \in V(D) - T \cup \{x_0\}$, if there exists a $(T \cup \{x_0\})z$-monochromatic path contained in D_1, then there exists a $z(T \cup \{x_0\})$-monochromatic path contained in D.

Case 1. There exists a Tz-monochromatic path contained in D_1.
Since $T \subseteq S$ and $S \in \mathcal{S}$, it follows that there exists a zS-monochromatic path contained in D. We may suppose that there exists a $z(S-T)$-monochromatic path contained in D. Let α_1 be a uz-monochromatic path contained in D_1 with $u \in T$, and let α_2 be a zw-monochromatic path with $w \in (S - T)$ contained in D. Since $w \in (S - T)$ it follows from the definition of T that there exists α_3 a wx_0-monochromatic path contained in D_2.

Moreover, color$(\alpha_1) \neq$ color(α_2) (color(α) denotes the color used in the arcs of α) otherwise there exists a uw-monochromatic path contained in $\alpha_1 \cup \alpha_2$, with $\{u, w\} \subseteq S$, in contradiction with the fact that S is independent by monochromatic paths. In addition, we will suppose that color $(\alpha_2) \neq$ color(α_3) since if color$(\alpha_2) =$ color(α_3) then $\alpha_2 \cup \alpha_3$ contains a zx_0-monochromatic path and Claim 2 is proved. Also color$(\alpha_1) \neq$ color(α_3) as color$(\alpha_1) \in C_1$ and color$(\alpha_3) \in C_2$.

So, we obtain that (u, z, w, x_0) is a rainbow P_3 in $\mathfrak{C}(D)$ involving colors of both C_1 and C_2, a contradiction.

Case 2. There exists a x_0z-monochromatic path contained in D_1.
Let α_1 be such a path, we may suppose that $z \notin X$. It follows from the definition of X that there exists some zS-monochromatic path contained in D, let α_2 be such path, say that α_2 ends in w. We will suppose that $w \in (S - T)$. Since $w \in (S - T)$, by the definition of T, we have that there exists a wx_0-monochromatic path contained in D_2, let α_3 be such a path.

Again, we have that color$(\alpha_1) \neq$ color(α_2) otherwise there exists a x_0w-monochromatic path contained in $\alpha_1 \cup \alpha_2$, contradicting that $x_0 \in X$ and $w \in S$. In addition, we will suppose that color$(\alpha_2) \neq$ color(α_3) since if color$(\alpha_2) =$ color(α_3) then $\alpha_2 \cup \alpha_3$ contains a zx_0-monochromatic path and Claim 2 is proved. Also color$(\alpha_1) \neq$ color(α_3) since $\alpha_1 \subseteq D_1$ and $\alpha_3 \subseteq D_2$.

Then (x_0, z, w, x_0) is a rainbow C_3 in $\mathfrak{C}(D)$ which involves colors of both C_1 and C_2, a contradiction.

We conclude from Claims 1 and 2 that $T \cup \{x_0\} \in \mathcal{S}$ and therefore $T \cup \{x_0\} \in V(D_{\mathcal{S}})$. We have that $(S, T \cup \{x_0\}) \in F(D_{\mathcal{S}})$ since $T \subseteq T \cup \{x_0\}$, and for each $s \in S - T$ there exists a sx_0-monochromatic path contained in D_2 and there is no x_0S-monochromatic path contained in D. But this contradicts the fact that $\delta^+_{D_S}(S) = 0$. Therefore S is a kernel by monochromatic paths in D and the Theorem is proved. ■

Remark 15. Notice that while in Theorem 12 it is asked for every cycle to be monochromatic, in the Theorem 1 there could exist non monochromatic cycles since the monochromatic cycles only are asked for each D_i, $i \in \{1, 2\}$.

Acknowledgement

The authors thank the anonymous referee for many suggestions which improve substantially the rewriting of this paper.

References

[1] C. Berge, Graphs (North-Holland, Amsterdam, 1985).

[2] P. Duchet, *Graphes Noyau – Parfaits*, Ann. Discrete Math. **9** (1980) 93–101.

[3] P. Duchet, *Classical Perfect Graphs, An introduction with emphasis on triangulated and interval graphs*, Ann. Discrete Math. **21** (1984) 67–96.

[4] P. Duchet and H. Meyniel, *A note on kernel-critical graphs*, Discrete Math. **33** (1981) 103–105.

[5] H. Galeana-Sánchez and V. Neumann-Lara, *On kernels and semikernels of digraphs*, Discrete Math. **48** (1984) 67–76.

[6] H. Galeana-Sánchez and V. Neumann-Lara, *On kernel-perfect critical digraphs*, Discrete Math. **59** (1986) 257–265.

[7] H. Galeana-Sánchez, *On monochromatic paths and monochromatics cycles in edge coloured tournaments*, Discrete Math. **156** (1996) 103–112.

[8] H. Galeana-Sánchez, *Kernels in edge-coloured digraphs*, Discrete Math. **184** (1998) 87–99.

[9] H. Galeana-Sánchez and J.J. García-Ruvalcaba, *Kernels in the closure of coloured digraphs*, Discuss. Math. Graph Theory **20** (2000) 103–110.

[10] H. Galeana-Sánchez and R. Rojas-Monroy, *A counterexample to a conjecture on edge-coloured tournaments*, Discrete Math. **282** (2004) 275–276.

[11] S. Minggang, *On monochromatic paths in m-coloured tournaments*, J. Combin. Theory (B) **45** (1988) 108–111.

[12] B. Sands, N. Sauer and R. Woodrow, *On monochromatic paths in edge-coloured digraphs*, J. Combin. Theory (B) **33** (1982) 271–275.

[13] I. Włoch, *On kernels by monochromatic paths in the corona of digraphs*, Cent. Eur. J. Math. **6** (2008) 537–542.

[14] I. Włoch, *On imp-sets and kernels by monochromatic paths in duplication*, Ars Combin. **83** (2007) 93–99.

Received 26 November 2009
Revised 18 December 2010
Accepted 19 December 2010