Discussiones Mathematicae
Graph Theory 31 (2011) 273–281

KERNELS BY MONOCHROMATIC PATHS AND THE COLOR-CLASS DIGRAPH

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Abstract

An m-colored digraph is a digraph whose arcs are colored with m colors. A directed path is monochromatic when its arcs are colored alike.

A set $S \subseteq V(D)$ is a kernel by monochromatic paths whenever the two following conditions hold:

1. For any $x, y \in S$, $x \neq y$, there is no monochromatic directed path between them.
2. For each $z \in (V(D) - S)$ there exists a zS-monochromatic directed path.

In this paper it is introduced the concept of color-class digraph to prove that if D is an m-colored strongly connected finite digraph such that:

(i) Every closed directed walk has an even number of color changes,

(ii) Every directed walk starting and ending with the same color has an even number of color changes, then D has a kernel by monochromatic paths.

This result generalizes a classical result by Sands, Sauer and Woodrow which asserts that any 2-colored digraph has a kernel by monochromatic paths, in case that the digraph D be a strongly connected digraph.

Keywords: kernel, kernel by monochromatic paths, the color-class digraph.

2010 Mathematics Subject Classification: 05C20.

1. Introduction

For general concepts we refer the reader to [12] and [3]. Let D be a digraph, a set of vertices $S \subseteq V(D)$ is dominating whenever for every $w \in V(D) - S$ there exists a wS-arc in D. (The topic of domination in graphs has been deeply studied by several authors, a very complete study of this topic can be found in [13] and [14]).

Dominating independent sets in digraphs (kernels in digraphs) have found many applications in several topics of Mathematics (see for example [1, 2, 5, 6] and [15]) and they have been studied by several authors, surveys of kernels in digraphs can be found in [4] and [6]. Clearly the concept of kernel by monochromatic paths is a generalization of that of kernel.

The study of the existence of kernels by monochromatic paths in edge-colored digraphs begins with the Theorem of Sands, Sauer and Woodrow proved in [16] which asserts that every 2-colored digraph has a kernel by monochromatic paths. Sufficient conditions for the existence of kernels by monochromatic paths in edge-colored digraphs have been obtained mainly in nearly tournaments and they ask for the monochromaticity or quasi-monochromaticity of small subdigraphs (due to the difficulty of the problem), see for example [8, 9, 10, 11, 7, 17] and [18].

In this paper we give a different aproach to obtain suffcient conditions for the existence of a kernel by monochromatic paths in an edge-colored digraph. We introduce the concept of color-class digraph of an m-colored digraph D and study some structural properties of that digraph which imply that D possesses a kernel by monochromatic paths. As a consequence it is obtained a wide generalization of the classical result of Sands, Sauer and Woodrow in the case that the digraph D be strongly connected.

2. The Color-class Digraph of an m-colored Digraph D

In this section the color-class digraph of an m-colored digraph D is defined; it is proved that some structural properties of this digraph allow us to consider that the m-colored digraph D is essentially 2-colored and we can conclude that D has a kernel by monochromatic paths.

Definition. Let D be an m-colored digraph. The color-class digraph of D denoted $\mathcal{C}_C(D)$ is defined as follows:
$V(\mathcal{C}_C(D)) = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$ where $\mathcal{C}_i$ is the subdigraph of D whose arcs

are the arcs of D colored i and the vertices of $\mathcal{C}$; are those vertices of D which are initial endpoints or terminal endpoints of the arcs colored i; $\mathcal{C}_i$ will be called the color-class i of D (Notice that since D is an m-colored digraph, we have $\mathcal{C}_i \neq \emptyset$ for each $1 \leq i \leq m$).

$(\mathcal{C}_i, \mathcal{C}_j) \in A(\mathcal{C}_C(D))$ if and only if there exists two arcs namely $f = (u, v) \in A(D)$ colored i and $g = (v, w) \in A(D)$ colored j.

Observe that $\mathcal{C}_C(D)$ may allow isolated vertices.

Lemma 2.1. *Let D be an m-colored digraph. If D is a strongly connected digraph, then $\mathcal{C}_C(D)$ is a strongly connected digraph.*

Proof. Let $\mathcal{C}_i, \mathcal{C}_j$ be two different vertices of $\mathcal{C}_C(D)$. Since D is an m-colored digraph, there exist $f = (u, v) \in A(\mathcal{C}_i)$ and $g = (z, w) \in A(\mathcal{C}_j)$. If $v = z$ then $(\mathcal{C}_i, \mathcal{C}_j)$ is a $\mathcal{C}_i\mathcal{C}_j$-directed path in $\mathcal{C}_C(D)$. If $v \neq z$ then we have that there exists a vz-directed path contained in D (because D is an strongly connected digraph). Let $T = (v = u_1, u_1, u_2, \ldots, u_{n-1} = z)$ and $P = (u_0 = u, v) \cup T \cup (u_{n-1} = z, u_n = w)$, $P = (u_0 = u, u_1 = v, u_2, u_3, \ldots, u_{n-1} = z, u_n = w)$. Take $u_{i_1}, u_{i_2}, \ldots, u_{i_k}$ the vertices of P where a color change occurs. So the walk P has k color changes and $(u, T, u_{i_1}) \subseteq \mathcal{C}_i$, $(u_{i_1}, T, u_{i_2}) \subseteq \mathcal{C}_{r_2}$, $(u_{i_2}, T, u_{i_3}) \subseteq \mathcal{C}_{r_3}, \ldots, (u_{i_{k-1}}, T, u_{i_k}) \subseteq \mathcal{C}_{r_k}$, $(u_{i_k}, P, w) \subseteq \mathcal{C}_j$ for some $\{r_2, \ldots, r_k\} \subseteq \{1, 2, \ldots, m\}$. Clearly we have that $\widehat{\mathcal{P}} = (\mathcal{C}_i, \mathcal{C}_{r_2}, \mathcal{C}_{r_3}, \mathcal{C}_{r_4}, \ldots, \mathcal{C}_{r_k}, \mathcal{C}_j)$ is a $\mathcal{C}_i\mathcal{C}_j$-directed walk in $\mathcal{C}_C(D)$. Therefore there exists a $\mathcal{C}_i\mathcal{C}_j$-directed path in $\mathcal{C}_C(D)$. ■

Lemma 2.2. *Let D be an m-colored digraph with color classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ such that neither the pair $(\mathcal{C}_1, \mathcal{C}_2)$ nor $(\mathcal{C}_1, \mathcal{C}_2)$ are arcs of the color-class digraph. And, let $\widehat{D}$ the $(m-1)$-colored digraph obtained from D by assigning color 1 to each arc of D colored 2 (Thus the arcs of D colored 2 are now colored 1 in $\widehat{D}$, the rest of the arcs of D remain the same). For any $u, v \in V(D) = V(\widehat{D})$, $u \neq v$; there exists a uv-monochromatic directed path in D if and only if there exists a uv-monochromatic directed path in $\widehat{D}$.*

Proof. First notice that the digraph $\widehat{D}$ is the same as D except that the arcs of $\mathcal{C}_2$ in D are now colored 1 in $\widehat{D}$; the color classes of $\widehat{D}$ are $\mathcal{C}'_1, \mathcal{C}'_2, \ldots, \mathcal{C}'_{m-1}$ where $\mathcal{C}'_1 = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}'_j = \mathcal{C}_{j+1}$.

First suppose that there exists a uv-monochromatic directed path contained in D and let P be such that a path. Thus $P \subseteq \mathcal{C}_i$ for some $i \in \{1, 2, \ldots, m\}$; for $i \in \{3, \ldots, m\}$ we have $\mathcal{C}_i = \mathcal{C}'_{i-1}$ and for $i \in \{1, 2\}$ we have $\mathcal{C}_i \subseteq \mathcal{C}'_1$ thus $P \subseteq \mathcal{C}'_j$ for some $j \in \{1, 2, \ldots, m-1\}$ which means that P is a uv-monochromatic directed path in $\widehat{D}$.

Now suppose that T is a uv-monochromatic directed path in $\widehat{D}$. Thus $T \subseteq \mathcal{C}'_j$ for some $j \in \{1, \ldots, m-1\}$. When $j \in \{2, \ldots, m-1\}$ we have $\mathcal{C}'_j = \mathcal{C}_{j+1}$ and T is a uv-monochromatic directed path in D. So, suppose $T \subseteq \mathcal{C}'_1 = \mathcal{C}_1 \cup \mathcal{C}_2$; when $T \subseteq \mathcal{C}_1$ or $T \subseteq \mathcal{C}_2$ we have that T is a uv-monochromatic directed path in D. Henceforth we have $T \subseteq \mathcal{C}_1 \cup \mathcal{C}_2$, $T \nsubseteq \mathcal{C}_1$ and $T \nsubseteq \mathcal{C}_2$; assume without loss of generality that T starts in $\mathcal{C}_1$. Let $T = (u_0, u_1, \ldots, u_n)$ and $g = (u_i, u_{i+1})$ the first arc of T belonging to $\mathcal{C}_2$; hence $f = (u_{i-1}, u_i) \in A(\mathcal{C}_1)$ and it follows from the definition of $\mathcal{C}_C(D)$ that $(\mathcal{C}_1, \mathcal{C}_2) \in A(\mathcal{C}_C(D))$ contradicting our assumption. We conclude that T is a uv-monochromatic directed path in D. ∎

Corollary 2.3. *Let D be an m-colored digraph and $\widehat{D}$ the $(m-1)$-colored digraph obtained from D as in the hypothesis of Lemma* 2.2. *A set $N \subseteq V(D) = V(\widehat{D})$ is a kernel by monochromatic paths of D if and only if it is a kernel by monochromatic paths of $\widehat{D}$.*

Theorem 2.4 (Sands, Sauer and Woodrow [16])**.** *If D is a* 1*-colored* (*monochromatic digraph*) *or D is a* 2*-colored digraph, then D has a kernel by monochromatic paths.*

This theorem will be usesul to prove the next theorem which is the main result of this section

Theorem 2.5. *Let D be an m-colored digraph. If $\mathcal{C}_C(D)$ is a bipartite digraph, then D has a kernel by monochromatic paths.*

Proof. We proceed by induction on $|V(\mathcal{C}_C(D))|$ (i.e., we proceed by induction on m).

For $m = 1$ or $m = 2$ the result follows directly from Theorem 2.4. Suppose that if D' is an $(m-1)$-colored digraph such that $\mathcal{C}_C(D')$ is bipartite (i.e., that $|V(\mathcal{C}_C(D'))| = m-1$), then D' has a kernel by monochromatic paths, for $m \geq 3$.

Let D be an m-colored digraph, and let V_1, V_2 the bipartition of $V(\mathcal{C}_C(D))$ which witnesses that $\mathcal{C}_C(D)$ is bipartite; so V_1 (resp. V_2) is an independent set in $\mathcal{C}_C(D)$. Since $m \geq 3$, $m = |V(\mathcal{C}_C(D))|$ we have $|V_1| \geq 2$ or $|V_2| \geq 2$; without loss of generality assume that $|V_1| \geq 2$ and let $\mathcal{C}_1, \mathcal{C}_2 \in V_1$. Consider $\widehat{D}$ the $(m-1)$-colored digraph obtained from D as in the hypothesis of Lemma 2.2. Clearly $\mathcal{C}_C(\widehat{D})$ is the digraph obtained from $\mathcal{C}_C(D)$ by identifying the vertices $\mathcal{C}_1$ and $\mathcal{C}_2$. Since $\mathcal{C}_C(D)$ is bipartite, we have that $\mathcal{C}_C(\widehat{D})$ is also bipartite. Thus it follows from the inductive hypothesis that

$\widehat{D}$ has a kernel by monochromatic paths; let N be such a kernel. Henceforth it follows from Corollary 2.3 that N is a kernel by monochromatic paths of D. ■

3. Kernels by Monochromatic Paths

In this section we study a condition on D which implies that $\mathcal{C}_C(D)$ is bipartite which from Theorem 2.5 implies that D has a kernel by monochromatic paths.

Theorem 3.1. *Let D be a strongly connected m-colored digraph. If D satisfies the two following conditions:*

(a) *Every closed directed walk in D possesses an even number of color changes.*

(b) *Every directed walk starting end ending in arcs of the same color has an even number of color changes.*

Then every directed cycle in $\mathcal{C}_C(D)$ has an even length.

Proof. Assume by contradiction that $\gamma = (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{2n}, \mathcal{C}_0)$ is an odd directed cycle in $\mathcal{C}_C(D)$. Where i is the color associated to $\mathcal{C}_i$. Then, from the definition of $\mathcal{C}_C(D)$ we have that there exists arcs $f_i = (x_i, y_i), f'_i = (x'_i, y'_i)$ both colored i for $i \in \{0, 1, \ldots, 2n\}$ such that $y'_i = x_{i+1}$, $y'_{2n} = x_0$. That means $f'_0 = (x'_0, y'_0)$ is colored 0 and $f_1 = (y'_0 = x_1, y_1)$ is colored 1; $f'_1 = (x'_1, y'_1)$ is colored 1 and $f_2 = (y'_1 = x_2, y_2)$ is colored 2; $f'_2 = (x'_2, y'_2)$ is colored 2 and $f_3 = (y'_2 = x_3, y_3)$ is colored 3; in general $f'_i = (x'_i, y'_i)$ is colored i and $f_{i+1} = (y'_i = x_{i+1}, y_{i+1})$ is colored $i + 1$ and $f'_{2n} = (x'_{2n}, y'_{2n})$ is colored $2n$ and $f_0 = (y'_{2n} = x_0, y_0)$ is colored 0. Since D is a strongly connected digraph; there exists a directed path, namely T_i from y_i to x'_i for each $i \in \{0, 1, \ldots, 2n\}$. Thus we have the directed walks $W_i = (x_i, y_i) \cup T_i \cup (x'_i, y'_i)$ starting in f_i and ending in f'_i; since f_i and f'_i are both colored i we have that W_i has an even number of color changes; for each $i \in \{0, 1, \ldots, 2n\}$. Now consider the closed directed walk $W = \bigcup_{i=0}^{2n} W_i$ clearly the color changes of W are those of each W_i and those that occur in x_i for each $i \in \{0, 1, \ldots, 2n\}$. Hence the number of color changes of W is odd contradicting our assumption. (See Figure 1.) ■

Theorem 3.2 [3]**.** *Let D be a strongly connected digraph; D is a bipartite digraph if and only if each directed cycle of D has an even length.*

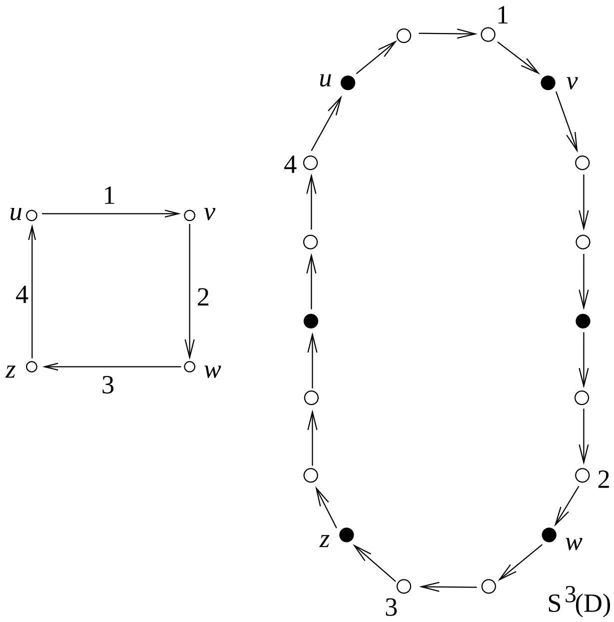

Figure 1.

Theorem 3.3. *Let D be a strongly connected m-colored digraph. If D satisfies the two following conditions:*

(a) *Every closed directed walk in D possesses an even number of color changes,*

(b) *Every directed walk starting and ending in arcs of the same color has an even number of color changes.*

Then D has a kernel by monochromatic paths.

Proof. It follows from Theorem 3.1 that very directed cycle of $\mathcal{C}_C(D)$ has an even length. From Lemma 2.1 $\mathcal{C}_C(D)$ is a strongly connected digraph. Thus from Theorem 3.2 we have that $\mathcal{C}_C(D)$ is a bipartite digraph. Hence we conlcude by Theorem 2.5 that D has a kernel by monochromatic paths. ■

As a direct consequence of Theorem 3.3 we have the following two corollaries.

Corollary 3.4. *Let D be a strongly connected m-colored digraph. If D satisfies the two following conditions:*

(a) *Every closed directed walk is* 1*-colored or* 2*-colored;*

(b) *Every directed walk starting and ending in arcs colored alike is* 1*-colored or* 2*-colored.*

Then D has a kernel by monochromatic paths.

Corollary 3.5. *If D is a strongly connected* 2*-colored digraph, then D has a kernel by monochromatic paths.*

Clearly Theorem 3.3 is a wide generalization of Theorem 2.4 in the case that D is a strongly connected digraph

4. Applications

Let D be an m-colored digraph, and let $C = \{c_1, c_2, \ldots, c_m\}$ the set of colors used to color $A(D)$.

Denote by $\xi(v) = \{c_i \in C|$ there exists an arc colored c_i incident with $v\}$ ($\xi(v)$ are the colors that appear in arcs incident from (or toward) v).

(I) Let D be an m-colored digraph such that:

(i) $|\xi(v)| \leq 2$ for each $v \in V(D)$.

(ii) There exists a fixed color c_i such that $c_i \in \xi(v)$ for each $v \in V(D)$.

Then D has a kernel by monochromatic paths.

Proof. Clearly the $\mathcal{C}_C(D)$ is bipartite. ∎

(II) Let D be an m-colored digraph such that:

(i) $|\xi(v)| \leq 2$ for each $v \in V(D)$.

(ii) There exist two fixed colors c_i, c_j such that $|\{c_i, c_j\} \cap \xi(v)| = 1$ for each $v \in V(D)$.

Then D has a kernel by monochromatic paths.

Proof. $\mathcal{C}_C(D)$ is bipartite. ∎

(III) Let H be a digraph possibly with loops and let D be a digraph whose arcs are colored with the vertices of H. A directed walk (path), W in D is an H-walk (H-path) if the consecutive color encountered on W form a directed walk in H. A set $N \subseteq V(D)$ is an H-kernel if no two vertices of N have an H-path between them and any $u \in V(D) \setminus N$ reaches some $v \in N$ on an H-path. The concept of H-walk was first introduced by Linek and Sands (1996). This concept was studied later by several authors.

Since $V(\mathcal{C}_C(D)) \subseteq V(H)$, the question that we can do us is the the next: What structure or substructures must $\mathcal{C}_C(D)$ have respect to the digraph H in order to ensure the existence of H-kernels in D?

This questions will be studied in a forthcoming paper (H-kernels, Hortensia Galeana-Sánchez and Rocío Sánchez López).

Acknowledgement

The author wish to thank the anonymous referees for many suggestions which improved the rewriting of this paper.

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Received 24 November 2009
Revised 2 December 2010
Accepted 27 January 2011