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ON THE CROSSING NUMBERS OF $G \square C_n$ FOR GRAPHS G ON SIX VERTICES

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Abstract

The crossing numbers of Cartesian products of paths, cycles or stars with all graphs of order at most four are known. The crossing numbers of $G \square C_n$ for some graphs G on five and six vertices and the cycle C_n are also given. In this paper, we extend these results by determining crossing numbers of Cartesian products $G \square C_n$ for some connected graphs G of order six with six and seven edges. In addition, we collect known results concerning crossing numbers of $G \square C_n$ for graphs G on six vertices.

Keywords: graph, cycle, drawing, crossing number, Cartesian product.

2010 Mathematics Subject Classification: 05C10.

1. Introduction

Let G be a simple graph with vertex set V and edge set E. The *crossing number* $cr(G)$ of a graph G is the minimum number of crossings of edges in a drawing of G in the plane such that no three edges cross in a point. It is easy to verify that a drawing with minimum number of crossings (an

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optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let D be a good drawing of the graph G. We denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote by $cr_D(G_i, G_j)$ the number of crossings between edges of G_i and edges of G_j, and by $cr_D(G_i)$ the number of crossings among edges of G_i in D.

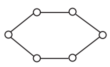
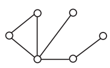
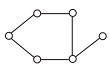
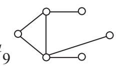
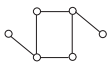
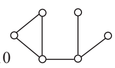
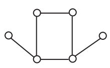
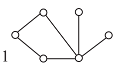
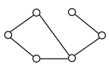
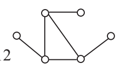
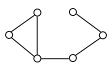
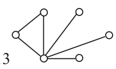
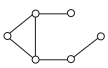
The investigation of crossing numbers of graphs is a classical and very difficult problem. Because of their structure, Cartesian products of special graphs are one of few graph classes for which the exact values of crossing numbers were obtained. (For a definition of Cartesian product, see [2].) Let C_n be the cycle of length n, P_n be the path of length n, and S_n be the star isomorphic to $K_{1,n}$. Harary *et al.* [7] conjectured that the crossing number of $C_m \square C_n$ is $(m-2)n$, for all m, n satisfying $3 \leq m \leq n$. This has been proved only for m, n satisfying $n \geq m$, $m \leq 7$. It was recently proved by Glebsky and Salazar [6] that the crossing number of $C_m \square C_n$ equals its long-conjectured value at least for $n \geq m(m+1)$. Beineke and Ringeisen in [2] and Jendroľ and Ščerbová in [8] determined the crossing numbers of the Cartesian products of all graphs on four vertices with cycles. Klešč in [9], [10, 11], Klešč, Richter and Stobert in [13], and Klešč and Kocúrová in [14] gave the crossing numbers of $G \square C_n$ for several graphs of order five. Except of the graph $K_5 \square C_n$, all known values of crossing numbers for the Cartesian products of cycles and graphs of order five are presented in [12]. It was proved in [18] that $cr(K_5 \square C_n) = 9n$. It seems natural to enquire about the crossing numbers of Cartesian products of cycles with other graphs. Except for the star S_5, the crossing numbers of Cartesian products of all connected graphs on six vertices and five edges with cycles were given in [4]. For the star on six vertices an upper bound is presented. In [5], the crossing number of the Cartesian product $G \square C_n$ for a specific 6-vertex graph containing seven edges is established. We extend these results by giving the crossing numbers of $G \square C_n$ for several graphs G of order six.

2. Graphs on Six Vertices and Six Edges

There are thirteen graphs G_j on six vertices and six edges (see Table 1 in this section). To establish crossing numbers of the graphs $G_j \square C_n$ for $j = 1, 2, \ldots, 10$, we will refer to the previous results. It was proved that $cr(C_3 \square C_n) = n$ for $n \geq 3$ [2], $cr(C_4 \square C_n) = 2n$ for $n \geq 4$ [3, 17], $cr(C_5 \square C_n) =$

$3n$ for $n \geq 5$ [13, 15], and $cr(C_6 \Box C_n) = 4n$ for $n \geq 6$ [1, 16]. Jendroľ and Ščerbová in [8] proved that $cr(S_3 \Box C_3) = 1$, $cr(S_3 \Box C_4) = 2$, $cr(S_3 \Box C_5) = 4$, and that $cr(S_3 \Box C_n) = n$ for $n \geq 6$. So, the crossing number of the graph $G_1 \Box C_n = C_6 \Box C_n$ is known. In this section we establish the crossing number for the Cartesian product $G_{10} \Box C_n$ and then we collect the crossing numbers of the graphs $G_j \Box C_n$ for all $j = 2, 3, \ldots, 9$. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map". We will use the following fact several times.

Table 1. The known values of crossing numbers for the graphs $G_j \Box C_n$.

G_i	$cr\,(G_i \Box C_n)$		G_i	$cr\,(G_i \Box C_n)$	
G_1 	$4n$ $(n > 5)$	6 $(n=3)$ 12 $(n=4)$ 18 $(n=5)$	G_8 	$2n$ $(n > 5)$	4 $(n=3)$ 6 $(n=4)$ 9 $(n=5)$
G_2 	$3n$ $(n > 4)$	5 $(n=3)$ 10 $(n=4)$	G_9 	$2n$ $(n > 5)$	4 $(n=3)$ 6 $(n=4)$ 9 $(n=5)$
G_3 	$2n$ $(n > 3)$	4 $(n=3)$	G_{10} 	$2n$ $(n > 5)$	4 $(n=3)$ 6 $(n=4)$ 9 $(n=5)$
G_4 	$2n$ $(n > 3)$	4 $(n=3)$	G_{11} 		
G_5 	$2n$ $(n > 3)$	4 $(n=3)$	G_{12} 		
G_6 	n		G_{13} 		
G_7 	n				

Lemma 2.1. *For $n \geq 4$, there is no good drawing of the graph $P_1 \Box C_n$ with one crossing.*

Proof. Assume that there is a good drawing of $P_1 \Box C_n$ with exactly one crossing. As no two edges incident with the same vertex cross in a good drawing, for $n \geq 4$ one can easily verify that in any good drawing of $P_1 \Box C_n$

the edges that cross each other must appear in two different edge-disjoint cycles. Two edge-disjoint cycles cannot cross only once. This contradiction completes the proof. ∎

2.1. The graph G_{10}

Assume $n \geq 3$ and consider the graph $G_{10} \square C_n$ in the following way: it has $6n$ vertices and edges that are the edges in the n copies G_{10}^i, $i = 0, 1, \ldots, n-1$, and in the six cycles of length n. For $i = 0, 1, \ldots, n-1$, let a_i and b_i be the vertices of G_{10}^i of degree two, c_i and d_i the vertices of degree three, and e_i and f_i the vertices of degree one (see Figure 1). Thus, for $x \in \{a, b, c, d, e, f\}$, the n-cycle C_n^x is induced by the vertices $x_0, x_1, \ldots, x_{n-1}$. Let T^x, $x = a, b$ ($x = e, f$), be the subgraph of the graph $G_{10} \square C_n$ consisting of the cycle C_n^x together with the vertices of C_n^c (C_n^d) and of the edges joining C_n^x with C_n^c (C_n^d). Let I^{xy} be the subgraph of $G_{10} \square C_n$ containing the vertices of two adjacent cycles C_n^x and C_n^y and the edges $\{x_i, y_i\}$ for all $i = 0, 1, \ldots, n-1$. It is not difficult to see that

$$G_{10} \square C_n = T^a \cup T^b \cup I^{ab} \cup C_n^c \cup I^{cd} \cup C_n^d \cup T^e \cup T^f.$$

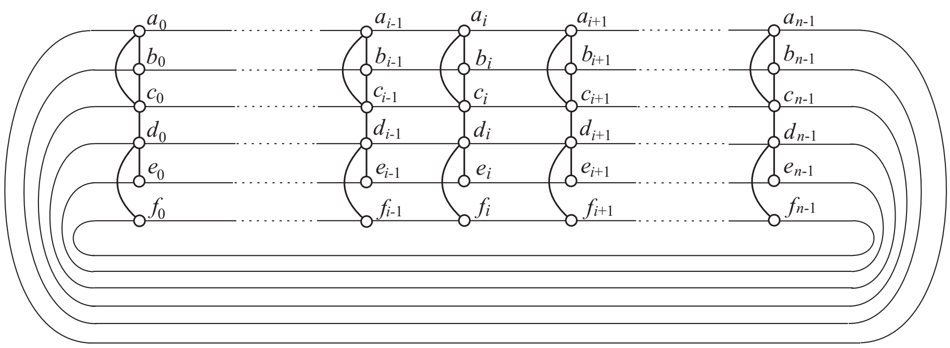


Figure 1. The graph $G_{10} \square C_n$.

Theorem 2.1. *$cr(G_{10} \square C_3) = 4$, $cr(G_{10} \square C_4) = 6$, $cr(G_{10} \square C_5) = 9$, and $cr(G_{10} \square C_n) = 2n$ for $n \geq 6$.*

Proof. It follows from Figure 2 that $cr(G_{10} \square C_5) \leq 9$. In the drawing of the graph $G_{10} \square C_5$ in Figure 2 there is one copy of G_{10} with three crossings on its edges. The removing of all edges of this copy of G_{10} results in the drawing of the graph homeomorphic to $G_{10} \square C_4$ with six crossings. Thus,

$cr(G_{10}\square C_4) \leq 6$. By deleting one copy of G_{10} with three crossings and one copy of G_{10} with two crossings from the drawing in Figure 2, the drawing of the graph homeomorphic to $G_{10}\square C_3$ with four crossings is obtained. Hence, $cr(G_{10}\square C_3) \leq 4$. To prove that $cr(G_{10}\square C_3) = 4$, $cr(G_{10}\square C_4) = 6$, and $cr(G_{10}\square C_5) = 9$, we need to confirm the reverse inequalities. The graph $G_{10}\square C_n$ consists of two subgraphs $C_3\square C_n$ and $S_3\square C_n$, where $C_3\square C_n$ is induced on the vertices a_i, b_i, and c_i and $S_3\square C_n$ is induced on the vertices c_i, d_i, e_i, and f_i for $i = 0, 1, \ldots, n-1$. The only edges of the cycle C_n^c belong to both subgraphs.

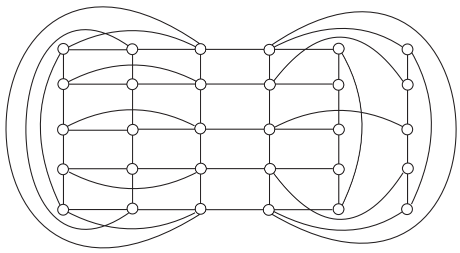


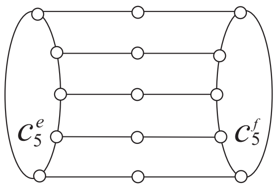
Figure 2. The drawing of $G_{10}\square C_5$ with nine crossings.

Consider a good drawing of the graph $G_{10}\square C_3$. The edges of the common 3-cycle C_3^c do not cross each other. Thus, as $cr(C_3\square C_3) = 3$ and $cr(S_3\square C_3) = 1$, the number of crossings in the drawing is at least $cr(C_3\square C_3)+cr(S_3\square C_3) = 3 + 1 = 4$. This confirms that $cr(G\square C_3) = 4$.

Assume now that there is a good drawing of the graph $G_{10}\square C_4$ with less than six crossings and let D be such a drawing. As $cr(C_3\square C_4) = 4$ and $cr(S_3\square C_4) = 2$, in D there is exactly one internal crossing on the edges of C_4^c. (The edges of C_4^c do not cross more than once in a good drawing.) Lemma 2.1 implies that $cr_D(C_4^c \cup I^{cd} \cup C_4^d) \geq 2$ and therefore, in D there are at least five crossings on the edges of $(C_3\square C_4) \cup I^{cd} \cup C_4^d$. This implies that no edge of the subgraph $T^e \cup T^f$ is crossed in D. In the subdrawing of $T^e \cup T^f$ induced from D there are at most two vertices of C_4^d on the boundary of a region, which enforces an additional crossing in D between the edges of $T^e \cup T^f$ and the edges of $C_4^c \cup I^{cd}$. This contradicts the assumption that D has less than six crossings. Hence, $cr(G_{10}\square C_4) = 6$.

If there is a good drawing D of the graph $G_{10}\square C_5$ with less than nine crossings, the facts $cr(C_3\square C_5) = 5$ and $cr(S_3\square C_5) = 4$ require that the edges of C_5^c cross each other at least once. (The edges of C_5^c cannot cross more than twice in a good drawing.) If $cr_D(C_5^c) = 1$, Lemma 2.1 implies that in the subdrawing of $C_5^c \cup I^{cd} \cup C_5^d$ there is at least one crossing on the edges of $I^{cd} \cup C_5^d$ which does not appear in $C_3 \Box C_5$. Hence, in D there are at most two crossings on the edges of $T^e \cup T^f$.

Assume first that $cr_D(T^e \cup T^f) = 0$. The planar subdrawing of $T^e \cup T^f$ induced from D divides the plane into two pentagonal and five hexagonal regions in such a way that there are at most two of the vertices $d_0, d_1, \dots, d_4$ on the boundary of a region, see Figure 3(a). So, if $cr_D(T^e \cup T^f, C_5^c) \neq 0$, then $cr_D(T^e \cup T^f, C_5^c) = 2$ and C_5^c is placed in D in two neighbouring regions of the subdrawing induced by $T^e \cup T^f$. In this case, as on the boundaries of two neighbouring regions there are at most three vertices of C_5^d, the edges of I^{cd} joining C_5^d with C_5^c cross the edges of $T^e \cup T^f$ and in D there are more than eight crossings, a contradiction. If $cr_D(T^e \cup T^f, C_5^c) = 0$, then C_5^c is placed in D in one region of the subdrawing induced by $T^e \cup T^f$ and the edges joining C_5^c with the vertices of C_5^d cross the edges of $T^e \cup T^f$ more than two times. This contradicts our assumption that the drawing D has less than nine crossings.



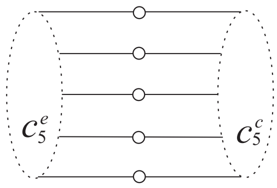


Figure 3. The subdrawings of $T^e \cup T^f$ and $T^e \cup I^{cd} \cup C_5^c$.

So, $cr_D(T^e \cup T^f) \neq 0$. In this case, $cr_D(T^e \cup T^f, C_5^c \cup I^{cd} \cup C_5^d) \leq 1$ and therefore, $cr_D(T^e, I^{cd} \cup C_5^c) = 0$ or $cr_D(T^f, I^{cd} \cup C_5^c) = 0$. Without loss of generality, let $cr_D(T^e, I^{cd} \cup C_5^c) = 0$. Consider now the subdrawing D' of the subgraph $T^e \cup I^{cd} \cup C_5^c$ induced by D. As $cr_D(T^e, I^{cd} \cup C_5^c) = 0$, D' divides the plane in such a way that on the boundary of a region there are at most two vertices of C_5^d and no two regions with a common boundary contain more than three vertices of C_5^d on their boundaries. Figure 3(b) shows the subdrawing D' in which possible crossings among the edges of T^e are inside the left disc bounded by the dotted cycle and possible crossings among the edges of $I^{cd} \cup C_5^c$ are inside the right disc bounded by the dotted cycle. We can suppose that if, in D, an edge of T^f passes through one of these two discs, then it crosses the edges of $T^e \cup I^{cd} \cup C_5^c$ at least twice. Then the same analysis as in the previous paragraph, for the case $cr_D(T^e \cup T^f) = 0$, confirms that $cr_D(T^e \cup I^{cd} \cup C_5^c, T^f) \geq 3$. This contradicts the assumption that D has less than nine crossings again.

The last possibility is that the edges of C_5^c cross each other two times. The subdrawing of such C_5^c is unique with one region containing all five vertices of C_5^c on its boundary. The ordering of the vertices along the boundary of this region is $c_i, c_{i+1}, c_{i+4}, c_{i+2}, c_{i+3}$, where indices are taken modulo 5. If the cycle C_5^d does not have a crossing on its edges in the subdrawing of $C_5^c \cup I^{cd} \cup C_5^d$, then the ordering of its vertices is d_0, d_1, d_2, d_3, d_4 and in the subdrawing of $C_5^c \cup I^{cd} \cup C_5^d$ there is a crossing on the edges of I^{cd}. Thus, in the subdrawing of $C_5^c \cup I^{cd} \cup C_5^d$ induces from D there is at least one crossing on the edges of $I^{cd} \cup C_5^d$. Now, the same analysis as for the case $cr_D(C_5^c) = 1$ gives the contradiction with the assumption that D has less than nine crossings. This confirms that $cr(G_{10}\square C_5) = 9$.

Let H_1 be the graph obtained from the graph G_{10} by deleting the edge $\{a, b\}$. It was proved in [4] that $cr(H_1\square C_n) = 2n$ for all $n \geq 6$. The graph $G_{10}\square C_n$ contains the graph $H_1\square C_n$ as a subgraph. So $cr(G_{10}\square C_n) \geq 2n$. On the hand, the drawing in Figure 1 gives the upper bound $2n$ for the crossing number of the graph $G_{10}\square C_n$. This completes the proof. ∎

2.2. The other graphs G_j

In Figure 4 there are segments of the graphs $G_j\square C_n$ for $j = 2, 3, \ldots, 9$. It is easy to see that $cr(G_2\square C_n) \leq 3n$, $cr(G_3\square C_n) \leq 2n$, $cr(G_4\square C_n) \leq 2n$, $cr(G_5\square C_n) \leq 2n$, $cr(G_6\square C_n) \leq n$, $cr(G_7\square C_n) \leq n$, $cr(G_8\square C_n) \leq 2n$, and $cr(G_9\square C_n) \leq 2n$. To establish the exact values of crossing numbers for all these graphs $G_j\square C_n$, we only need to find lower bounds for their crossing numbers. This we will do by finding the suitable subgraphs with known crossing numbers. For some of these graphs we also use special drawings for small values of n.

In Figure 5(a) there is the drawing of the graph $G_2\square C_4$ with ten crossings. The deleting the edges of one copy of the graph G_2 with five crossings from this drawing results in the drawing of the subdivision of $G_2\square C_3$ with five crossings. Hence, $cr(G_2\square C_3) \leq 5$ and $cr(G_2\square C_4) \leq 10$. On the other hand, $cr(G_2\square C_3) \geq 5$, because the graph $G_2\square C_3$ contains the graph $C_5\square C_3$ as a subgraph. Similarly, $cr(G_2\square C_4) \geq 10$, because the graph $G_2\square C_4$ contains the subgraph $C_5\square C_4$. As the graph $G_2\square C_n$ contains the graph $C_5\square C_n$ as a subgraph and $cr(C_5\square C_n) = 3n$ for all $n \geq 5$, the crossing number of the graph $G_2 \square C_n$ is at least $3n$. This, together with $cr(G_2 \square C_n) \leq 3n$, confirms that $cr(G_2 \square C_n) = 3n$ for all $n \geq 5$.

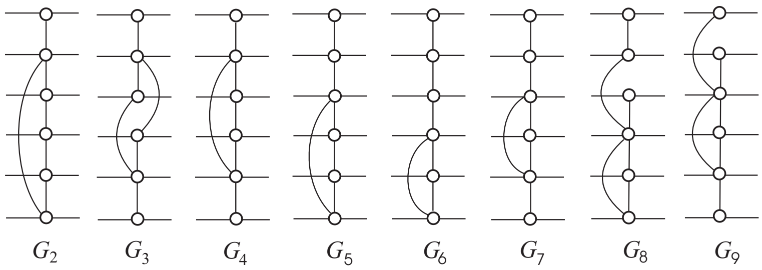


Figure 4. The segments of one copy of G_j for all graphs $G_j \square C_n$, $j = 2, 3, \ldots, 9$.

The drawings of the graphs $G_3 \square C_3$, $G_4 \square C_3$, and $G_5 \square C_3$ in Figure 5(b), 5(c), and 5(d) show that $cr(G_3 \square C_3) \leq 4$, $cr(G_4 \square C_3) \leq 4$, and $cr(G_5 \square C_3) \leq 4$. Every of the graphs $G_j \square C_n$, $j = 3, 4, 5$, contains the graph $C_4 \square C_n$ as a subgraph. As $cr(C_4 \square C_3) = 4$, $cr(G_j \square C_3) \geq 4$ for all $j = 3, 4, 5$. Thus, $cr(G_3 \square C_3) = cr(G_4 \square C_3) = cr(G_5 \square C_3) = 4$. We can generalize this idea and to state that $cr(G_3 \square C_n) = cr(G_4 \square C_n) = cr(G_5 \square C_n) = 2n$ for $n \geq 4$.

Both graphs $G_6 \square C_n$ and $G_7 \square C_n$ contain the graph $C_3 \square C_n$ as a subgraph. The fact $cr(C_3 \square C_n) = n$ and the drawings in Figure 4 for the graphs G_6 and G_7 confirm that $cr(G_6 \square C_n) = cr(G_7 \square C_n) = n$ for $n \geq 3$.

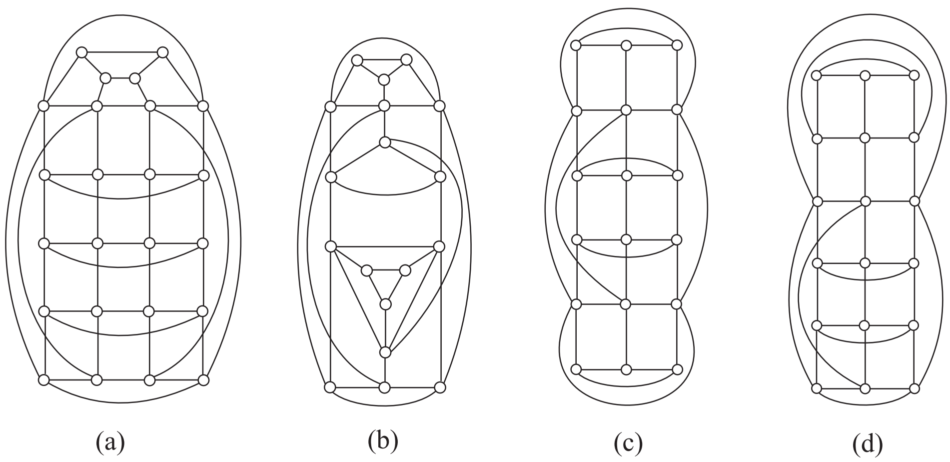
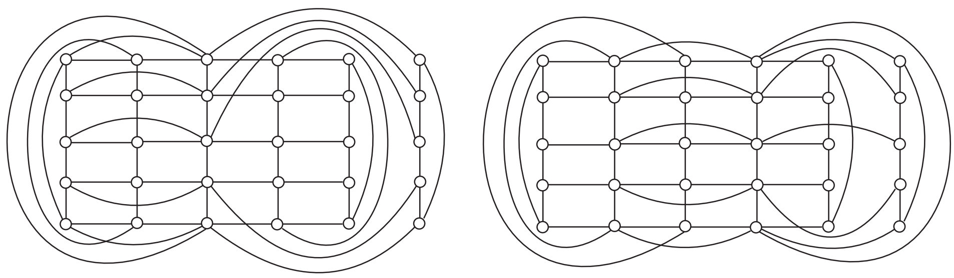


Figure 5. The graphs $G_2 \square C_4$, $G_3 \square C_3$, $G_4 \square C_3$, and $G_5 \square C_3$.

Let H_2 be the graph obtained from the complete bipartite graph $K_{1,4}$ by adding one new edge. It was shown in [12] that $cr(H_2\Box C_n) = 2n$ for $n \geq 6$, and that $cr(H_2\Box C_3) = 4$, $cr(H_2\Box C_4) = 6$, and $cr(H_2\Box C_5) = 9$. As both graphs $G_8\Box C_n$ and $G_9\Box C_n$ contain the graph $H_2\Box C_n$ as a subgraph, we have the lower bounds for crossing numbers of the graphs $G_j\Box C_n$, $j = 8, 9$. In Figure 6(a) there is the drawing of the graph $G_8\Box C_5$ with nine crossings. Hence, $cr(G_8\Box C_5) \leq 9$. Deleting the edges of one copy of the graph G_8 with three crossings from this drawing results in the subdivision of the graph $G_8\Box C_4$ with six crossings. By deleting the edges of one copy of G_8 with three crossings and of one copy of G_8 with two crossings, the subdivision of the graph $G_8\Box C_3$ with four crossings is obtained. So, $cr(G_8\Box C_4) \leq 6$ and $cr(G_8\Box C_3) \leq 4$. The same we can do in the drawing of the graph $G_9\Box C_5$ with nine crossings in Figure 6(b). Hence, $cr(G_9\Box C_5) \leq 9$, $cr(G_9\Box C_4) \leq 6$, and $cr(G_9\Box C_3) \leq 4$. Figure 4 shows that the crossing number of both graphs $G_8\Box C_n$ and $G_9\Box C_n$ is at most $2n$ for $n \geq 6$. These lower and upper bounds confirm that $cr(G_8\Box C_3) = cr(G_9\Box C_3) = 4$, $cr(G_8\Box C_4) = cr(G_9\Box C_4) = 6$, $cr(G_8\Box C_5) = cr(G_9\Box C_5) = 9$, and that $cr(G_8\Box C_n) = cr(G_9\Box C_n) = 2n$ for $n \geq 6$.



(a) (b)

Figure 6. The graphs $G_8\Box C_5$ and $G_9\Box C_5$.

3. Graphs on Six Vertices and Seven Edges

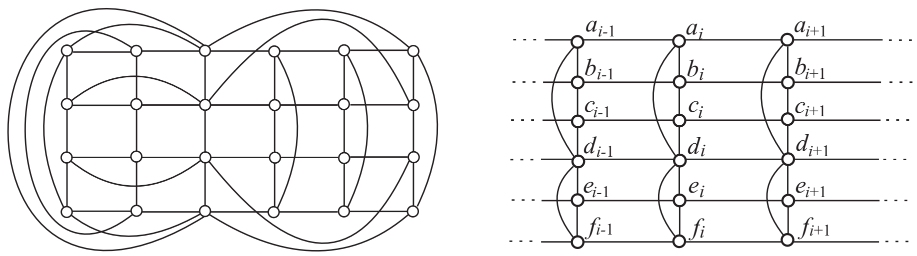
For one specific graph G of order six with seven edges, the crossing number of the Cartesian product $G\Box C_n$ is given in [5]. In this section, we find the crossing number of the Cartesian product of one other graph of the same size with the cycle C_n. Let F be the graph on six vertices consisting of edge disjoint cycles C_4 and C_3 with one common vertex. Denote the common vertex of both cycles by d. Let a and c be the vertices of C_4 adjacent with the vertex d, and let b be the vertex of C_4 adjacent to a and c. Let us denote by e and f the vertices of degree two in the cycle C_3. Assume $n \geq 3$ and consider the graph $F \square C_n$ in the following way: it has $6n$ vertices and edges that are the edges in the n copies F^i, $i = 0, 1, \dots, n-1$, and in the six cycles of length n (see segment in Figure 7(b)). Thus, for $x \in \{a, b, c, d, e, f\}$, the n-cycle C_n^x is induced by the vertices $x_0, x_1, \dots, x_{n-1}$. For $i = 0, 1, \dots, n-1$, let P^i denote the subgraph of $F \square C_n$ containing the vertices of F^i and F^{i+1} and six edges joining F^i to F^{i+1}, i taken modulo n. Let T^x, $x = a, c, e, f$, be the subgraph of the graph $F \square C_n$ consisting of the cycle C_n^x together with the vertices of C_n^d and of the edges joining C_n^x with C_n^d. For $x, y \in \{a, b, c, d, e, f\}, x \neq y$, let I^{xy} be the subgraph of $F \square C_n$ consisting of the vertices in the adjacent cycles C_n^x and C_n^y and of the edges $\{x_i, y_i\}$ for all $i = 0, 1, \dots, n-1$.

It is easy to see that

$$F \square C_n = T^a \cup T^c \cup I^{ab} \cup C_n^b \cup I^{bc} \cup C_n^d \cup T^e \cup T^f \cup I^{ef},$$

and also

$$F \square C_n = (C_4 \square C_n) \cup (C_3 \square C_n), \quad \text{where} \quad (C_4 \square C_n) \cap (C_3 \square C_n) = C_n^d.$$



(a) (b)

Figure 7. The graph $F \square C_4$ and the segment Q^i of the graph $F \square C_n$.

We say that a good drawing of the graph $F \square C_n$ is *coherent* if for each F^i holds that all vertices of the subgraph $(F \square C_n) \setminus V(F^i)$ lie in the same region in the view of the subdrawing of F^i.

Lemma 3.1. *$cr(F \square C_3) = 7$ and $cr(F \square C_4) = 12$.*

Proof. The drawing in Figure 7(a) shows that $cr(F \square C_4) \leq 12$. The deleting all edges of one copy of the subgraph F with five crossings results in the subdrawing of the graph homeomorphic to $F \square C_3$ with seven crossings. So, $cr(F \square C_3) \leq 7$ and $cr(F \square C_4) \leq 12$. As $F \square C_3 = (C_4 \square C_3) \cup (C_3 \square C_3)$ and in a good drawing the 3-cycle C_3^d does not have an internal crossing, $cr(F \square C_3) \geq cr(C_4 \square C_3) + cr(C_3 \square C_3) = 4 + 3 = 7$, and the proof is done for $n = 3$. It remains to prove the reverse inequality for the case $n = 4$.

Assume that there is a good drawing of the graph $F \square C_4$ with less than 12 crossings and let D be such a drawing. As $cr(C_4 \square C_4) = 8$ and $cr(C_3 \square C_4) = 4$, in D there is at least one crossing among the edges of the cycle C_4^d. The edges of a 4-cycle can not cross each other more than once in a good drawing, and therefore $cr_D(C_4^d) = 1$. As in D there are at most eleven crossings, the edges of $C_4 \square C_4 = T^c \cup I^{cb} \cup C_4^b \cup I^{ba} \cup T^a$ do not cross the edges of $C_3 \square C_4 = T^e \cup I^{ef} \cup T^f$. This implies that $cr_D(T^a, T^e) = cr_D(T^a, T^f) = 0$. As the edges of the cycle C_4^d cross once, using Lemma 2.1, $cr_D(C_4^d \cup T^e) \geq 2$ and $cr_D(C_4^d \cup T^f) \geq 2$. Hence, in the subdrawing of $C_4^d \cup T^e$ there is a crossing on the edges of T^e and also in the subdrawing of $C_4^d \cup T^f$ there is a crossing on the edges of T^f. This implies that in D there is at most one crossing between the edges of T^e and T^f. Consider now the subdrawing D' of the subgraph $T^a \cup T^e$ induced by D. As $cr_D(T^a, T^e) = 0$, D' divides the plane in such a way that on the boundary of a region there are at most two vertices of C_4^d. As $cr_D(T^a, T^f) = 0$ and $cr_D(T^e, T^f) \leq 1$, the cycle C_4^f is placed in D in one region of D' and the edges of I^{df} cross in D the edges of $T^a \cup T^e$ at least two times. This enforces at least twelve crossings in D, and therefore $cr(F \square C_4) = 12$. ∎

Lemma 3.2. *If D is a good drawing of $F \square C_n$, $n \geq 4$, in which every F^i has at most two crossings on its edges, then D has at least $3n$ crossings.*

Proof. First we show that the drawing D is coherent. The indices are considered modulo n in the proof. If some F^i, $i \in \{0, 1, \ldots, n-1\}$, separates vertices of the 3-connected subgraph induced by the vertices $V(F^{i+1}) \cup \cdots \cup V(F^{i-1})$, then its edges are crossed at least three times. So, all subgraphs F^j, $j \neq i$, lie in D in the same region in the view of the subdrawing of F^i. Moreover, two different F^i and F^j do not cross each other, otherwise one of them separates the vertices of the other.

For $i = 0, 1, \ldots, n-1$, let Q^i denote the subgraph of $F \square C_n$ induced by $V(F^{i-1}) \cup V(F^i) \cup V(F^{i+1})$ (see Figure 7(b)), where i is taken modulo n. Thus, $Q^i = F^{i-1} \cup P^{i-1} \cup F^i \cup P^i \cup F^{i+1}$. Let us denote by Q_4^i the subgraph of Q^i obtained from Q^i by removing six vertices e_j and f_j for $j = i-1, i, i+1$ and let Q_3^i be the subgraph of Q^i obtained by removing nine vertices a_j, b_j, c_j for $j = i-1, i, i+1$.

Let us consider the following types of crossings on the edges of Q^i in a drawing of the graph $F \Box C_n$:

(1) a crossing of an edge in $P^{i-1} \cup P^i$ with an edge in F^i,

(2) a self-intersection in F^i,

(3) a crossing of an edge in $F^{i-1} \cup P^{i-1}$ with an edge in $F^{i+1} \cup P^i$.

It is readily seen that every crossing of types (1), (2), and (3) appears in a drawing of the graph $F \Box C_n$ only on the edges of the subgraph Q^i. In a good drawing of $F \Box C_n$, we define the *force* $f(Q^i)$ of Q^i in the following way: every crossing of type (1), (2) or (3) contributes the value 1 to $f(Q^i)$. The *total force* of the drawing is the sum of $f(Q^i)$. It is easy to see that the number of crossings in the drawing is not less than the total force of the drawing. The aim of our proof is to show that if every F^i has at most two crossings on its edges, then $f(Q^i) \geq 3$ for all $i = 0, 1, \dots, n-1$.

Consider the subdrawing D_3^i of Q_3^i induced from D. As the drawing D is coherent, the cycles C_3^{i-1} and C_3^{i+1} lie in D_3^i in the same region in the view of the subdrawing induced by C_3^i. If $cr_{D_3^i}(P^{i-1}, C_3^i) \neq 0$, then $f(Q_3^i) \geq 1$. Otherwise the subdrawing of $C_3^{i-1} \cup P^{i-1} \cup C_3^i$ induced from D_3^i divides the plane in such a way that there are at most two vertices of C_3^i on the boundary of a region and $cr_{D_3^i}(C_3^{i-1} \cup P^{i-1} \cup C_3^i, P^i \cup C_3^{i+1}) \geq 1$. Hence, $f(Q_3^i) \geq 1$ again.

Consider now the subdrawing D_4^i of Q_4^i induced by D. If, in D_4^i, both P^{i-1} and P^i cross the edges of C_4^i, then $f(Q_4^i) \geq 2$. Assume, that $cr_{D_4^i}(P^{i-1}, C_4^i) = 0$. Regardless of whether or not the edges of the cycle C_4^i cross each other, the subdrawing of $C_4^{i-1} \cup P^{i-1} \cup C_4^i$ induced from D_4^i divides the plane in such a way that on the boundary of a region there are at most two vertices of C_4^i. This requires that, in D_4^i, the edges of $C_4^{i+1} \cup P^i$ cross the edges of $C_4^{i-1} \cup P^{i-1} \cup P^i$ at least twice. Hence $f(Q_4^i) \geq 2$.

As the only edges which belong to both subgraphs Q_3^i and Q_4^i are two edges $\{d_{i-1}, d_i\}$ and $\{d_i, d_{i+1}\}$, the only crossing which contributes to both $f(Q_3^i)$ and $f(Q_4^i)$ is the crossing between these two edges. But the edges incident with the vertex d_i do not cross in the good drawing D. This implies that $f(Q^i) \geq f(Q_3^i) + f(Q_4^i) \geq 3$ for every i. Since i runs through $0, 1, \dots,$ $n-1$, the drawing D has at least $3n$ crossings. ∎

Theorem 3.1. *$cr(F\Box C_n) = 3n$ for $n \geq 4$.*

Proof. The drawing in Figure 7(b) shows that $cr(F\Box C_n) \leq 3n$ for $n \geq 4$. We prove the reverse inequality by the induction on n. By Lemma 3.1, $cr(F\Box C_4) = 12$, so the result is true for $n = 4$. Assume it is true for $n = k$, $k \geq 4$, and suppose that there is a good drawing of $F\Box C_{k+1}$ with fewer than $3(k+1)$ crossings. By Lemma 3.2, some F^i must be crossed at least three times. By the removal of all edges of this F^i, we obtain a subdivision of $F\Box C_k$ with fewer than $3k$ crossings. This contradiction completes the proof. ■

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