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GENERALIZED TOTAL COLORINGS OF GRAPHS

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Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphism. Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary properties of graphs. A $(\mathcal{P}, \mathcal{Q})$-*total coloring*

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of a simple graph G is a coloring of the vertices $V(G)$ and edges $E(G)$ of G such that for each color i the vertices colored by i induce a subgraph of property $\mathcal{P}$, the edges colored by i induce a subgraph of property $\mathcal{Q}$ and incident vertices and edges obtain different colors. In this paper we present some general basic results on $(\mathcal{P}, \mathcal{Q})$-total colorings. We determine the $(\mathcal{P}, \mathcal{Q})$-total chromatic number of paths and cycles and, for specific properties, of complete graphs. Moreover, we prove a compactness theorem for $(\mathcal{P}, \mathcal{Q})$-total colorings.

Keywords: hereditary properties, generalized total colorings, paths, cycles, complete graphs.

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1. Introduction

We denote the class of all finite simple graphs by $\mathcal{I}$ (see [1]). A *graph property* $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$.

We use the following standard notations for specific hereditary properties:

$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$,
$\mathcal{O}^k = \{G \in \mathcal{I} : \chi(G) \leq k\}$,
$\mathcal{D}_k = \{G \in \mathcal{I} : \text{each subgraph of } G \text{ contains a vertex of degree at most } k\}$,
$\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}$,
$\mathcal{J}_k = \{G \in \mathcal{I} : \chi'(G) \leq k\}$,
$\mathcal{O}_k = \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\}$,
$\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\}$,

where $\chi(G)$ is the *chromatic number*, $\chi'(G)$ the *chromatic index* and $\Delta(G)$ the *maximum degree* of the graph $G = (V, E)$.

A *total coloring* of a graph G is a coloring of the vertices and edges (together called the *elements* of G) such that all pairs of adjacent or incident elements obtain distinct colors. The minimum number of colors of a total coloring of G is called the *total chromatic number* $\chi''(G)$ of G.

Let $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ be two additive and hereditary graph properties. Then a $(\mathcal{P}, \mathcal{Q})$*-total coloring* of a graph G is a coloring of the vertices and edges of G such that for any color i all vertices of color i induce a subgraph of property $\mathcal{P}$, all edges of color i induce a subgraph of property $\mathcal{Q}$ and vertices and incident edges are colored differently. The minimum number of colors of a $(\mathcal{P}, \mathcal{Q})$-total coloring of G is called the $(\mathcal{P}, \mathcal{Q})$-*total chromatic number* $\chi''_{\mathcal{P},\mathcal{Q}}(G)$ of G.

If G contains edges then $\chi''_{\mathcal{P},\mathcal{Q}}(G)$ is only defined if $K_2 \in \mathcal{Q}$ and therefore $\mathcal{O}_1 \subseteq \mathcal{Q}$. Since $\mathcal{O} \subseteq \mathcal{P}$ for all additive hereditary properties we obtain $\chi''_{\mathcal{P},\mathcal{Q}}(G) \leq |V|+|E|$ which guarantees the existence of $(\mathcal{P}, \mathcal{Q})$-total chromatic numbers.

$(\mathcal{P}, \mathcal{Q})$-total colorings are *generalized total colorings* since $\chi''_{\mathcal{O},\mathcal{O}_1}(G) = \chi''(G)$ for all graphs G.

Generalized $\mathcal{P}$-vertex colorings and *$\mathcal{P}$-chromatic numbers* $\chi_{\mathcal{P}}(G)$ as well as *generalized $\mathcal{Q}$-edge colorings* and *$\mathcal{Q}$-chromatic indices* $\chi'_{\mathcal{Q}}(G)$ are analogously defined (see [3, 9] for some results). Evidently, these are generalizations of proper vertex colorings and proper edge colorings since $\chi_{\mathcal{O}}(G) = \chi(G)$ and $\chi'_{\mathcal{O}_1}(G) = \chi'(G)$.

The $\mathcal{P}$-chromatic number and the $\mathcal{Q}$-chromatic index of G provide general lower and upper bounds for $\chi''_{\mathcal{P},\mathcal{Q}}(G)$.

Theorem 1.

(a) $\max\{\chi_{\mathcal{P}}(G), \chi'_{\mathcal{Q}}(G)\} \leq \chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi_{\mathcal{P}}(G) + \chi'_{\mathcal{Q}}(G)$,

(b) $\chi_{\mathcal{P}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi_{\mathcal{P}}(G) + 1$ *if* $G \in \mathcal{Q}$,

(c) $\chi'_{\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi'_{\mathcal{Q}}(G) + 1$ *if* $G \in \mathcal{P}$,

(d) $\chi''_{\mathcal{P},\mathcal{Q}}(G) = 1$ *iff* $G \in \mathcal{O}$,

(e) $\chi''_{\mathcal{P},\mathcal{Q}}(G) = 2$ *iff* $G \in (\mathcal{P} \cap \mathcal{Q}) \setminus \mathcal{O}$,

(f) $\chi''_{\mathcal{P},\mathcal{Q}}(G) \geq 3$ *iff* $G \in \mathcal{I} \setminus (\mathcal{P} \cap \mathcal{Q})$.

Proof. Since a $(\mathcal{P}, \mathcal{Q})$-total coloring induces a $\mathcal{P}$-vertex coloring and a $\mathcal{Q}$-edge coloring it follows that $\chi_{\mathcal{P}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G)$ and $\chi'_{\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G)$. A $\mathcal{P}$-vertex coloring of G with $\chi_{\mathcal{P}}(G)$ colors and a $\mathcal{Q}$-edge coloring with $\chi'_{\mathcal{Q}}(G)$ additional colors induce a $(\mathcal{P}, \mathcal{Q})$-total coloring of G with $\chi_{\mathcal{P}}(G) + \chi'_{\mathcal{Q}}(G)$ colors.

If $G \in \mathcal{Q}$ or $G \in \mathcal{P}$, respectively, then all edges or all vertices can obtain the same additional color which implies $\chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi_{\mathcal{P}}(G) + 1$ or $\chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi'_{\mathcal{Q}}(G) + 1$, respectively.

If G has no edges then $G \in \mathcal{O} \subseteq \mathcal{P}$ and therefore all vertices can obtain the same color which implies $\chi''_{\mathcal{P},\mathcal{Q}}(G) = 1$. If G has edges then $G \notin \mathcal{O}$ and therefore at least two colors are needed to color a vertex and an incident edge which implies $\chi''_{\mathcal{P},\mathcal{Q}}(G) \geq 2$.

It holds $\chi''_{\mathcal{P},\mathcal{Q}}(G) = 2$ if and only if G contains edges and for each non-trivial component of G all vertices as well as all edges can be colored with one color each, that is, if and only if $G \in (\mathcal{P} \cap \mathcal{Q}) \setminus \mathcal{O}$.

Obviously, if $G \notin \mathcal{P} \cap \mathcal{Q}$ then $\chi''_{\mathcal{P},\mathcal{Q}}(G) \geq 3$. ■

The following monotonicity and additivity results are obvious.

Lemma 1. *If $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$, then $\chi''_{\mathcal{P}_2,\mathcal{Q}_2}(G) \leq \chi''_{\mathcal{P}_1,\mathcal{Q}_1}(G)$.*

Proof. If $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ then each $(\mathcal{P}_1, \mathcal{Q}_1)$-total coloring is a $(\mathcal{P}_2, \mathcal{Q}_2)$-total coloring. ■

It follows $\chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{O},\mathcal{O}_1}(G) = \chi''(G)$ since $\mathcal{O} \subseteq \mathcal{P}$ and $\mathcal{O}_1 \subseteq \mathcal{Q}$, that is, the total chromatic number is an upper bound for the $(\mathcal{P}, \mathcal{Q})$-total chromatic number of a graph G.

Lemma 2. *If $H \subseteq G$, then $\chi''_{\mathcal{P},\mathcal{Q}}(H) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G)$.*

Proof. The restriction of a $(\mathcal{P}, \mathcal{Q})$-total coloring of G to the elements of H is a $(\mathcal{P}, \mathcal{Q})$-total coloring of H. ■

The following lemma implies that one can restrict oneself to connected graphs, in general.

Lemma 3. *If G and H are disjoint, then $\chi''_{\mathcal{P},\mathcal{Q}}(G \cup H) = \max\{\chi''_{\mathcal{P},\mathcal{Q}}(G), \chi''_{\mathcal{P},\mathcal{Q}}(H)\}$.*

Proof. $(\mathcal{P}, \mathcal{Q})$-total colorings of G and of H provide a $(\mathcal{P}, \mathcal{Q})$-total coloring of $G \cup H$ since G and H are disjoint which implies $\chi''_{\mathcal{P},\mathcal{Q}}(G \cup H) \leq \max\{\chi''_{\mathcal{P},\mathcal{Q}}(G), \chi''_{\mathcal{P},\mathcal{Q}}(H)\}$. Lemma 2 implies equality. ■

If one of the properties is the class $\mathcal{I}$ of all finite simple graphs then the $(\mathcal{P}, \mathcal{Q})$-total chromatic number of G attains one of two possible values by Theorem 1:

$$(1) \quad \chi_{\mathcal{P}}(G) \leq \chi''_{\mathcal{P},\mathcal{I}}(G) \leq \chi_{\mathcal{P}}(G) + 1, \quad \chi'_{\mathcal{Q}}(G) \leq \chi''_{\mathcal{I},\mathcal{Q}}(G) \leq \chi'_{\mathcal{Q}}(G) + 1.$$

If $\mathcal{P} = \mathcal{Q} = \mathcal{I}$ then $\chi''_{\mathcal{I},\mathcal{I}}(G) = 1$ if $G \in \mathcal{O}$ and $\chi''_{\mathcal{I},\mathcal{I}}(G) = 2$ otherwise by Theorem 1.

If $G \in \mathcal{Q}$ then $\chi''_{\mathcal{P},\mathcal{Q}}(G)$ and therefore $\chi''_{\mathcal{P},\mathcal{I}}(G)$ for all graphs G can be determined as follows.

Theorem 2. *If $G \in \mathcal{Q}$, then*

$$\chi''_{\mathcal{P},\mathcal{Q}}(G) = \begin{cases} \chi_{\mathcal{P}}(G) & \text{if } G \in \mathcal{O} \text{ or } \chi_{\mathcal{P}}(G) \geq 3, \\ \chi_{\mathcal{P}}(G) + 1 & \text{if } G \in \mathcal{P} \setminus \mathcal{O} \text{ or } \chi_{\mathcal{P}}(G) = 2. \end{cases}$$

Proof. By Theorem 1, $\chi_{\mathcal{P}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi_{\mathcal{P}}(G) + 1$.

If $\chi_{\mathcal{P}}(G) = 1$ then $G \in \mathcal{P}$ which implies $\chi''_{\mathcal{P},\mathcal{Q}}(G) = 1$ for $G \in \mathcal{O}$ and $\chi''_{\mathcal{P},\mathcal{Q}}(G) = 2$ for $G \in \mathcal{P} \setminus \mathcal{O}$ by Theorem 1.

If $\chi_{\mathcal{P}}(G) = 2$ then $G \notin \mathcal{P}$ and therefore $\chi''_{\mathcal{P},\mathcal{Q}}(G) \geq 3$ by Theorem 1. On the other hand, $\chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi_{\mathcal{P}}(G) + 1 = 3$.

If $\chi_{\mathcal{P}}(G) \geq 3$ then $\chi''_{\mathcal{P},\mathcal{Q}}(G) \geq \chi_{\mathcal{P}}(G)$. Consider a $\mathcal{P}$-vertex coloring of G with $\chi_{\mathcal{P}}(G)$ colors. Each edge can be colored with a color different to the colors of its end-vertices. This is a $(\mathcal{P},\mathcal{Q})$-total coloring of G with $\chi_{\mathcal{P}}(G)$ colors since $H \in \mathcal{Q}$ for all $H \subseteq G$. ■

2. $\mathcal{P} = \mathcal{O}$ or $\mathcal{Q} = \mathcal{O}_1$

Since $\mathcal{O} \subseteq \mathcal{P} \subseteq \mathcal{I}$ and $\mathcal{O}_1 \subseteq \mathcal{Q} \subseteq \mathcal{I}$, Lemma 1 provides the following bounds:

$$\chi''_{\mathcal{I},\mathcal{I}}(G) \leq \chi''_{\mathcal{P},\mathcal{I}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{O}_1}(G) \leq \chi''_{\mathcal{O},\mathcal{O}_1}(G) = \chi''(G), \tag{2}$$

$$\chi''_{\mathcal{I},\mathcal{I}}(G) \leq \chi''_{\mathcal{I},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{O},\mathcal{Q}}(G) \leq \chi''_{\mathcal{O},\mathcal{O}_1}(G) = \chi''(G), \tag{3}$$

$$\chi''_{\mathcal{P},\mathcal{I}}(G) \leq \chi''_{\mathcal{O},\mathcal{I}}(G) \leq \chi''_{\mathcal{O},\mathcal{Q}}(G), \tag{4}$$

$$\chi''_{\mathcal{I},\mathcal{Q}}(G) \leq \chi''_{\mathcal{I},\mathcal{O}_1}(G) \leq \chi''_{\mathcal{P},\mathcal{O}_1}(G). \tag{5}$$

$(\mathcal{O},\mathcal{I})$- and $(\mathcal{I},\mathcal{O}_1)$-total coloring are certain $[r,s,t]$-colorings which also are generalizations of ordinary colorings.

Given non-negative integers r, s, and t with $\max\{r,s,t\} \geq 1$, an $[r,s,t]$-*coloring* of a finite and simple graph G with vertex set $V(G)$ and edge set $E(G)$ is a mapping c from $V(G) \cup E(G)$ to the color set $\{0,1,\ldots,k-1\}$, $k \in \mathbb{N}$, such that $|c(v_i) - c(v_j)| \geq r$ for every two adjacent vertices v_i, v_j, $|c(e_i) - c(e_j)| \geq s$ for every two adjacent edges e_i, e_j, and $|c(v_i) - c(e_j)| \geq t$ for all pairs of incident vertices and edges, respectively. The $[r,s,t]$-*chromatic number* $\chi_{r,s,t}(G)$ of G is defined to be the minimum k such that G admits an $[r,s,t]$-coloring (see [10, 11]).

By this definition we obtain $\chi''_{\mathcal{I},\mathcal{I}}(G) = \chi_{0,0,1}(G)$, $\chi''_{\mathcal{O},\mathcal{I}}(G) = \chi_{1,0,1}(G)$, $\chi''_{\mathcal{I},\mathcal{O}_1}(G) = \chi_{0,1,1}(G)$ and $\chi''_{\mathcal{O},\mathcal{O}_1}(G) = \chi_{1,1,1}(G)$. The first three of these $[r,s,t]$-chromatic numbers were determined in [10].

Theorem 3.

(a) $\chi''_{\mathcal{O},\mathcal{I}}(G) = \chi_{1,0,1}(G) = \begin{cases} \chi(G) & \textit{if } \chi(G) \neq 2, \\ 3 = \chi(G)+1 & \textit{if } \chi(G) = 2, \end{cases}$

(b) $\chi''_{\mathcal{I},\mathcal{O}_1}(G) = \chi_{0,1,1}(G) = \Delta(G)+1.$

Proof. (a) By Theorem 2 we obtain for $\mathcal{P} = \mathcal{O}$ that $\chi''_{\mathcal{O},\mathcal{I}}(G) = \chi_{\mathcal{O}}(G) = \chi(G)$ if $G \in \mathcal{O}$ or $\chi(G) \geq 3$ and $\chi''_{\mathcal{O},\mathcal{I}}(G) = \chi(G)+1$ if $\chi(G) = 2$.

(b) If $\chi'(G) = \Delta(G)$ then $\chi''_{\mathcal{I},\mathcal{O}_1}(G) \geq \Delta(G)+1$ since an additional color is necessary to color a vertex of maximum degree. If $\chi'(G) = \Delta(G)+1$ then $\chi''_{\mathcal{I},\mathcal{O}_1}(G) \geq \chi'(G) = \Delta(G)+1$ by Theorem 1.

On the other hand, we have $\chi''_{\mathcal{I},\mathcal{O}_1}(G) \leq \Delta(G)+1$ since the edges can be colored with at most $\Delta(G)+1$ colors by Vizing's Theorem and at each vertex there is a missing edge color which can be used to color this vertex. ■

To illustrate the results we consider as examples paths P_n, cycles C_n and complete graphs K_n.

Examples 1.

1. Theorem 3 implies $\chi''_{\mathcal{O},\mathcal{I}}(P_1) = \chi''_{\mathcal{I},\mathcal{O}_1}(P_1) = 1$, $\chi''_{\mathcal{O},\mathcal{I}}(P_2) = 3$, $\chi''_{\mathcal{I},\mathcal{O}_1}(P_2) = 2$ and $\chi''_{\mathcal{O},\mathcal{I}}(P_n) = \chi''_{\mathcal{I},\mathcal{O}_1}(P_n) = 3$ for $n \geq 3$.
2. We have $\chi_{\mathcal{O}}(C_n) = \chi(C_n) = \chi'_{\mathcal{O}_1}(C_n) = \chi'(C_n)$ and $\chi(C_n) = 2$ if n is even and $\chi(C_n) = 3$ if n is odd. Moreover, we have $\chi''_{\mathcal{O},\mathcal{I}}(C_n) = \chi''_{\mathcal{I},\mathcal{O}_1}(C_n) = 3$ by Theorem 3. Therefore, the lower and upper bounds of (1) are attained for cycles C_n.
3. Theorem 3 implies $\chi''_{\mathcal{I},\mathcal{O}_1}(K_n) = n$ and $\chi''_{\mathcal{O},\mathcal{I}}(K_n) = \begin{cases} n & \text{if } n \neq 2, \\ n+1 & \text{if } n = 2. \end{cases}$

   If n is odd then $n = \chi''_{\mathcal{I},\mathcal{O}_1}(K_n) \leq \chi''_{\mathcal{P},\mathcal{O}_1}(K_n) \leq \chi''_{\mathcal{O},\mathcal{O}_1}(K_n) = \chi''(K_n) = n$ and $n = \chi''_{\mathcal{O},\mathcal{I}}(K_n) \leq \chi''_{\mathcal{O},\mathcal{Q}}(K_n) \leq \chi''_{\mathcal{O},\mathcal{O}_1}(K_n) = \chi''(K_n) = n$ by Lemma 1. Therefore, if n is odd then $\chi''_{\mathcal{P},\mathcal{O}_1}(K_n) = \chi''_{\mathcal{O},\mathcal{Q}}(K_n) = n$ for all additive and hereditary properties $\mathcal{P}$ and $\mathcal{Q}$.

In Theorems 4 and 5 we also consider complete graphs of even order.

Theorem 4. $\chi''_{\mathcal{O},\mathcal{Q}}(K_n) = \begin{cases} n & \textit{if } n \textit{ odd or } (n \geq 4 \textit{ even and } \mathcal{O}_1 \subset \mathcal{Q}), \\ n+1 & \textit{if } n = 2 \textit{ or } (n \textit{ even and } \mathcal{Q} = \mathcal{O}_1). \end{cases}$

Proof. The case that n is odd is considered in the above example and the case $n = 2$ is obvious.

If n is even and $\mathcal{Q} = \mathcal{O}_1$ then $\chi''_{\mathcal{O},\mathcal{Q}}(K_n) = \chi''(K_n) = n + 1$.

If $n \geq 4$ is even and $\mathcal{O}_1 \neq \mathcal{Q}$ then $P_3 \in \mathcal{Q}$. We partition the elements of K_n with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ in n color classes as follows:

Class F_i, $0 \leq i \leq n-1$, contains the vertex v_i, the edges $v_{i-1}v_{i+1}, v_{i-2}v_{i+2}, \ldots, v_{i-y+1}v_{i+y-1}$ as well as the edges $v_{i+n/2}v_{i+n/2+1}, v_{i+n/2-1}v_{i+n/2+2}, \ldots, v_{i+y+1}v_{i-y}$ where $y = \lceil n/4 \rceil$ and the indices are reduced modulo n (see Figure 1).

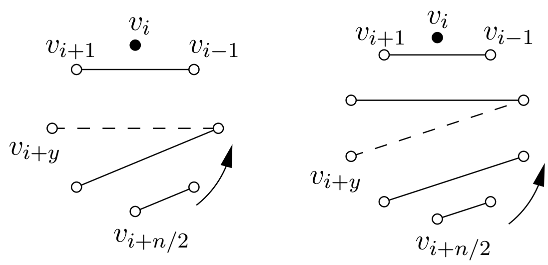

Figure 1. Color class F_i of K_n for $n = 8$ and $n = 10$.

In each of the color classes F_i the vertex v_{i+y} is unmatched. Therefore, we can add the edge $v_{i+y}v_{i-\lfloor n/4 \rfloor}$ in each F_i, $0 \leq i \leq n/2 - 1$ (represented as a dashed line in Figure 1).

Each vertex and each edge of K_n is contained in exactly one of these color classes. The induced subgraphs of this partition consist of K_1, K_2, and P_3. Therefore, this is an $(\mathcal{O}, \mathcal{Q})$-total coloring of the complete graph K_n with n colors. ■

Theorem 5. $\chi''_{\mathcal{P},\mathcal{O}_1}(K_n) = \begin{cases} n & \textit{if } \mathcal{P} \neq \mathcal{O} \textit{ or } n \textit{ odd}, \\ n+1 & \textit{if } \mathcal{P} = \mathcal{O} \textit{ and } n \textit{ even}. \end{cases}$

Proof. The case that n is odd is treated in the above example, the case $\mathcal{P} = \mathcal{O}$ and n even in Theorem 4.

If n is even and $\mathcal{P} \neq \mathcal{O}$ then $K_2 \in \mathcal{P}$. First note that $\chi''_{\mathcal{P},\mathcal{O}_1}(K_n) \geq \chi''_{\mathcal{I},\mathcal{O}_1}(K_n) = n$ by Lemma 1 and Theorem 3.

In the following we provide a $(\mathcal{P}, \mathcal{O}_1)$-total coloring of K_n with n colors which implies $\chi''_{\mathcal{P},\mathcal{O}_1}(K_n) = n$.

For $n = 2$ and $n = 4$ see Figure 2.

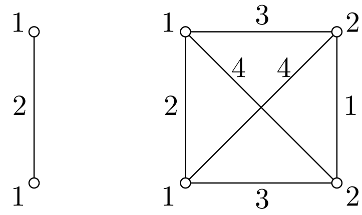

Figure 2. $(\mathcal{P}, \mathcal{O}_1)$-total colorings of K_2 and K_4.

If $n \geq 6$ then there exists an edge coloring of K_n with $n-1$ colors such that there are $n/2$ independent edges with pairwise distinct colors. This can be seen as follows. Consider a drawing of $K_n - v \cong K_{n-1}$ with vertex set $\{v_0, \ldots, v_{n-2}\}$ as a regular $(n-1)$-gon. Color parallel edges of K_{n-1} with one color and the edges vv_i, $0 \leq i \leq n-2$, with the missing color at v_i. If $n \equiv 2 \pmod 4$ then the edges $v_0v_1, v_2v_3, \ldots, v_{n-4}v_{n-3}, v_{n-2}v$ are independent with mutually distinct colors. If $n \equiv 0 \pmod 4$ then the edges $v_0v_1, v_2v_4, v_3v_6, v_5v$ and if $n \geq 12$ also $v_7v_8, v_9v_{10}, \ldots, v_{n-3}v_{n-2}$ are independent with pairwise distinct colors.

Assign the color of each of these edges to its end-vertices and then replace the colors of all these edges by the nth color (see Figure 3 for an example). ■

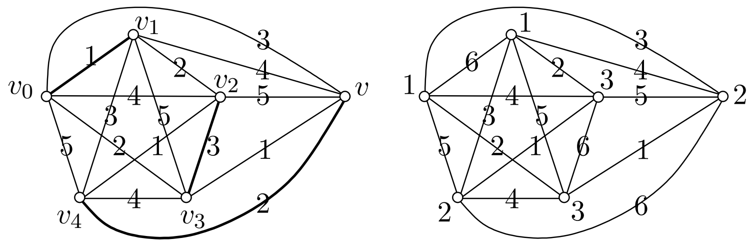

Figure 3. Edge coloring and $(\mathcal{P}, \mathcal{O}_1)$-total colorings of K_6.

The corresponding results concerning $(\mathcal{O}, \mathcal{Q})$- and $(\mathcal{P}, \mathcal{O}_1)$-total colorings of paths and cycles are special cases of the following theorems.

Theorem 6. $\chi''_{\mathcal{P},\mathcal{Q}}(P_n) = \begin{cases} 1 & \textit{if } n = 1, \\ 2 & \textit{if } P_n \in (\mathcal{P} \cap \mathcal{Q}) \setminus \mathcal{O}, \\ 3 & \textit{otherwise.} \end{cases}$

Proof. The result follows from Theorem 1 and from $\chi''_{\mathcal{P},\mathcal{Q}}(P_n) \le \chi''(P_n) \le 3$ (see Lemma 1). ■

Theorem 7. $\chi''_{\mathcal{P},\mathcal{Q}}(C_n) = \begin{cases} 2 & \textit{if } C_n \in \mathcal{P} \cap \mathcal{Q}, \\ 4 & \textit{if } (\mathcal{P}=\mathcal{O}, \mathcal{Q} = \mathcal{O}_1, n \not\equiv 0 \pmod 3) \textit{ or } (n=5, \\ & \mathcal{P} = \mathcal{O}, P_4 \notin \mathcal{Q}) \textit{or } (n = 5, \mathcal{P} = \mathcal{Q} = \mathcal{O}_1), \\ 3 & \textit{otherwise.} \end{cases}$

Proof. If $C_n \in \mathcal{P} \cap \mathcal{Q}$ then $\chi''_{\mathcal{P},\mathcal{Q}}(C_n) = 2$ by Theorem 1 and if $C_n \notin \mathcal{P} \cap \mathcal{Q}$ then $3 \le \chi''_{\mathcal{P},\mathcal{Q}}(C_n) \le 4$ by Theorem 1, Lemma 1, and the fact that $\chi''(C_n) \le 4$.

If $n \equiv 0 \pmod 3$ then $\chi''(C_n) = 3$ and therefore $\chi''_{\mathcal{P},\mathcal{Q}}(C_n) = 3$.

Let $n \not\equiv 0 \pmod 3$. If $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \mathcal{O}_1$ then $\chi''_{\mathcal{O},\mathcal{O}_1}(C_n) = 4$. If $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} \supset \mathcal{O}_1$ then color the successive vertices $v_0, v_1, \ldots, v_{n-1}$ of C_n by colors $1, 2, 3, 1, 2, 3, \ldots, 1, 2, 3, 2$ if $n \equiv 1 \pmod 3$ and by colors $1, 2, 3, 1, 2, 3, \ldots, 1, 2, 3, 2, 1, 2, 3, 2$ if $n \equiv 2 \pmod 3$, $n \ge 8$, and the edges with the at their end-vertices missing color of $\{1, 2, 3\}$. This is an $(\mathcal{O}, \mathcal{Q})$-total coloring of C_n since $P_3 \in \mathcal{Q}$. If $n = 5$ then color the vertices with colors $1, 2, 1, 2, 3$ (unique up to permutation) and the edges again with the at their end-vertices missing color of the set $\{1, 2, 3\}$. This is an $(\mathcal{O}, \mathcal{Q})$-total coloring of C_5 if $P_4 \in \mathcal{Q}$. If $P_4 \notin \mathcal{Q}$ then $\chi''_{\mathcal{O},\mathcal{Q}}(C_5) = 4$.

By switching the colors of vertices and edges one obtains $\chi''_{\mathcal{P},\mathcal{O}_1}(C_n) = 3$ if $\mathcal{P} \supset \mathcal{O}$ with the exception of $\chi''_{\mathcal{P},\mathcal{O}_1}(C_5) = 4$ if $P_3 \notin \mathcal{P}$.

If $\mathcal{P} \supset \mathcal{O}$ and $\mathcal{Q} \supset \mathcal{O}_1$ then color the elements $v_0, v_0v_1, v_1, v_1v_2, \ldots$ successively with colors $1, 2, 3, 1, 2, 3, \ldots$ if $n \not\equiv 2 \pmod 3$ and with colors $1, 2, 3, 1, 2, 3, \ldots, 1, 2, 3, 2, 1, 3, 2$ if $n \equiv 2 \pmod 3$ to obtain a $(\mathcal{P}, \mathcal{Q})$-total coloring of C_n with 3 colors. ■

3. TOTAL ACYCLIC COLORINGS ($\mathcal{P} = \mathcal{Q} = \mathcal{D}_1$)

Total acyclic colorings are $(\mathcal{D}_1, \mathcal{D}_1)$-total colorings where $\mathcal{D}_1$ contains the 1-degenerate graphs which are the acyclic graphs. The $\mathcal{D}_1$-vertex chromatic number is the *vertex arboricity* $a(G) = \chi_{\mathcal{D}_1}(G)$ and the $\mathcal{D}_1$-edge chromatic number is the (*edge*) *arboricity* $a'(G) = \chi'_{\mathcal{D}_1}(G)$.

We mention some known results on the vertex and edge arboricity: $\chi_{\mathcal{D}_1}(G) = \chi'_{\mathcal{D}_1}(G) = 1$ if and only if G is acyclic, $\chi_{\mathcal{D}_1}(C_n) = \chi'_{\mathcal{D}_1}(C_n) = 2$, $\chi_{\mathcal{D}_1}(K_n) = \chi'_{\mathcal{D}_1}(K_n) = \lceil n/2 \rceil$, $\chi_{\mathcal{D}_1}(K_{m,n}) = 1$ if $m = 1$ or $n = 1$,

$\chi_{\mathcal{D}_1}(K_{m,n}) = 2$ if $m \neq 1 \neq n$, $\chi'_{\mathcal{D}_1}(K_{m,n}) = \lceil mn/(m+n-1) \rceil$ (see [13], e.g.).

We denote induced subgraphs H of G by $H \leq G$. Proved upper bounds are $\chi_{\mathcal{D}_1}(G) \leq \max_{H \leq G}\{\lfloor \delta(H)/2 \rfloor + 1\}$ [7] which implies $\chi_{\mathcal{D}_1}(G) \leq \lfloor \Delta(G)/2 \rfloor + 1$ and $\chi'_{\mathcal{D}_1}(G) \leq \lfloor \Delta(G)/2 \rfloor + 1$. The latter is an implication of

$$\chi'_{\mathcal{D}_1}(G) = \max_{\substack{H \leq G \\ |V(H)|>1}} \{\lceil |E(H)| / (|V(H)| - 1) \rceil\} \tag{6}$$

which is due to Nash-Williams [13]. Moreover, $\chi_{\mathcal{D}_1}(G) \leq \chi'_{\mathcal{D}_1}(G)$ (see [5]).

Observe that we have an analogous situation for ordinary colorings: $\chi(G) \leq \Delta(G) + 1$, $\chi'(G) \leq \Delta(G) + 1$ (Vizing [14]) and $\chi(G) \leq \chi'(G)$ (Brooks [4]).

Theorem 1 implies that $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) = 1$ if and only if $G \in \mathcal{O}$ and $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) = 2$ if and only if $G \in \mathcal{D}_1 \setminus \mathcal{O}$ (acyclic graphs with edges). For cycles C_n we have $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(C_n) = 3$ by Theorem 7 since $C_n \notin \mathcal{D}_1$.

Theorem 8. $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(K_1) = 1$, $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(K_2) = 2$, $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(K_n) = \lfloor n/2 \rfloor + 2$ *for* $n \geq 3$.

Proof. The results for $n = 1$ and $n = 2$ follow from Theorem 1.

Let $n \geq 3$. Each color class of a $(\mathcal{D}_1, \mathcal{D}_1)$-total coloring of K_n with c colors contains 0, 1, or 2 vertices and at most $n-1$, $n-2$, or $n-3$ edges, respectively. If x_i denotes the number of color classes with i vertices we obtain $x_0 + x_1 + x_2 = c$ (number of color classes), $x_1 + 2x_2 = n$ (number of vertices) and $(n-1)x_0 + (n-2)x_1 + (n-3)x_2 \geq \binom{n}{2}$ (number of edges). It follows $(n-1)(c-1) - 1 \geq \binom{n}{2}$ and therefore $c \geq \lceil n/2 + 1 + 1/(n-1) \rceil$. If n is even then $c \geq n/2 + 2$; if $n \geq 3$ is odd then $1/(n-1) \leq 1/2$ and therefore $c \geq \lceil n/2 \rceil + 1 = \lfloor n/2 \rfloor + 2$ which implies $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(K_n) \geq \lfloor n/2 \rfloor + 2$ if $n \geq 3$.

On the other hand, it holds $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(K_n) \leq \lfloor n/2 \rfloor + 2$ which can be seen by the following partition of the elements of K_n in $\lfloor n/2 \rfloor + 2$ classes.

If n is even then class F_i, $0 \leq i \leq \frac{n}{2} - 1$, contains vertices v_i and $v_{i+n/2}$ and the $n-3$ edges of the path $(v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \ldots, v_{i+n/2-1}, v_{i-n/2+1})$ where all indices are reduced modulo n. The remaining edges $v_0v_1, v_1v_2, \ldots, v_{n-1}v_0$ induce a cycle which can be colored with two additional colors (see Figure 4, upper part).

If n is odd then class F_i, $0 \leq i \leq \frac{n-3}{2}$, contains vertices v_i and $v_{i-(n-1)/2}$ and the $n-3$ edges of the path $(v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \ldots, v_{i+(n-1)/2})$.

Moreover, the remaining elements of K_n can be colored using two additional colors:

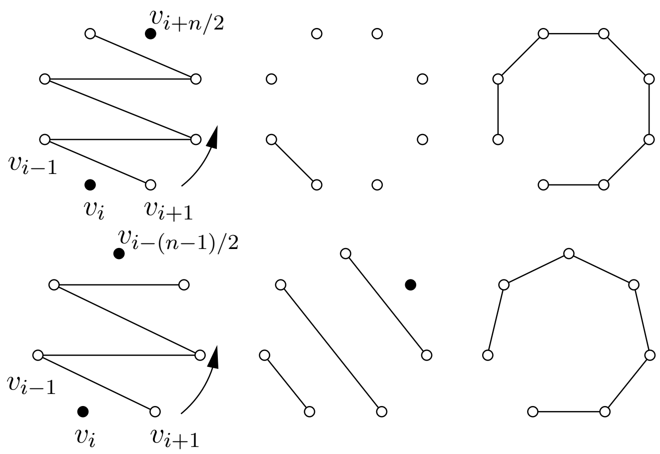

Figure 4. Color classes of K_n if n is even (above) or odd (below).

vertex $v_{(n-1)/2}$ and edges $v_{(n-1)/2-j}v_{(n-1)/2+j}$, $j = 1, \ldots, (n-1)/2$ with one new color and the edges of the path $(v_0, v_1, \ldots, v_{n-1})$ with the second new color (see Figure 4, lower part). ■

The results for acyclic graphs, cycles and complete graphs suggest the following general conjecture.

Conjecture 1. $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq \left\lfloor \frac{\Delta(G)+1}{2} \right\rfloor + 2.$

This conjecture is an analogy to the *total coloring conjecture* which says that $\chi''(G) \leq \Delta(G) + 2$ for all graphs G.

Since $m \leq 3n - 6$ for planar graphs G of order $n \geq 3$ and size m we obtain $\chi_{\mathcal{D}_1}(G) \leq \chi'_{\mathcal{D}_1}(G) \leq 3$ by (6) which implies $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq 6$. We can improve this to $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq 5$ but we do not know whether $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq 4$ is true for all planar graphs. For outerplanar graphs G it holds $\chi''_{\mathcal{D}_1,\mathcal{D}_1}(G) \leq 3$.

4. $(\mathcal{P}, \mathcal{Q})$-total Colorings of Infinite Graphs — a Compactness Theorem

All our considerations hold for arbitrary simple infinite graphs. Let us denote by $\mathcal{I}^*$ the class of all simple infinite graphs. A graph property $\mathcal{P}$ is any isomorphism-closed nonempty subclass of $\mathcal{I}^*$.

In 1951, de Bruijn and Erdős [8] proved that an infinite graph G is k-colorable if and only if every finite subgraph of G is k-colorable. Analogous compactness theorems for generalized colorings were proved in [6]. They all have been based on the "Set Partition Compactness Theorem" (see [6]), where the key concept is that of a property being of *finite character*. A graph property $\mathcal{P}$ is of *finite character* if a graph in $\mathcal{I}^*$ has property $\mathcal{P}$ if and only if each of its finite induced subgraphs has property $\mathcal{P}$. It is easy to see that if $\mathcal{P}$ is of finite character and a graph has property $\mathcal{P}$ then so does every induced subgraph. A property $\mathcal{P}$ is said to be *induced-hereditary* if $G \in \mathcal{P}$ and $H \leq G$ implies $H \in \mathcal{P}$, that is, $\mathcal{P}$ is closed under taking induced subgraphs. Thus properties of finite character are induced-hereditary. However, not all induced-hereditary properties are of finite character. For example, the graph property of not containing a vertex of infinite degree is induced-hereditary but not of finite character. Let us also remark that every property which is hereditary with respect to every subgraph (we say simply *hereditary*) is induced-hereditary as well. The properties of being edgeless, of maximum degree at most k, K_n-free, acyclic, complete, perfect, etc. are properties of finite character. Each additive hereditary graph property $\mathcal{P}$ of finite character can be characterized (see, e.g., [12]) by the set of *connected minimal forbidden graphs* of $\mathcal{P}$, which is defined as follows:

$$\mathbf{F}(\mathcal{P}) = \big\{G : G \text{ connected, } G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\big\}.$$

In the paper [6] also a compactness result for generalized colorings of hypergraphs has been presented. A *simple hypergraph* $H = (X, E)$ is a hypergraph on a vertex set X where all hyperedges $e \in E$ are different finite subsets of the vertex set X. Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be properties of simple hypergraphs (i.e. classes of simple hypergraphs closed under isomorphism). A hypergraph $H = (X, E)$ is $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-*colorable* if the vertex set X of H can be partitioned into sets $X_1, \ldots, X_m$ such that the induced subhypergraphs $H[X_i] = (X_i, E(X_i))$ of H, where $E(X_i)$ consists of all hyperedges of H all of whose vertices belong to X_i, has property $\mathcal{P}_i$, $i = 1, 2, \ldots, m$. A property

$\mathcal{P}$ of hypergraphs is of *finite vertex character* if a hypergraph has property $\mathcal{P}$ if and only if every finite induced subhypergraph has property $\mathcal{P}$. Then, using the Set Partition Compactness Theorem, it holds:

Theorem 9. *Let H be a simple hypergraph and suppose $\mathcal{P}_1, \ldots, \mathcal{P}_m$ are properties of hypergraphs of finite vertex character. Then H is $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-colorable if every finite induced subhypergraph of H is $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-colorable.*

In particular, if $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_m = \mathcal{O}_H$, where $\mathcal{O}_H$ denotes the property of a hypergraph "to be hyperedgeless", i.e., $E = \emptyset$, we have a compactness theorem for the regular hypergraph coloring, since $\mathcal{O}_H$ is of finite character. Now we will use this result to prove the compactness theorem for $(\mathcal{P}, \mathcal{Q})$-total colorings:

Theorem 10. *Let $G \in \mathcal{I}^*$ be a simple infinite graph and suppose $\mathcal{P}$ and $\mathcal{Q} \neq \mathcal{O}$ are additive properties of finite character. Then G is $(\mathcal{P}, \mathcal{Q})$-totally k-colorable if and only if every finite induced subgraph of G is $(\mathcal{P}, \mathcal{Q})$-totally k-colorable.*

Proof. Let $G = (V(G), E(G))$ be a simple infinite graph and let $\mathcal{P}, \mathcal{Q}, \mathcal{Q} \neq \mathcal{O}$ be additive hereditary properties of finite character. Let $\mathbf{F}(\mathcal{P})$ and $\mathbf{F}(\mathcal{Q})$ be the sets of minimal forbidden graphs of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Let us define a hypergraph $H(G) = (V^*, E^*)$ so that $V^* = V(G) \cup E(G)$ and a set $e \subset V^*$ is an hyperedge of $H(G)$ if and only if

(1) $e = \{v, h\}, v \in V(G), h \in E(G), v \in h$, or

(2) $G[e] \in \mathbf{F}(\mathcal{P}), e \subset V(G)$, or

(3) $G[e] \in \mathbf{F}(\mathcal{Q}), e \subset E(G)$.

By the definition of the hypergraph $H(G)$ of G, a graph G is $(\mathcal{P}, \mathcal{Q})$-totally k-colorable if the hypergraph $H(G)$ is regularly k-colorable. By Theorem 9, $H(G)$ is regularly k-colorable if every finite induced subhypergraph of $H(G)$ is regularly k-colorable. However, if every finite induced subgraph of G is $(\mathcal{P}, \mathcal{Q})$-totally k-colorable, then obviously every finite induced subhypergraph of $H(G)$ is regularly k-colorable. ∎

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