# GENERALIZED TOTAL COLORINGS OF GRAPHS 

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#### Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphism. Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary properties of graphs. A $(\mathcal{P}, \mathcal{Q})$-total coloring


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#### Abstract

of a simple graph $G$ is a coloring of the vertices $V(G)$ and edges $E(G)$ of $G$ such that for each color $i$ the vertices colored by $i$ induce a subgraph of property $\mathcal{P}$, the edges colored by $i$ induce a subgraph of property $\mathcal{Q}$ and incident vertices and edges obtain different colors. In this paper we present some general basic results on $(\mathcal{P}, \mathcal{Q})$-total colorings. We determine the $(\mathcal{P}, \mathcal{Q})$-total chromatic number of paths and cycles and, for specific properties, of complete graphs. Moreover, we prove a compactness theorem for $(\mathcal{P}, \mathcal{Q})$-total colorings.


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## 1. Introduction

We denote the class of all finite simple graphs by $\mathcal{I}$ (see [1]). A graph property $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is called additive if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. A property $\mathcal{P}$ is called hereditary if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$.

We use the following standard notations for specific hereditary properties:

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\(\mathcal{O}=\{G \in \mathcal{I}: E(G)=\emptyset\}\),
\(\mathcal{O}^{k}=\{G \in \mathcal{I}: \chi(G) \leq k\}\),
\(\mathcal{D}_{k}=\{G \in \mathcal{I}\) : each subgraph of \(G\) contains a vertex of degree at most \(k\}\),
\(\mathcal{I}_{k}=\left\{G \in \mathcal{I}: G\right.\) does not contain \(\left.K_{k+2}\right\}\),
\(\mathcal{J}_{k}=\left\{G \in \mathcal{I}: \chi^{\prime}(G) \leq k\right\}\),
\(\mathcal{O}_{k}=\{G \in \mathcal{I}:\) each component of \(G\) has at most \(k+1\) vertices \(\}\),
\(\mathcal{S}_{k}=\{G \in \mathcal{I}: \Delta(G) \leq k\}\),
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where $\chi(G)$ is the chromatic number, $\chi^{\prime}(G)$ the chromatic index and $\Delta(G)$ the maximum degree of the graph $G=(V, E)$.

A total coloring of a graph $G$ is a coloring of the vertices and edges (together called the elements of $G$ ) such that all pairs of adjacent or incident elements obtain distinct colors. The minimum number of colors of a total coloring of $G$ is called the total chromatic number $\chi^{\prime \prime}(G)$ of $G$.

Let $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_{1}$ be two additive and hereditary graph properties. Then a $(\mathcal{P}, \mathcal{Q})$-total coloring of a graph $G$ is a coloring of the vertices and edges of $G$ such that for any color $i$ all vertices of color $i$ induce a subgraph of property $\mathcal{P}$, all edges of color $i$ induce a subgraph of property $\mathcal{Q}$ and
vertices and incident edges are colored differently. The minimum number of colors of a $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$ is called the $(\mathcal{P}, \mathcal{Q})$-total chromatic number $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$ of $G$.

If $G$ contains edges then $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$ is only defined if $K_{2} \in \mathcal{Q}$ and therefore $\mathcal{O}_{1} \subseteq \mathcal{Q}$. Since $\mathcal{O} \subseteq \mathcal{P}$ for all additive hereditary properties we obtain $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq|V|+|E|$ which guarantees the existence of $(\mathcal{P}, \mathcal{Q})$-total chromatic numbers.
$(\mathcal{P}, \mathcal{Q})$-total colorings are generalized total colorings since $\chi_{\mathcal{O}, \mathcal{O}_{1}}^{\prime \prime}(G)=$ $\chi^{\prime \prime}(G)$ for all graphs $G$.

Generalized $\mathcal{P}$-vertex colorings and $\mathcal{P}$-chromatic numbers $\chi_{\mathcal{P}}(G)$ as well as generalized $\mathcal{Q}$-edge colorings and $\mathcal{Q}$-chromatic indices $\chi_{\mathcal{Q}}^{\prime}(G)$ are analogously defined (see [3, 9] for some results). Evidently, these are generalizations of proper vertex colorings and proper edge colorings since $\chi_{\mathcal{O}}(G)=$ $\chi(G)$ and $\chi_{\mathcal{O}_{1}}^{\prime}(G)=\chi^{\prime}(G)$.

The $\mathcal{P}$-chromatic number and the $\mathcal{Q}$-chromatic index of $G$ provide general lower and upper bounds for $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$.

## Theorem 1.

(a) $\max \left\{\chi_{\mathcal{P}}(G), \chi_{\mathcal{Q}}^{\prime}(G)\right\} \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}}(G)+\chi_{\mathcal{Q}}^{\prime}(G)$,
(b) $\chi_{\mathcal{P}}(G) \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}}(G)+1$ if $G \in \mathcal{Q}$,
(c) $\chi_{\mathcal{Q}}^{\prime}(G) \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{Q}}^{\prime}(G)+1$ if $G \in \mathcal{P}$,
(d) $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)=1$ iff $G \in \mathcal{O}$,
(e) $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)=2$ iff $G \in(\mathcal{P} \cap \mathcal{Q}) \backslash \mathcal{O}$,
(f) $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \geq 3$ iff $G \in \mathcal{I} \backslash(\mathcal{P} \cap \mathcal{Q})$.

Proof. Since a $(\mathcal{P}, \mathcal{Q})$-total coloring induces a $\mathcal{P}$-vertex coloring and a $\mathcal{Q}$-edge coloring it follows that $\chi_{\mathcal{P}}(G) \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$ and $\chi_{\mathcal{Q}}^{\prime}(G) \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$. A $\mathcal{P}$-vertex coloring of $G$ with $\chi_{\mathcal{P}}(G)$ colors and a $\mathcal{Q}$-edge coloring with $\chi_{\mathcal{Q}}^{\prime}(G)$ additional colors induce a $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$ with $\chi_{\mathcal{P}}(G)+\chi_{\mathcal{Q}}^{\prime}(G)$ colors.

If $G \in \mathcal{Q}$ or $G \in \mathcal{P}$, respectively, then all edges or all vertices can obtain the same additional color which implies $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}}(G)+1$ or $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{Q}}^{\prime}(G)+1$, respectively.

If $G$ has no edges then $G \in \mathcal{O} \subseteq \mathcal{P}$ and therefore all vertices can obtain the same color which implies $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)=1$. If $G$ has edges then $G \notin \mathcal{O}$ and therefore at least two colors are needed to color a vertex and an incident edge which implies $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \geq 2$.

It holds $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)=2$ if and only if $G$ contains edges and for each non-trivial component of $G$ all vertices as well as all edges can be colored with one color each, that is, if and only if $G \in(\mathcal{P} \cap \mathcal{Q}) \backslash \mathcal{O}$.

Obviously, if $G \notin \mathcal{P} \cap \mathcal{Q}$ then $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \geq 3$.
The following monotonicity and additivity results are obvious.
Lemma 1. If $\mathcal{P}_{1} \subseteq \mathcal{P}_{2}$ and $\mathcal{Q}_{1} \subseteq \mathcal{Q}_{2}$, then $\chi_{\mathcal{P}_{2}, \mathcal{Q}_{2}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}_{1}, \mathcal{Q}_{1}}^{\prime \prime}(G)$.
Proof. If $\mathcal{P}_{1} \subseteq \mathcal{P}_{2}$ and $\mathcal{Q}_{1} \subseteq \mathcal{Q}_{2}$ then each $\left(\mathcal{P}_{1}, \mathcal{Q}_{1}\right)$-total coloring is a ( $\mathcal{P}_{2}, \mathcal{Q}_{2}$ )-total coloring.

It follows $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{O}, \mathcal{O}_{1}}^{\prime \prime}(G)=\chi^{\prime \prime}(G)$ since $\mathcal{O} \subseteq \mathcal{P}$ and $\mathcal{O}_{1} \subseteq \mathcal{Q}$, that is, the total chromatic number is an upper bound for the $(\mathcal{P}, \mathcal{Q})$-total chromatic number of a graph $G$.

Lemma 2. If $H \subseteq G$, then $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(H) \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$.
Proof. The restriction of a $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$ to the elements of $H$ is a $(\mathcal{P}, \mathcal{Q})$-total coloring of $H$.

The following lemma implies that one can restrict oneself to connected graphs, in general.

Lemma 3. If $G$ and $H$ are disjoint, then $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G \cup H)=\max \left\{\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)\right.$, $\left.\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(H)\right\}$.

Proof. $(\mathcal{P}, \mathcal{Q})$-total colorings of $G$ and of $H$ provide a $(\mathcal{P}, \mathcal{Q})$-total coloring of $G \cup H$ since $G$ and $H$ are disjoint which implies $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G \cup H) \leq$ $\max \left\{\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G), \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(H)\right\}$. Lemma 2 implies equality.

If one of the properties is the class $\mathcal{I}$ of all finite simple graphs then the $(\mathcal{P}, \mathcal{Q})$-total chromatic number of $G$ attains one of two possible values by Theorem 1:

$$
\begin{equation*}
\chi_{\mathcal{P}}(G) \leq \chi_{\mathcal{P}, \mathcal{I}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}}(G)+1, \quad \chi_{\mathcal{Q}}^{\prime}(G) \leq \chi_{\mathcal{I}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{Q}}^{\prime}(G)+1 \tag{1}
\end{equation*}
$$

If $\mathcal{P}=\mathcal{Q}=\mathcal{I}$ then $\chi_{\mathcal{I}, \mathcal{I}}^{\prime \prime}(G)=1$ if $G \in \mathcal{O}$ and $\chi_{\mathcal{I}, \mathcal{I}}^{\prime \prime}(G)=2$ otherwise by Theorem 1.

If $G \in \mathcal{Q}$ then $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)$ and therefore $\chi_{\mathcal{P}, \mathcal{I}}^{\prime \prime}(G)$ for all graphs $G$ can be determined as follows.

Theorem 2. If $G \in \mathcal{Q}$, then

$$
\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)= \begin{cases}\chi_{\mathcal{P}}(G) & \text { if } G \in \mathcal{O} \text { or } \chi_{\mathcal{P}}(G) \geq 3, \\ \chi_{\mathcal{P}}(G)+1 & \text { if } G \in \mathcal{P} \backslash \mathcal{O} \text { or } \chi_{\mathcal{P}}(G)=2\end{cases}
$$

Proof. By Theorem 1, $\chi_{\mathcal{P}}(G) \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}}(G)+1$.
If $\chi_{\mathcal{P}}(G)=1$ then $G \in \mathcal{P}$ which implies $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)=1$ for $G \in \mathcal{O}$ and $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G)=2$ for $G \in \mathcal{P} \backslash \mathcal{O}$ by Theorem 1 .

If $\chi_{\mathcal{P}}(G)=2$ then $G \notin \mathcal{P}$ and therefore $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \geq 3$ by Theorem 1 . On the other hand, $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}}(G)+1=3$.

If $\chi_{\mathcal{P}}(G) \geq 3$ then $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \geq \chi_{\mathcal{P}}(G)$. Consider a $\mathcal{P}$-vertex coloring of $G$ with $\chi_{\mathcal{P}}(G)$ colors. Each edge can be colored with a color different to the colors of its end-vertices. This is a $(\mathcal{P}, \mathcal{Q})$-total coloring of $G$ with $\chi_{\mathcal{P}}(G)$ colors since $H \in \mathcal{Q}$ for all $H \subseteq G$.

$$
\text { 2. } \quad \mathcal{P}=\mathcal{O} \text { or } \mathcal{Q}=\mathcal{O}_{1}
$$

Since $\mathcal{O} \subseteq \mathcal{P} \subseteq \mathcal{I}$ and $\mathcal{O}_{1} \subseteq \mathcal{Q} \subseteq \mathcal{I}$, Lemma 1 provides the following bounds:

$$
\begin{align*}
\chi_{\mathcal{I}, \mathcal{I}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}, \mathcal{I}}^{\prime \prime}(G) & \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}, \mathcal{O}_{1}}^{\prime \prime}(G) \leq \chi_{\mathcal{O}, \mathcal{O}_{1}}^{\prime \prime}(G)=\chi^{\prime \prime}(G),  \tag{2}\\
\chi_{\mathcal{I}, \mathcal{I}}^{\prime \prime}(G) \leq \chi_{\mathcal{I}, \mathcal{Q}}^{\prime \prime}(G) & \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{O}, \mathcal{Q}}^{\prime \prime}(G) \leq \chi_{\mathcal{O}, \mathcal{O}_{1}}^{\prime \prime}(G)=\chi^{\prime \prime}(G), \\
\chi_{\mathcal{P}, \mathcal{I}}^{\prime \prime}(G) & \leq \chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}(G) \leq \chi_{\mathcal{O}, \mathcal{Q}}^{\prime \prime}(G), \\
\chi_{\mathcal{I}, \mathcal{Q}}^{\prime \prime}(G) & \leq \chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}(G) \leq \chi_{\mathcal{P}, \mathcal{O}_{1}}^{\prime \prime}(G) .
\end{align*}
$$

$(\mathcal{O}, \mathcal{I})$ - and $\left(\mathcal{I}, \mathcal{O}_{1}\right)$-total coloring are certain $[r, s, t]$-colorings which also are generalizations of ordinary colorings.

Given non-negative integers $r, s$, and $t$ with $\max \{r, s, t\} \geq 1$, an $[r, s, t]$ coloring of a finite and simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$ is a mapping $c$ from $V(G) \cup E(G)$ to the color set $\{0,1, \ldots, k-1\}$, $k \in \mathbb{N}$, such that $\left|c\left(v_{i}\right)-c\left(v_{j}\right)\right| \geq r$ for every two adjacent vertices $v_{i}, v_{j}$, $\left|c\left(e_{i}\right)-c\left(e_{j}\right)\right| \geq s$ for every two adjacent edges $e_{i}, e_{j}$, and $\left|c\left(v_{i}\right)-c\left(e_{j}\right)\right| \geq t$ for all pairs of incident vertices and edges, respectively. The $[r, s, t]$-chromatic number $\chi_{r, s, t}(G)$ of $G$ is defined to be the minimum $k$ such that $G$ admits an $[r, s, t]$-coloring (see [10, 11]).

By this definition we obtain $\chi_{\mathcal{I}, \mathcal{I}}^{\prime \prime}(G)=\chi_{0,0,1}(G), \chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}(G)=\chi_{1,0,1}(G)$, $\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}(G)=\chi_{0,1,1}(G)$ and $\chi_{\mathcal{O}, \mathcal{O}_{1}}^{\prime \prime}(G)=\chi_{1,1,1}(G)$. The first three of these $[r, s, t]$-chromatic numbers were determined in [10].

## Theorem 3.

(a) $\chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}(G)=\chi_{1,0,1}(G)= \begin{cases}\chi(G) & \text { if } \chi(G) \neq 2, \\ 3=\chi(G)+1 & \text { if } \chi(G)=2,\end{cases}$
(b) $\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}(G)=\chi_{0,1,1}(G)=\Delta(G)+1$.

Proof. (a) By Theorem 2 we obtain for $\mathcal{P}=\mathcal{O}$ that $\chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}(G)=\chi_{\mathcal{O}}(G)=$ $\chi(G)$ if $G \in \mathcal{O}$ or $\chi(G) \geq 3$ and $\chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}(G)=\chi(G)+1$ if $\chi(G)=2$.
(b) If $\chi^{\prime}(G)=\Delta(G)$ then $\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}(G) \geq \Delta(G)+1$ since an additional color is necessary to color a vertex of maximum degree. If $\chi^{\prime}(G)=\Delta(G)+1$ then $\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}(G) \geq \chi^{\prime}(G)=\Delta(G)+1$ by Theorem 1 .

On the other hand, we have $\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}(G) \leq \Delta(G)+1$ since the edges can be colored with at most $\Delta(G)+1$ colors by Vizing's Theorem and at each vertex there is a missing edge color which can be used to color this vertex.

To illustrate the results we consider as examples paths $P_{n}$, cycles $C_{n}$ and complete graphs $K_{n}$.

## Examples 1.

1. Theorem 3 implies $\chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}\left(P_{1}\right)=\chi_{\mathcal{I} \mathcal{O}_{1}}^{\prime \prime}\left(P_{1}\right)=1, \chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}\left(P_{2}\right)=3, \chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}\left(P_{2}\right)$ $=2$ and $\chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}\left(P_{n}\right)=\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}\left(P_{n}\right)=3$ for $n \geq 3$.
2. We have $\chi_{\mathcal{O}}\left(C_{n}\right)=\chi\left(C_{n}\right)=\chi_{\mathcal{O}_{1}}^{\prime}\left(C_{n}\right)=\chi^{\prime}\left(C_{n}\right)$ and $\chi\left(C_{n}\right)=2$ if $n$ is even and $\chi\left(C_{n}\right)=3$ if $n$ is odd. Moreover, we have $\chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}\left(C_{n}\right)=$ $\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}\left(C_{n}\right)=3$ by Theorem 3 . Therefore, the lower and upper bounds of (1) are attained for cycles $C_{n}$.
3. Theorem 3 implies $\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right)=n$ and $\chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}\left(K_{n}\right)= \begin{cases}n & \text { if } n \neq 2, \\ n+1 & \text { if } n=2 .\end{cases}$

If $n$ is odd then $n=\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right) \leq \chi_{\mathcal{P}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right) \leq \chi_{\mathcal{O}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right)=\chi^{\prime \prime}\left(K_{n}\right)=$ $n$ and $n=\chi_{\mathcal{O}, \mathcal{I}}^{\prime \prime}\left(K_{n}\right) \leq \chi_{\mathcal{O}, \mathcal{Q}}^{\prime \prime}\left(K_{n}\right) \leq \chi_{\mathcal{O}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right)=\chi^{\prime \prime}\left(K_{n}\right)=n$ by Lemma 1. Therefore, if $n$ is odd then $\chi_{\mathcal{P}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right)=\chi_{\mathcal{O}, \mathcal{Q}}^{\prime \prime}\left(K_{n}\right)=n$ for all additive and hereditary properties $\mathcal{P}$ and $\mathcal{Q}$.

In Theorems 4 and 5 we also consider complete graphs of even order.
Theorem 4. $\chi_{\mathcal{O}, \mathcal{Q}}^{\prime \prime}\left(K_{n}\right)= \begin{cases}n & \text { if } n \text { odd or }\left(n \geq 4 \text { even and } \mathcal{O}_{1} \subset \mathcal{Q}\right) \text {, } \\ n+1 & \text { if } n=2 \text { or }\left(n \text { even and } \mathcal{Q}=\mathcal{O}_{1}\right) \text {. }\end{cases}$
Proof. The case that $n$ is odd is considered in the above example and the case $n=2$ is obvious.

If $n$ is even and $\mathcal{Q}=\mathcal{O}_{1}$ then $\chi_{\mathcal{O}, \mathcal{Q}}^{\prime \prime}\left(K_{n}\right)=\chi^{\prime \prime}\left(K_{n}\right)=n+1$.
If $n \geq 4$ is even and $\mathcal{O}_{1} \neq \mathcal{Q}$ then $P_{3} \in \mathcal{Q}$. We partition the elements of $K_{n}$ with vertex set $\left\{v_{0}, v_{1}, \ldots, v_{n-1}\right\}$ in $n$ color classes as follows:

Class $F_{i}, 0 \leq i \leq n-1$, contains the vertex $v_{i}$, the edges $v_{i-1} v_{i+1}, v_{i-2} v_{i+2}$, $\ldots, v_{i-y+1} v_{i+y-1}$ as well as the edges $v_{i+n / 2} v_{i+n / 2+1}, v_{i+n / 2-1} v_{i+n / 2+2}, \ldots$, $v_{i+y+1} v_{i-y}$ where $y=\lceil n / 4\rceil$ and the indices are reduced modulo $n$ (see Figure 1).


Figure 1. Color class $F_{i}$ of $K_{n}$ for $n=8$ and $n=10$.

In each of the color classes $F_{i}$ the vertex $v_{i+y}$ is unmatched. Therefore, we can add the edge $v_{i+y} v_{i-\lfloor n / 4\rfloor}$ in each $F_{i}, 0 \leq i \leq n / 2-1$ (represented as a dashed line in Figure 1).

Each vertex and each edge of $K_{n}$ is contained in exactly one of these color classes. The induced subgraphs of this partition consist of $K_{1}, K_{2}$, and $P_{3}$. Therefore, this is an $(\mathcal{O}, \mathcal{Q})$-total coloring of the complete graph $K_{n}$ with $n$ colors.

Theorem 5. $\chi_{\mathcal{P}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right)= \begin{cases}n & \text { if } \mathcal{P} \neq \mathcal{O} \text { or } n \text { odd }, \\ n+1 & \text { if } \mathcal{P}=\mathcal{O} \text { and } n \text { even. }\end{cases}$
Proof. The case that $n$ is odd is treated in the above example, the case $\mathcal{P}=\mathcal{O}$ and $n$ even in Theorem 4.

If $n$ is even and $\mathcal{P} \neq \mathcal{O}$ then $K_{2} \in \mathcal{P}$. First note that $\chi_{\mathcal{P}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right) \geq$ $\chi_{\mathcal{I}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right)=n$ by Lemma 1 and Theorem 3.

In the following we provide a $\left(\mathcal{P}, \mathcal{O}_{1}\right)$-total coloring of $K_{n}$ with $n$ colors which implies $\chi_{\mathcal{P}, \mathcal{O}_{1}}^{\prime \prime}\left(K_{n}\right)=n$.

For $n=2$ and $n=4$ see Figure 2.


Figure 2. $\left(\mathcal{P}, \mathcal{O}_{1}\right)$-total colorings of $K_{2}$ and $K_{4}$.

If $n \geq 6$ then there exists an edge coloring of $K_{n}$ with $n-1$ colors such that there are $n / 2$ independent edges with pairwise distinct colors. This can be seen as follows. Consider a drawing of $K_{n}-v \cong K_{n-1}$ with vertex set $\left\{v_{0}, \ldots, v_{n-2}\right\}$ as a regular $(n-1)$-gon. Color parallel edges of $K_{n-1}$ with one color and the edges $v v_{i}, 0 \leq i \leq n-2$, with the missing color at $v_{i}$. If $n \equiv 2(\bmod 4)$ then the edges $v_{0} v_{1}, v_{2} v_{3}, \ldots, v_{n-4} v_{n-3}, v_{n-2} v$ are independent with mutually distinct colors. If $n \equiv 0(\bmod 4)$ then the edges $v_{0} v_{1}, v_{2} v_{4}, v_{3} v_{6}, v_{5} v$ and if $n \geq 12$ also $v_{7} v_{8}, v_{9} v_{10}, \ldots, v_{n-3} v_{n-2}$ are independent with pairwise distinct colors.

Assign the color of each of these edges to its end-vertices and then replace the colors of all these edges by the $n$th color (see Figure 3 for an example).


Figure 3. Edge coloring and $\left(\mathcal{P}, \mathcal{O}_{1}\right)$-total colorings of $K_{6}$.
The corresponding results concerning $(\mathcal{O}, \mathcal{Q})$ - and $\left(\mathcal{P}, \mathcal{O}_{1}\right)$-total colorings of paths and cycles are special cases of the following theorems.

Theorem 6. $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}\left(P_{n}\right)= \begin{cases}1 & \text { if } n=1, \\ 2 & \text { if } P_{n} \in(\mathcal{P} \cap \mathcal{Q}) \backslash \mathcal{O}, \\ 3 & \text { otherwise }\end{cases}$

Proof. The result follows from Theorem 1 and from $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}\left(P_{n}\right) \leq \chi^{\prime \prime}\left(P_{n}\right) \leq$ 3 (see Lemma 1).

Theorem 7. $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}\left(C_{n}\right)=\left\{\begin{array}{l}2 \text { if } C_{n} \in \mathcal{P} \cap \mathcal{Q}, \\ 4 \text { if }\left(\mathcal{P}=\mathcal{O}, \mathcal{Q}=\mathcal{O}_{1}, n \not \equiv 0(\bmod 3)\right) \text { or }(n=5, \\ \left.\mathcal{P}=\mathcal{O}, P_{4} \notin \mathcal{Q}\right) \text { or }\left(n=5, \mathcal{P}=\mathcal{Q}=\mathcal{O}_{1}\right), \\ 3 \text { otherwise. }\end{array}\right.$
Proof. If $C_{n} \in \mathcal{P} \cap \mathcal{Q}$ then $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}\left(C_{n}\right)=2$ by Theorem 1 and if $C_{n} \notin$ $\mathcal{P} \cap \mathcal{Q}$ then $3 \leq \chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}\left(C_{n}\right) \leq 4$ by Theorem 1, Lemma 1, and the fact that $\chi^{\prime \prime}\left(C_{n}\right) \leq 4$.

If $n \equiv 0(\bmod 3)$ then $\chi^{\prime \prime}\left(C_{n}\right)=3$ and therefore $\chi_{\mathcal{P}, \mathcal{Q}}^{\prime \prime}\left(C_{n}\right)=3$.
Let $n \not \equiv 0(\bmod 3)$. If $\mathcal{P}=\mathcal{O}$ and $\mathcal{Q}=\mathcal{O}_{1}$ then $\chi_{\mathcal{O}, \mathcal{O}_{1}}^{\prime \prime}\left(C_{n}\right)=4$. If $\mathcal{P}=\mathcal{O}$ and $\mathcal{Q} \supset \mathcal{O}_{1}$ then color the successive vertices $v_{0}, v_{1}, \ldots, v_{n-1}$ of $C_{n}$ by colors $1,2,3,1,2,3, \ldots, 1,2,3,2$ if $n \equiv 1(\bmod 3)$ and by colors $1,2,3,1,2,3, \ldots, 1,2,3,2,1,2,3,2$ if $n \equiv 2(\bmod 3), n \geq 8$, and the edges with the at their end-vertices missing color of $\{1,2,3\}$. This is an $(\mathcal{O}, \mathcal{Q})$ total coloring of $C_{n}$ since $P_{3} \in \mathcal{Q}$. If $n=5$ then color the vertices with colors $1,2,1,2,3$ (unique up to permutation) and the edges again with the at their end-vertices missing color of the set $\{1,2,3\}$. This is an $(\mathcal{O}, \mathcal{Q})$-total coloring of $C_{5}$ if $P_{4} \in \mathcal{Q}$. If $P_{4} \notin \mathcal{Q}$ then $\chi_{\mathcal{O}, \mathcal{Q}}^{\prime \prime}\left(C_{5}\right)=4$.

By switching the colors of vertices and edges one obtains $\chi_{\mathcal{P}, \mathcal{O}_{1}}^{\prime \prime}\left(C_{n}\right)=3$ if $\mathcal{P} \supset \mathcal{O}$ with the exception of $\chi_{\mathcal{P}, \mathcal{O}_{1}}^{\prime \prime}\left(C_{5}\right)=4$ if $P_{3} \notin \mathcal{P}$.

If $\mathcal{P} \supset \mathcal{O}$ and $\mathcal{Q} \supset \mathcal{O}_{1}$ then color the elements $v_{0}, v_{0} v_{1}, v_{1}, v_{1} v_{2}, \ldots$ successively with colors $1,2,3,1,2,3, \ldots$ if $n \not \equiv 2(\bmod 3)$ and with colors $1,2,3,1,2,3, \ldots, 1,2,3,2,1,3,2$ if $n \equiv 2(\bmod 3)$ to obtain a $(\mathcal{P}, \mathcal{Q})$-total coloring of $C_{n}$ with 3 colors.

## 3. Total Acyclic Colorings $\left(\mathcal{P}=\mathcal{Q}=\mathcal{D}_{1}\right)$

Total acyclic colorings are ( $\mathcal{D}_{1}, \mathcal{D}_{1}$ )-total colorings where $\mathcal{D}_{1}$ contains the 1 -degenerate graphs which are the acyclic graphs. The $\mathcal{D}_{1}$-vertex chromatic number is the vertex arboricity $a(G)=\chi_{\mathcal{D}_{1}}(G)$ and the $\mathcal{D}_{1}$-edge chromatic number is the (edge) arboricity $a^{\prime}(G)=\chi_{\mathcal{D}_{1}}^{\prime}(G)$.

We mention some known results on the vertex and edge arboricity: $\chi_{\mathcal{D}_{1}}(G)=\chi_{\mathcal{D}_{1}}^{\prime}(G)=1$ if and only if $G$ is acyclic, $\chi_{\mathcal{D}_{1}}\left(C_{n}\right)=\chi_{\mathcal{D}_{1}}^{\prime}\left(C_{n}\right)=$ $2, \chi_{\mathcal{D}_{1}}\left(K_{n}\right)=\chi_{\mathcal{D}_{1}}^{\prime}\left(K_{n}\right)=\lceil n / 2\rceil, \chi_{\mathcal{D}_{1}}\left(K_{m, n}\right)=1$ if $m=1$ or $n=1$,
$\chi_{\mathcal{D}_{1}}\left(K_{m, n}\right)=2$ if $m \neq 1 \neq n, \chi_{\mathcal{D}_{1}}^{\prime}\left(K_{m, n}\right)=\lceil m n /(m+n-1)\rceil$ (see [13], e.g.).

We denote induced subgraphs $H$ of $G$ by $H \leq G$. Proved upper bounds are $\chi_{\mathcal{D}_{1}}(G) \leq \max _{H \leq G}\{\lfloor\delta(H) / 2\rfloor+1\}[7]$ which implies $\chi_{\mathcal{D}_{1}}(G) \leq$ $\lfloor\Delta(G) / 2\rfloor+1$ and $\chi_{\mathcal{D}_{1}}^{\prime}(G) \leq\lfloor\Delta(G) / 2\rfloor+1$. The latter is an implication of

$$
\begin{equation*}
\chi_{\mathcal{D}_{1}}^{\prime}(G)=\max _{\substack{H \leq G \\|V(H)|>1}}\{\lceil|E(H)| /(|V(H)|-1)\rceil\} \tag{6}
\end{equation*}
$$

which is due to Nash-Williams [13]. Moreover, $\chi_{\mathcal{D}_{1}}(G) \leq \chi_{\mathcal{D}_{1}}^{\prime}(G)$ (see [5]).
Observe that we have an analogous situation for ordinary colorings: $\chi(G) \leq \Delta(G)+1, \chi^{\prime}(G) \leq \Delta(G)+1$ (Vizing [14]) and $\chi(G) \leq \chi^{\prime}(G)$ (Brooks [4]).

Theorem 1 implies that $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}(G)=1$ if and only if $G \in \mathcal{O}$ and $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}(G)=2$ if and only if $G \in \mathcal{D}_{1} \backslash \mathcal{O}$ (acyclic graphs with edges). For cycles $C_{n}$ we have $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}\left(C_{n}\right)=3$ by Theorem 7 since $C_{n} \notin \mathcal{D}_{1}$.

Theorem 8. $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}\left(K_{1}\right)=1$, $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}\left(K_{2}\right)=2$, $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}\left(K_{n}\right)=\lfloor n / 2\rfloor+2$ for $n \geq 3$.

Proof. The results for $n=1$ and $n=2$ follow from Theorem 1 .
Let $n \geq 3$. Each color class of a $\left(\mathcal{D}_{1}, \mathcal{D}_{1}\right)$-total coloring of $K_{n}$ with $c$ colors contains 0,1 , or 2 vertices and at most $n-1$, $n-2$, or $n-3$ edges, respectively. If $x_{i}$ denotes the number of color classes with $i$ vertices we obtain $x_{0}+x_{1}+x_{2}=c$ (number of color classes), $x_{1}+2 x_{2}=n$ (number of vertices) and $(n-1) x_{0}+(n-2) x_{1}+(n-3) x_{2} \geq\binom{ n}{2}$ (number of edges). It follows $(n-1)(c-1)-1 \geq\binom{ n}{2}$ and therefore $c \geq\lceil n / 2+1+1 /(n-1)\rceil$. If $n$ is even then $c \geq n / 2+2$; if $n \geq 3$ is odd then $1 /(n-1) \leq 1 / 2$ and therefore $c \geq\lceil n / 2\rceil+1=\lfloor n / 2\rfloor+2$ which implies $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}\left(K_{n}\right) \geq\lfloor n / 2\rfloor+2$ if $n \geq 3$.

On the other hand, it holds $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}\left(K_{n}\right) \leq\lfloor n / 2\rfloor+2$ which can be seen by the following partition of the elements of $K_{n}$ in $\lfloor n / 2\rfloor+2$ classes.

If $n$ is even then class $F_{i}, 0 \leq i \leq \frac{n}{2}-1$, contains vertices $v_{i}$ and $v_{i+n / 2}$ and the $n-3$ edges of the path $\left(v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \ldots, v_{i+n / 2-1}, v_{i-n / 2+1}\right)$ where all indices are reduced modulo $n$. The remaining edges $v_{0} v_{1}, v_{1} v_{2}$, $\ldots, v_{n-1} v_{0}$ induce a cycle which can be colored with two additional colors (see Figure 4, upper part).

If $n$ is odd then class $F_{i}, 0 \leq i \leq \frac{n-3}{2}$, contains vertices $v_{i}$ and $v_{i-(n-1) / 2}$ and the $n-3$ edges of the path $\left(v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \ldots, v_{i+(n-1) / 2}\right)$.

Moreover, the remaining elements of $K_{n}$ can be colored using two additional colors:


Figure 4. Color classes of $K_{n}$ if $n$ is even (above) or odd (below).
vertex $v_{(n-1) / 2}$ and edges $v_{(n-1) / 2-j} v_{(n-1) / 2+j}, j=1, \ldots,(n-1) / 2$ with one new color and the edges of the path ( $v_{0}, v_{1}, \ldots, v_{n-1}$ ) with the second new color (see Figure 4, lower part).

The results for acyclic graphs, cycles and complete graphs suggest the following general conjecture.

Conjecture 1. $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}(G) \leq\left\lfloor\frac{\Delta(G)+1}{2}\right\rfloor+2$.
This conjecture is an analogy to the total coloring conjecture which says that $\chi^{\prime \prime}(G) \leq \Delta(G)+2$ for all graphs $G$.

Since $m \leq 3 n-6$ for planar graphs $G$ of order $n \geq 3$ and size $m$ we obtain $\chi_{\mathcal{D}_{1}}(G) \leq \chi_{\mathcal{D}_{1}}^{\prime}(G) \leq 3$ by (6) which implies $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}(G) \leq 6$. We can improve this to $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}(G) \leq 5$ but we do not know whether $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}(G) \leq 4$ is true for all planar graphs. For outerplanar graphs $G$ it holds $\chi_{\mathcal{D}_{1}, \mathcal{D}_{1}}^{\prime \prime}(G) \leq 3$.

## 4. $(\mathcal{P}, \mathcal{Q})$-total Colorings of Infinite Graphs - a Compactness Theorem

All our considerations hold for arbitrary simple infinite graphs. Let us denote by $\mathcal{I}^{*}$ the class of all simple infinite graphs. A graph property $\mathcal{P}$ is any isomorphism-closed nonempty subclass of $\mathcal{I}^{*}$.

In 1951, de Bruijn and Erdős [8] proved that an infinite graph $G$ is $k$ colorable if and only if every finite subgraph of $G$ is $k$-colorable. Analogous compactness theorems for generalized colorings were proved in [6]. They all have been based on the "Set Partition Compactness Theorem" (see [6]), where the key concept is that of a property being of finite character. A graph property $\mathcal{P}$ is of finite character if a graph in $\mathcal{I}^{*}$ has property $\mathcal{P}$ if and only if each of its finite induced subgraphs has property $\mathcal{P}$. It is easy to see that if $\mathcal{P}$ is of finite character and a graph has property $\mathcal{P}$ then so does every induced subgraph. A property $\mathcal{P}$ is said to be induced-hereditary if $G \in \mathcal{P}$ and $H \leq G$ implies $H \in \mathcal{P}$, that is, $\mathcal{P}$ is closed under taking induced subgraphs. Thus properties of finite character are induced-hereditary. However, not all induced-hereditary properties are of finite character. For example, the graph property of not containing a vertex of infinite degree is induced-hereditary but not of finite character. Let us also remark that every property which is hereditary with respect to every subgraph (we say simply hereditary) is induced-hereditary as well. The properties of being edgeless, of maximum degree at most $k, K_{n}$-free, acyclic, complete, perfect, etc. are properties of finite character. Each additive hereditary graph property $\mathcal{P}$ of finite character can be characterized (see, e.g., [12]) by the set of connected minimal forbidden graphs of $\mathcal{P}$, which is defined as follows:

$$
\begin{aligned}
\mathbf{F}(\mathcal{P})=\{ & G: G \text { connected, } G \notin \mathcal{P} \text { but each proper subgraph } H \text { of } G \\
& \text { belongs to } \mathcal{P}\} .
\end{aligned}
$$

In the paper [6] also a compactness result for generalized colorings of hypergraphs has been presented. A simple hypergraph $H=(X, E)$ is a hypergraph on a vertex set $X$ where all hyperedges $e \in E$ are different finite subsets of the vertex set $X$. Let $\mathcal{P}_{1}, \ldots, \mathcal{P}_{m}$ be properties of simple hypergraphs (i.e. classes of simple hypergraphs closed under isomorphism). A hypergraph $H=(X, E)$ is $\left(\mathcal{P}_{1}, \ldots, \mathcal{P}_{m}\right)$-colorable if the vertex set $X$ of $H$ can be partitioned into sets $X_{1}, \ldots, X_{m}$ such that the induced subhypergraphs $H\left[X_{i}\right]=\left(X_{i}, E\left(X_{i}\right)\right)$ of $H$, where $E\left(X_{i}\right)$ consists of all hyperedges of $H$ all of whose vertices belong to $X_{i}$, has property $\mathcal{P}_{i}, i=1,2, \ldots, m$. A property
$\mathcal{P}$ of hypergraphs is of finite vertex character if a hypergraph has property $\mathcal{P}$ if and only if every finite induced subhypergraph has property $\mathcal{P}$. Then, using the Set Partition Compactness Theorem, it holds:

Theorem 9. Let $H$ be a simple hypergraph and suppose $\mathcal{P}_{1}, \ldots, \mathcal{P}_{m}$ are properties of hypergraphs of finite vertex character. Then $H$ is $\left(\mathcal{P}_{1}, \ldots, \mathcal{P}_{m}\right)$ colorable if every finite induced subhypergraph of $H$ is $\left(\mathcal{P}_{1}, \ldots, \mathcal{P}_{m}\right)$-colorable.

In particular, if $\mathcal{P}_{1}=\mathcal{P}_{2}=\cdots=\mathcal{P}_{m}=\mathcal{O}_{H}$, where $\mathcal{O}_{H}$ denotes the property of a hypergraph "to be hyperedgeless", i.e., $E=\emptyset$, we have a compactness theorem for the regular hypergraph coloring, since $\mathcal{O}_{H}$ is of finite character. Now we will use this result to prove the compactness theorem for $(\mathcal{P}, \mathcal{Q})$ total colorings:

Theorem 10. Let $G \in \mathcal{I}^{*}$ be a simple infinite graph and suppose $\mathcal{P}$ and $\mathcal{Q} \neq \mathcal{O}$ are additive properties of finite character. Then $G$ is $(\mathcal{P}, \mathcal{Q})$-totally $k$-colorable if and only if every finite induced subgraph of $G$ is $(\mathcal{P}, \mathcal{Q})$-totally $k$-colorable.

Proof. Let $G=(V(G), E(G))$ be a simple infinite graph and let $\mathcal{P}, \mathcal{Q}, \mathcal{Q} \neq$ $\mathcal{O}$ be additive hereditary properties of finite character. Let $\mathbf{F}(\mathcal{P})$ and $\mathbf{F}(\mathcal{Q})$ be the sets of minimal forbidden graphs of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Let us define a hypergraph $H(G)=\left(V^{*}, E^{*}\right)$ so that $V^{*}=V(G) \cup E(G)$ and a set $e \subset V^{*}$ is an hyperedge of $H(G)$ if and only if
(1) $e=\{v, h\}, v \in V(G), h \in E(G), v \in h$, or
(2) $G[e] \in \mathbf{F}(\mathcal{P}), e \subset V(G)$, or
(3) $G[e] \in \mathbf{F}(\mathcal{Q}), e \subset E(G)$.

By the definition of the hypergraph $H(G)$ of $G$, a graph $G$ is $(\mathcal{P}, \mathcal{Q})$-totally $k$-colorable if the hypergraph $H(G)$ is regularly $k$-colorable. By Theorem 9 , $H(G)$ is regularly $k$-colorable if every finite induced subhypergraph of $H(G)$ is regularly $k$-colorable. However, if every finite induced subgraph of $G$ is ( $\mathcal{P}, \mathcal{Q}$ )-totally $k$-colorable, then obviously every finite induced subhypergraph of $H(G)$ is regularly $k$-colorable.

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