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THE FORCING STEINER NUMBER OF A GRAPH

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Abstract

For a connected graph $G = (V, E)$, a set $W \subseteq V$ is called a Steiner set of G if every vertex of G is contained in a Steiner W-tree of G. The Steiner number $s(G)$ of G is the minimum cardinality of its Steiner sets and any Steiner set of cardinality $s(G)$ is a minimum Steiner set of G. For a minimum Steiner set W of G, a subset $T \subseteq W$ is called a forcing subset for W if W is the unique minimum Steiner set containing T. A forcing subset for W of minimum cardinality is a minimum forcing subset of W. The forcing Steiner number of W, denoted by $f_s(W)$, is the cardinality of a minimum forcing subset of W. The forcing Steiner number of G, denoted by $f_s(G)$, is $f_s(G) = \min\{f_s(W)\}$, where the minimum is taken over all minimum Steiner sets W in G. Some general properties satisfied by this concept are studied. The forcing Steiner numbers of certain classes of graphs are determined. It is shown for every pair a, b of integers with $0 \leq a < b$, $b \geq 2$, there exists a connected graph G such that $f_s(G) = a$ and $s(G) = b$.

Keywords: geodetic number, Steiner number, forcing geodetic number, forcing Steiner number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an *$u - v$ geodesic*. It is known that the distance is a metric on the vertex set of G. For basic graph theoretic terminology, we refer to [1]. A *geodetic set* of G is a set S of vertices such that every vertex of G is contained in a geodesic joining some pair of vertices of S. The geodetic number $g(G)$ of G is the minimum cardinality of its geodetic sets and any geodetic set of cardinality $g(G)$ is a minimum geodetic set or simply a *g-set* of G. A vertex v is said to be a *geodetic vertex* if v belongs to every g-set of G. The geodetic number of a graph was introduced in [6] and further studied in [4, 7]. It was shown in [7] that determining the geodetic number of a graph is an NP-hard problem. A subset $T \subseteq S$ is called a *forcing subset for S* if S is the unique minimum geodetic set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing geodetic number of S*, denoted by $f(S)$, is the cardinality of a minimum forcing subset of S. The *forcing geodetic number of G*, denoted by $f(G)$, is $f(G) = \min\{f(S)\}$, where the minimum is taken over all minimum geodetic sets S in G. The forcing geodetic number of a graph was introduced and studied in [2]. The forcing dimension of a graph was discussed in [3]. Santhakumaran *et al.* studied the connected geodetic number of a graph in [9] and also the upper connected geodetic number and the forcing connected geodetic number of a graph in [10].

For a nonempty set W of vertices in a connected graph G, the *Steiner distance* $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a *Steiner tree* with respect to W or a *Steiner W-tree*. It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. The set of all vertices of G that lie on some Steiner W-tree is denoted by $S(W)$. If $S(W) = V$, then W is called a *Steiner set* for G. A Steiner set of minimum cardinality is a *minimum Steiner set* or simply a *s-set* of G and this cardinality is the *Steiner number* $s(G)$ of G. We observe that if W is a proper Steiner set of G, then $\langle W \rangle$, the subgraph induced by W is disconnected. The Steiner number of a graph was introduced and studied in [5]. It was proved in [5] that every Steiner set of G is a geodetic set of G. However, this was proved to be wrong in [7].

For the graph G given in Figure 1.1(a), $W = \{v_1, v_5, v_9\}$ is the unique s-set of G so that $s(G) = 3$. Also $S_1 = \{v_1, v_5, v_7, v_9\}$ and $S_2 = \{v_1, v_5, v_6, v_9\}$ are the only two g-sets of G so that $g(G) = 4$ and $f(G) = 1$. For the graph G given in Figure 1.1(b), $W = \{v_1, v_2, v_5, v_6\}$ is the unique s-set of G so that $s(G) = 4$. Also $S_1 = \{v_1, v_5, v_6\}$ and $S_2 = \{v_2, v_5, v_6\}$ are the only two g-sets of G so that $g(G) = 3$ and $f(G) = 1$. For the graph G given in Figure 1.1(c), $W = \{v_1, v_5\}$ is the unique g-set as well as the unique s-set of G so that $g(G) = s(G) = 2$ and $f(G) = 0$.

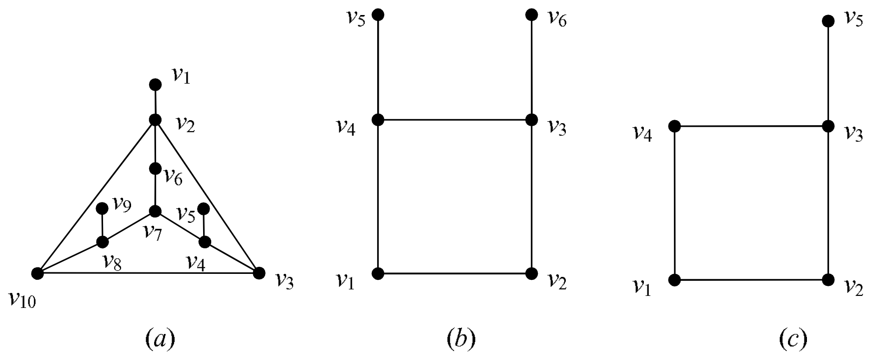

Figure 1.1

A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. The following theorems are used in the sequel.

Theorem 1.1 [5]. *Each extreme vertex of a connected graph G belongs to every Steiner set of G.*

Theorem 1.2 [5]. *For a connected graph G, $s(G) = p$ if and only if $G = K_p$.*

Throughout the following G denotes a connected graph with at least two vertices.

2. THE FORCING STEINER NUMBER OF A GRAPH

Even though every connected graph contains a minimum Steiner set, some connected graphs may contain several minimum Steiner sets. For each minimum Steiner set W in a connected graph G, there is always some subset T

of W that uniquely determines W as the minimum Steiner set containing T. Such "forcing subsets" will be considered in this section.

Definition 2.1. Let G be a connected graph and W a minimum Steiner set of G. A subset $T \subseteq W$ is called a *forcing subset for* W if W is the unique minimum Steiner set containing T. A forcing subset for W of minimum cardinality is a *minimum forcing subset of* W. The *forcing Steiner number of* W, denoted by $f_s(W)$, is the cardinality of a minimum forcing subset of W. The *forcing Steiner number of* G, denoted by $f_s(G)$, is $f_s(G) = \min\{f_s(W)\}$, where the minimum is taken over all minimum Steiner sets W in G.

Example 2.2. For the graph G given in Figure 1.1(a), $W = \{v_1, v_5, v_9\}$ is the unique minimum Steiner set of G so that $f_s(G) = 0$ and for the graph G given in Figure 2.1, $W_1 = \{v_1, v_5, v_7\}$ and $W_2 = \{v_1, v_5, v_6\}$ are the only two s-sets of G. It is clear that $f_s(W_1) = f_s(W_2) = 1$ so that $f_s(G) = 1$.

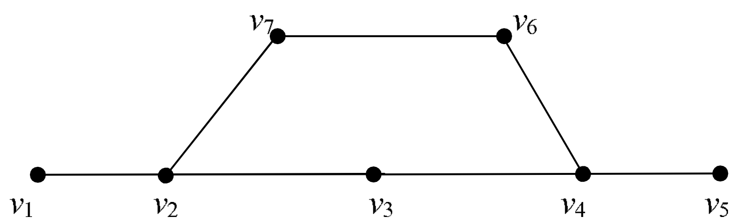

Figure 2.1. A graph G with $s(G) = 3$ and $f_s(G) = 1$.

The following theorem was proved in [2].

Theorem A. For a connected graph G, $0 \leq f(G) \leq g(G)$.

The next theorem is similar to this.

Theorem 2.3. *For a connected graph G, $0 \leq f_s(G) \leq s(G)$.*

The following observation is an easy consequence of the definition of forcing Steiner number of a graph.

Observation 2.4. *Let G be a connected graph. Then*

(a) *$f_s(G) = 0$ if and only if G has a unique minimum Steiner set.*

(b) $f_s(G) = 1$ *if and only if* G *has at least two minimum Steiner sets, one of which is a unique minimum Steiner set containing one of its elements, and*

(c) $f_s(G) = s(G)$ *if and only if no minimum Steiner set of* G *is the unique minimum Steiner set containing any of its proper subsets.*

Definition 2.5. A vertex v of a graph G is said to be a *Steiner vertex* if v belongs to every minimum Steiner set of G.

Example 2.6. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_3, v_4\}$ and $S_2 = \{v_1, v_3, v_5\}$ are the only two s-sets of G so that v_1 and v_3 are Steiner vertices of G.

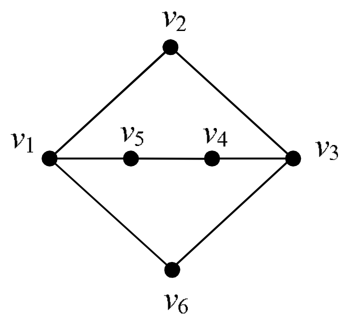

Figure 2.2. A graph G with Steiner vertices v_1 and v_3.

Theorem 2.7. *Let* G *be a connected graph and let* $\Im$ *be the set of relative complements of the minimum forcing subsets in their respective minimum Steiner sets in* G. *Then* $\bigcap_{F\in\Im} F$ *is the set of Steiner vertices of* G.

Proof. Let W denote the set of Steiner vertices of G. We show that $W = \bigcap_{F\in\Im} F$. Let $v \in W$. Then v belongs to every minimum Steiner set of G. Let $T \subseteq S$ be any minimum forcing subset for any minimum Steiner set S of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S is the unique minimum Steiner set containing T' so that T' is a forcing subset for S with $|T'| < |T|$, which is a contradiction to T a minimum forcing subset for S. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S. Hence $v \in \bigcap_{F\in\Im} F$ so that $W \subseteq \bigcap_{F\in\Im} F$.

Conversely, let $v \in \bigcap_{F\in\Im} F$. Then v belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is a minimum forcing subset for S. Since F is the relative complement of T in S, we have

$F \subseteq S$ and thus $v \in S$ for every S, which implies that v is a Steiner vertex of G. Thus $v \in W$ and so $\bigcap_{F \in \Im} F \subseteq W$. Hence $W = \bigcap_{F \in \Im} F$. ■

Corollary 2.8. *Let G be a connected graph and S a minimum Steiner set of G. Then no Steiner vertex of G belongs to any minimum forcing set of S.*

The following observation is clear from the definitions of forcing Steiner number and the Steiner vertex of a graph.

Observation 2.9. *Let G be a connected graph and W be the set of all Steiner vertices of G. Then $f_s(G) \leq s(G) - |W|$.*

It is clear from Theorem 1.1 and Observation 2.9 that for a connected graph with k extreme vertices, $f_s(G) \leq s(G) - k$. The bound in Observation 2.9 is sharp. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_3, v_4\}$ and $S_2 = \{v_1, v_3, v_5\}$ are the only two s-sets so that $s(G) = 3$ and $f_s(G) = 1$. Also, $W = \{v_1, v_3\}$ is the set of all Steiner vertices of G and so $f_s(G) = s(G) - |W|$. The inequality in Observation 2.9 can also be strict. For the graph G given in Figure 2.3, $S_1 = \{v_1, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_6\}$ and $S_3 = \{v_1, v_3, v_5\}$ are the only three s-sets of G so that $s(G) = 3$ and $f_s(G) = 1$. Since v_1 is the only Steiner vertex of G, we have $f_s(G) < s(G) - |W|$.

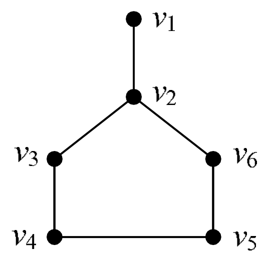

Figure 2.3. G

In the following we determine the forcing Steiner numbers of certain standard graphs. It is proved in [2] that the forcing number of a cycle C_p is 1 if p is even; and 2 if p is odd. The proof for the forcing Steiner number of a cycle C_p follows in line with the proof of the corresponding theorem in [2]. However, we give an outline of the proof to highlight Steiner concepts. We observe that for an even cycle C_p, an s-set is a g-set and consists of precisely a pair of antipodal vertices of C_p and so it follows from Observation 2.4(b) that $f_s(G_p) = 1$. If p is odd with $p = 2n + 1$, let the cycle be

$C_p : v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n+1}, v_1$. If $S = \{u, v\}$ is any set of two vertices of C_p, then no vertex of the $u-v$ longest path lies on the Steiner S-tree in C_p and so no two element subset of C_p is a Steiner set of C_p. Now, it is clear that the sets $S_1 = \{v_1, v_{n+1}, v_{n+2}\}, S_2 = \{v_2, v_{n+2}, v_{n+3}\}, \ldots, S_{n+2} = \{v_{n+2}, v_1, v_2\}, \ldots$ and $S_{2n+1} = \{v_{2n+1}, v_n, v_{n+1}\}$ are s-sets of C_p. (Note that there are more s-sets of C_p, for example, $S = \{v_1, v_{n+1}, v_{n+3}\}$ is a s-set different from these). It is clear from the s-sets S_i $(1 \leq i \leq 2n+1)$ that each $\{v_i\}(1 \leq i \leq 2n+1)$ is a subset of more than one s-set S_i. Hence it follows from Observation 2.4 (a) and (b) that $f_s(C_p) \geq 2$. Now, since v_{n+1} and v_{n+2} are antipodal to v_1, it is clear that S_1 is the unique s-set containing $\{v_{n+1}, v_{n+2}\}$ and so $f_s(C_p) = 2$. Thus we have the following result.

Theorem 2.10. *For a cycle C_p $(p \geq 4)$, $f_s(C_p) = \begin{cases} 1 & \text{if } p \text{ is even,} \\ 2 & \text{if } p \text{ is odd.} \end{cases}$*

Theorem 2.11. *If G is a complete graph or a tree, then $f_s(G) = 0$.*

Proof. Since the set of all vertices of a complete graph is the unique minimum Steiner set; and the set of all end vertices of a tree is the unique minimum Steiner set, the result follows from Theorem 1.1 and Observation 2.4(a). ∎

Theorem 2.12. *For the complete bipartite graph $G = K_{m,n}$ $(m, n \geq 2)$,*
$$f_s(G) = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases}$$

Proof. First assume that $m < n$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G. Let $S = U$. We prove that S is a s-set of G. Any Steiner S-tree T is a star centered at w_j $(1 \leq j \leq n)$ with u_i $(1 \leq i \leq m)$ as end vertices of T. Hence every vertex of G lies on a Steiner S-tree of G so that S is a Steiner set of G. Let X be any set of vertices such that $|X| < |S|$. Then there exists a vertex $u_i \in U$ such that $u_i \notin X$. Since any Steiner X-tree is a star centered at w_j $(1 \leq j \leq n)$, whose end-vertices are elements of X, the vertex u_i does not lie on any Steiner X-tree of G. Thus X is not a Steiner set of G. Hence S is a s-set so that $s(G) = |S| = m$. We show that S is the unique s-set of G. Now, let S_1 be a set of vertices such that $|S_1| = m$. If S_1 is a subset of W, then since $m < n$, there exists a vertex $w_j \in W$ such that $w_j \notin S_1$. Then the vertex w_j does not lie on any Steiner S_1-tree of G, as earlier. If $S_1 \subsetneq U \cup W$ such that S_1 contains at least one vertex from each of U and W, then since $S_1 \neq U$, there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin S_1$ and $w_j \notin S_1$. Then, as earlier, the vertices u_i, w_j do not lie on any Steiner S_1-tree of G so that S_1 is not a Steiner set of G. Hence U is the unique s-set of G and it follows from Observation 2.4(a) that $f_s(G) = 0$. Now, let $m = n$. Then, as in the proof of the first part of this theorem, both U and W are s-sets of G. Let S' be any set of vertices such that $|S'| = m$ and $S' \neq U, W$. Then there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin S'$ and $w_j \notin S'$. Since any Steiner S'-tree is a spanning tree containing only the vertices of S', it follows that S' is not a Steiner set of G and hence it follows that U and W are the only two s-sets of G. Since U is the unique minimum Steiner set containing $\{u_i\}$, it follows from Observation 2.4(b) that $f_s(G) = 1$. ■

Theorem 2.13. *For the wheel $W_p = K_1 + C_{p-1} (p \geq 5)$, $s(W_p) = p - 3$ and $f_s(W_p) = p - 4$.*

Proof. Let v be the vertex of K_1 and let $v_1, v_2, \ldots, v_{p-1}, v_1$ be the cycle C_{p-1}. First, we observe that v does not belong to any proper Steiner set of W_p. For $p = 5$, $W_1 = \{v_1, v_3\}$ and $W_2 = \{v_2, v_4\}$ are the only two s-sets of W_p so that $s(W_p) = 2 = p - 3$ and $f_s(W_p) = 1 = p - 4$. Let $p \geq 6$. Let W be any subset of vertices of C_{p-1} of cardinality $p - 3$ obtained by deleting two non-adjacent vertices of C_{p-1}. We may assume without loss of generality that $W = \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{p-1}\}$, where $1 \leq i < j \leq p - 1$ and $j \geq i + 2$. It is easily seen that W is a minimum Steiner set of G so that $s(W_p) = |W| = p - 3$. Since the subgraph induced by a proper Steiner set of G is disconnected, it follows that any s-set is of the form $W = \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_{p-1}\}$, where v_i and v_j are non-adjacent. Let T be a subset of W with $|T| \leq p - 5$. Since $p \geq 6$, there exist distinct vertices $x, y \in W$ such that $x, y \notin T$. If x and y are adjacent, then x is non-adjacent to at least one of v_i and v_j, say v_j. Then $W_1 = V(C_{p-1}) - \{x, v_j\}$ is a s-set such that $W_1 \neq W$ and W_1 properly contains T. If x and y are non-adjacent, then $W_2 = V(C_{p-1}) - \{x, y\}$ is a s-set such that $W_2 \neq W$ and W_2 properly contains T. Thus T is not a forcing subset for W. Now, we show that there exists a forcing subset of W of cardinality $p - 4$. For convenience, let $W = \{v_2, v_4, v_5, v_6, \ldots, v_{p-1}\}$. We show that $T_1 = \{v_4, v_5, v_6, \ldots, v_{p-1}\}$ is a forcing subset for W. If T_1 is not a forcing subset for W, then there exists a s-set $W' \neq W$ such that $T_1 \subseteq W'$. Since $W' \neq W$, $|W'| = p - 3$ and $|T_1| = p - 4$, W' must contain exactly one of v_1 or v_3. In any case, $\langle W' \rangle$ is connected and so W' is not a Steiner set of G, which is a contradiction. Hence it follows that $f_s(W_p) = p - 4$. ■

It is proved in [2] that if G is a connected graph with $g(G) = 2$, then $f(G) \leq 1$. It is not hard to prove that if a set $S = \{u, v\}$ is a s-set of G, then u and v are antipodal vertices of G. The next theorem follows immediately from this result and is similar to the one in [2].

Theorem 2.14. *If G is a connected graph with $s(G) = 2$, then $f_s(G) \leq 1$.*

Corollary 2.15. *Let G be a connected graph with $s(G) = 2$. If G contains an extreme vertex, then $f_s(G) = 0$.*

Proof. Let v be an extreme vertex of G. If $f_s(G) = 1$, then there exist distinct vertices u, w such that $\{u, v\}$ and $\{w, v\}$ are s-sets. Then it follows that w is an internal vertex of a $u - v$ geodesic and u is an internal vertex of a $w - v$ geodesic. Hence $d(u, v) > d(v, w)$ and $d(v, w) > d(u, v)$, which is not possible. Since $f_s(G) \geq 0$, it follows from Theorem 2.14 that $f_s(G) = 0$. ■

In view of Theorem 2.3, the following theorem gives a realization of the forcing Steiner number and the Steiner number of a graph.

Theorem 2.16. *For every pair a, b of integers with $0 \leq a < b$, $b \geq 2$, there exists a connected graph G such that $f_s(G) = a$ and $s(G) = b$.*

Proof. If $a = 0$, let $G = K_b$. Then by Theorems 2.11 and 1.2, $f_s(G) = 0$ and $s(G) = b$. Now, assume that $a \geq 1$. For $b = a + 1$, let $G = K_1 + C_{a+3}$ $(a \geq 1)$. By Theorem 2.13, $s(G) = a + 1 = b$ and $f_s(G) = a$. For $b \neq a + 1$, let $F_i : s_i, t_i, u_i, v_i, r_i, s_i$ $(1 \leq i \leq a)$ be a copy of the cycle C_5. Let G be the graph obtained from F_i's by first identifying the vertices r_{i-1} of F_{i-1} and t_i of F_i $(2 \leq i \leq a)$ and then adding $b - a$ new vertices $z_1, z_2, \ldots, z_{b-a-1}, u$ and joining the $b - a$ edges $t_1 z_i$ $(1 \leq i \leq b - a - 1)$ and $r_a u$. The graph G is given in Figure 2.4. Let $Z = \{z_1, z_2, \ldots, z_{b-a-1}, u\}$ be the set of end-vertices of G. By Theorem 1.1, every s-set of G contains Z. Let $H_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$. First, we show that $s(G) = b$. Since the vertices u_i, v_i do not lie on the unique Steiner Z- tree of G, it is clear that Z is not a Steiner set of G. We observe that every s-set of G must contain exactly one vertex from each H_i $(1 \leq i \leq a)$ and so $s(G) \geq b - a + a = b$. On the other hand, since the set $W = Z \cup \{v_1, v_2, \ldots, v_a\}$ is a Steiner set of G, it follows that $s(G) \leq |W| = b$.

Thus, $s(G) = b$. Next, we show that $f_s(G) = a$. By Theorem 1.1, every Steiner set of G contains Z and so it follows from Observation 2.9 that $f_s(G) \leq s(G) - |Z| = a$. Now, since $s(G) = b$ and every s-set of G contains Z, it is easily seen that every s-set S is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then there is a vertex c_j $(1 \leq j \leq a)$ such that $c_j \notin T$. Let d_j be a vertex of H_j distinct from c_j. Then $S_2 = (S - \{c_j\}) \cup \{d_j\}$ is a s-set properly containing T. Thus S is not the unique s-set containing T and so T is not a forcing subset of S. This is true for all s-sets of G and so $f_s(G) = a$. ■

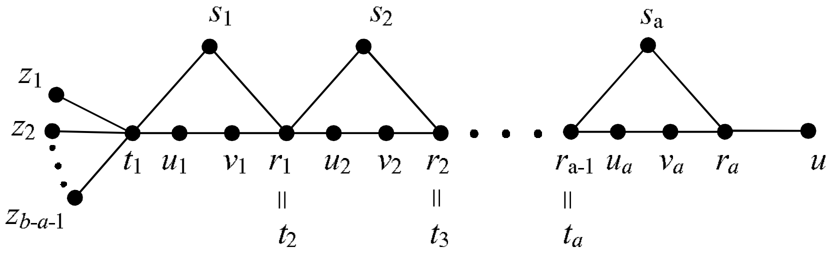

Figure 2.4. The graph G in Theorem 2.16 for $1 \leq a < b$.

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