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COLORING RECTANGULAR BLOCKS IN 3-SPACE

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Abstract

If rooms in an office building are allowed to be any rectangular solid, how many colors does it take to paint any configuration of rooms so that no two rooms sharing a wall or ceiling/floor get the same color? In this work, we provide a new construction which shows this number can be arbitrarily large.

Keywords: chromatic number, channel assignment problem, 3 dimensional rectangular blocks.

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1. Introduction

At the 53rd European Study Group with Industry, BAE Systems introduced a channel assignment problem. We give a brief restatement of the problem as follows. If rooms in an office building are allowed to be any rectangular solid, can we paint any configuration of rooms with a bounded number of colors so that no two rooms sharing a wall or ceiling/floor get the same color?

It turns out that the answer to this question is no, as proven by Reed and Allwright [4]. In their work, they showed that the chromatic number

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is unbounded by producing a construction with chromatic number at least k for any integer k. In the present paper we give an alternative proof of this fact. Our construction is closely related to an old construction of a k-chromatic graph by Descartes (for more information on Descartes' problem, see Chapter 7 of [2] or [3]). The advantage of our construction is that it is guaranteed to have chromatic number exactly k. Moreover, we only use intersections of the type bottom/top while in [4], blocks can also intersect on the sides.

We also would like to mention that a similar problem was considered by Burling in [1]. The main difference is that he allows blocks to share interior points. It turns out his construction does not work when we forbid blocks from sharing interior points.

2. Preliminaries

In order to state the problem formally, we introduce the following definitions.

Definition. A *block* is the cartesian product of three closed finite nontrivial intervals of the real line. Given a block B, we denote its component intervals by X_B, Y_B, and Z_B, so that $B = X_B \times Y_B \times Z_B$.

Definition. A set of blocks is said to be *valid* if no two blocks in the set share an interior point. They are, however, allowed to share boundary points.

Definition. Given a valid set of blocks $\mathcal{B}$, we define the *graph of* $\mathcal{B}$, denoted by $G(\mathcal{B})$, to be the graph whose vertex set is $\mathcal{B}$, and whose edge set is

$$\big\{\{A, B\} \subseteq \mathcal{B} \colon A \cap B \neq \emptyset\big\}.$$

A graph is called a *block graph* if it is the graph of some valid set of blocks. If G is a block graph and $\mathcal{B}$ is a valid set of blocks with $G \cong G(\mathcal{B})$, we say that $\mathcal{B}$ is a *realization of* G.

Using these definitions, the question loosely stated above becomes the following.

Question 1. Is there an absolute constant k such that every block graph has chromatic number at most k, or is it true that for every k there exists a valid set of blocks whose graph has chromatic number at least k?

We prove that the chromatic number of block graphs is not bounded. In order to present our construction, we will need a few more definitions.

Definition. Given a valid set of blocks $\mathcal{B}$, we define an equivalence relation on $\mathcal{B}$ in which blocks A and B are equivalent if and only if $\max Z_A = \max Z_B$. This relation on $\mathcal{B}$ is called the *ceiling relation*, and its equivalence classes are called the *ceiling classes* of $\mathcal{B}$. To each ceiling class $\mathcal{C} \subseteq \mathcal{B}$ we associate a number, called the *ceiling label* of $\mathcal{C}$, which is precisely $\max Z_A$ for any $A \in \mathcal{C}$. Given a ceiling label z of $\mathcal{B}$, we denote the corresponding ceiling class by $\mathcal{B}(z)$.

Definition. A valid set of blocks $\mathcal{B}$ with ceiling labels $z_1 < z_2 < \cdots < z_\ell$ is said to be *graded* if there exist $x_1 < x_2 < \cdots < x_\ell$, and $\varepsilon > 0$ such that for every $1 \le i \le \ell$, and for all $B \in \mathcal{B}(z_i)$, we have $[x_i - \varepsilon, x_i + \varepsilon] \subseteq X_B$. Moreover, $x_i + \varepsilon < \min X_A$ for all $A \in \mathcal{B}(z_j)$ whenever $j > i$. We say that numbers $x_1 < \cdots < x_\ell$ and ε satisfying the above properties are a *certificate of gradation for* $\mathcal{B}$. The number ε is called the *thickness*.

Definition. Given a valid set of blocks $\mathcal{B}$, we define the *stretching of* $\mathcal{B}$ *by* $\Delta x \ge 0$ *at* x_0 as the operation of replacing each block $B \in \mathcal{B}$ with the block $X \times Y_B \times Z_B$, where

$$X = \begin{cases} X_B & \text{if } \max X_B \le x_0, \\ [\min X_B, \max X_B + \Delta x] & \text{if } \min X_B \le x_0 < \max X_B, \\ [\min X_B + \Delta x, \max X_B + \Delta x] & \text{if } \min X_B > x_0. \end{cases}$$

Similarly, we define stretching along the y and z axes.

Definition. Given a valid set of blocks $\mathcal{B}$, we define the *translation of* $\mathcal{B}$ *by* Δx (with Δx possibly negative) as the operation of replacing each block $B \in \mathcal{B}$, with the block $X \times Y_B \times Z_B$, where $X = [\min X_B + \Delta x, \max X_B + \Delta x]$. Similarly, we define translation along the y and z axes.

We now mention a few properties of the stretching and translation operations, from now on simply called *transformations*. First, observe that the set of blocks resulting from a transformation of a valid set of blocks is also valid. Furthermore, if a valid set of blocks is graded, any transformation of it is also graded. Below, we briefly discuss how to obtain a certificate of gradation for the set of blocks resulting from a transformation, and we introduce a convenient notation.

Let $\mathcal{B}$ be a graded (and therefore also a valid) set of blocks. Let $z_1, \dots, z_\ell$ be its ceiling labels, and let $x_1 < \cdots < x_\ell$ and $\varepsilon > 0$ be a certificate of gradation for $\mathcal{B}$. Assume a stretching of $\mathcal{B}$ by $\Delta x \geq 0$ at x_0 is performed, and denote the resulting set of blocks by $\mathcal{B}'$. Since translations are relatively easier than stretchings, we leave that case to the reader. We can easily find a certificate of gradation $x'_1 < \cdots < x'_\ell$ and ε' for $\mathcal{B}'$ as follows. Define

$$x'_i = \begin{cases} x_i & \text{if } x_i \leq x_0, \\ x_i + \Delta x & \text{if } x_i > x_0, \end{cases}$$

and let $\varepsilon' = \varepsilon$. In fact, we will refer to $x'_1 < \cdots < x'_\ell$ and ε' as the *inherited* certificate under the performed transformation. Similarly, if one performs a translation of Δz on $\mathcal{B}$, the inherited ceiling labels would be $z_1 + \Delta z, \dots, z_\ell + \Delta z$, and the inherited certificate would be the same, since a translation on the z-axis does not affect the projection of $\mathcal{B}$ onto the x-axis. This notation will become particularly useful in the proof of the main theorem, when several consecutive transformations are performed on a particular set of blocks and we need to keep track of the values z_i and x_i.

3. The Construction

Lemma 2. *Let G be a $(k-1)$-chromatic graph. Let H be a graph constructed in the following way. Start with a set T consisting of at least $(k-1)(\ell-1)+1$ vertices. For every ℓ-subset $S \subset T$, consider a new copy G_S of G, and add edges from G_S to S such that every vertex of G_S is adjacent to at least one vertex in S. Then H is k-chromatic.*

Proof. The chromatic number is clearly at most k, since we can color each copy of G with $(k-1)$ colors, and then color all vertices of T with a new color. Now, let us consider any $(k-1)$-coloring φ of the vertices of H. By the pigeon hole principle, since T has at least $(k-1)(\ell-1)+1$ vertices, there exists an ℓ-subset $S \subset T$ whose vertices are all colored the same. Then the subgraph of H induced by $V(G_S) \cup S$ is not properly colored by φ. ■

We point out that Lemma 2 admits a natural generalization. Let G be a $(k-1)$-chromatic graph. Consider a k-chromatic ℓ-uniform hypergraph $\mathscr{H}$. Construct a graph H as follows. Start with the vertices of $V(\mathscr{H})$ and no edges. For every edge $E \in \mathscr{H}$, consider a new copy G_E of G, and add edges from G_E to vertices in E such that every vertex of G_E is adjacent to at least one vertex in E. The constructed graph H is k-chromatic.

Theorem 3. *For each positive integer n, there exists a graded valid set of blocks $\mathcal{B}_n$ for which $G(\mathcal{B}_n)$ has chromatic number precisely n.*

Proof. The proof is by induction. For the case $n = 1$, we take $\mathcal{B}_1$ to be a single block. Now suppose $n > 1$ and assume the existence of a graded valid set of blocks $\mathcal{B}_{n-1}$ with $G(\mathcal{B}_{n-1})$ being $(n-1)$-chromatic. Furthermore, assume that $\mathcal{B}_{n-1}$ fits in a cube of edge length less than one (i.e., $\max_{B\in\mathcal{B}_{n-1}} \max X_B - \min_{B\in\mathcal{B}_{n-1}} \min X_B < 1$, and similarly for y and z). Note that such an assumption is possible by a trivial rescaling of $\mathcal{B}_{n-1}$. The induction step will be divided into three parts: the actual construction of $\mathcal{B}_n$, the proof that the constructed $\mathcal{B}_n$ is valid and graded, and the proof that $G(\mathcal{B}_n)$ is indeed n-chromatic.

3.1. Overview

Before we get into the formal proof of the induction step, we give an outline of the main idea behind the construction. We assume that $\mathcal{B}_{n-1}$ has ℓ ceiling classes. Set $t = (n-1)(\ell-1)+1$. First, we will define a set of blocks $\mathcal{T} = \{T_1, \dots, T_t\}$ which will function as a set of "parallel rails" depicted in Figure 1 (for now, ignore the numbers marked along the axes).

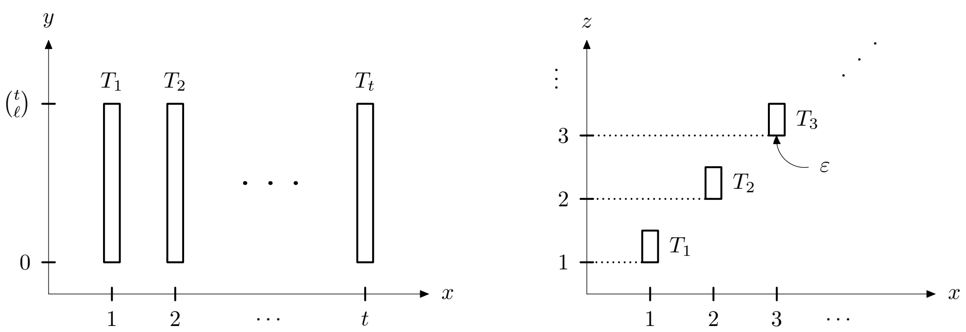

Figure 1. Placement of the blocks $T_1, T_2, \dots, T_t$.

Next, for every ℓ-subset $\mathcal{S} \subseteq \mathcal{T}$, we will consider a new copy of $\mathcal{B}_{n-1}$. We then deform this copy (stretching and translating it multiple times) in order to obtain a graded valid set of blocks $\mathcal{B}_\mathcal{S}$ with the property that each one of its blocks is adjacent to at least one block in $\mathcal{S}$. More specifically, every block in the i-th ceiling class of $\mathcal{B}_\mathcal{S}$ (in increasing order of labels) will be adjacent precisely to the i-th block of $\mathcal{S}$ (in the order induced by the x-axis). To prevent distinct copies of $\mathcal{B}_{n-1}$ from intersecting each other, we allocate, for each ℓ-subset $\mathcal{S}$ of $\mathcal{T}$, an exclusive band of the y-dimension into which $\mathcal{B}_\mathcal{S}$ will be placed (see Figure 2). We then let

$$\mathcal{B}_n = \mathcal{T} \cup \Big(\bigcup_{\mathcal{S}} \mathcal{B}_\mathcal{S} \Big)$$

be the desired set of blocks.

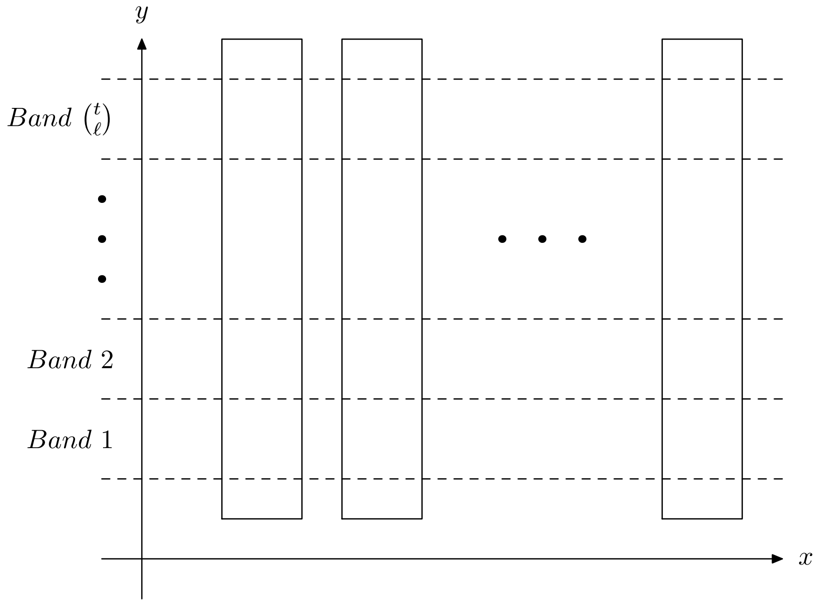

Figure 2. Exclusive bands.

3.2. Construction Details

Let $z_1 < z_2 < \cdots < z_\ell$ be the ceiling labels of $\mathcal{B}_{n-1}$. Let $x_1 < x_2 < \cdots < x_\ell$, and $\epsilon > 0$ be a certificate that $\mathcal{B}_{n-1}$ is graded. Set $t = (n-1)(\ell-1)+1$. We define a set of blocks $\mathcal{T} = \{T_1, \ldots, T_t\}$ as follows

$$T_j = [j - \tfrac{\epsilon}{2}, j + \tfrac{\epsilon}{2}] \times [0, \tbinom{t}{\ell}] \times [j, j + \tfrac{1}{2}]. \tag{1}$$

Figure 1 illustrates where each block in $\mathcal{T}$ has been placed.

Next, fix an ordering of the ℓ-subsets of $\mathcal{T}$. For each ℓ-subset $\mathcal{S} \subseteq \mathcal{T}$, let $\mathrm{idx}(\mathcal{S})$ denote the index of $\mathcal{S}$ according to this ordering (the smallest being 0). Let $\mathcal{S} = \{T_{j_1}, \dots, T_{j_\ell}\}$ be an arbitrary ℓ-subset of $\mathcal{T}$. In order to obtain $\mathcal{B}_\mathcal{S}$ from $\mathcal{B}_{n-1}$, we will define a sequence of graded valid sets of blocks $\mathcal{B}_{n-1} = \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_\ell = \mathcal{B}_\mathcal{S}$, where $\mathcal{A}_i$ is obtained from $\mathcal{A}_{i-1}$ by exactly two stretchings (except for A_1, which is obtained from $\mathcal{A}_0$ by three translations). We let $z_{i,1} < \cdots < z_{i,\ell}$ denote the ceiling classes of $\mathcal{A}_i$, with $z_{0,j} = z_j$. Since each $\mathcal{A}_i$ is graded, it has a certificate of gradation $x_{i,1} < \cdots < x_{i,\ell}$ (the thickness ε is always the same). When $i = 0$, the certificate is given by $x_{0,j} = x_j$. For $i > 0$, we define $x_{i,1} < \cdots < x_{i,\ell}$ (and $z_{i,1} < \cdots < z_{i,\ell}$) as the inherited images of $x_{i-1,1} < \cdots < x_{i-1,\ell}$ (respectively $z_{i-1,1} < \cdots < z_{i-1,\ell}$) under the transformations that are performed to obtain $\mathcal{A}_i$ from $\mathcal{A}_{i-1}$.

Now we provide a formal algorithm to obtain $\mathcal{B}_\mathcal{S}$ from a copy of $\mathcal{B}_{n-1}$. Recall that $\mathcal{B}_{n-1}$ fits in a cube of size less than 1. In order to simplify the notation, we further suppose that $\min_{B \in \mathcal{B}_{n-1}} \min Y_B = 0$.

Algorithm 1: Generating $\mathcal{B}_\mathcal{S}$

Input: a copy $\mathcal{A}_0$ of $\mathcal{B}_{n-1}$, and a set $\mathcal{S} = \{T_{j_1}, \dots, T_{j_\ell}\}$, with $j_1 < \cdots < j_\ell$

Output: $\mathcal{B}_\mathcal{S}$

temp $\leftarrow$ **translation of** $\mathcal{A}_0$ **by** $\Delta y = \mathrm{idx}(\mathcal{S})$;
temp $\leftarrow$ **translation of** *temp* **by** $\Delta z = j_1 - z_{0,1}$;
$\mathcal{A}_1 \leftarrow$ **translation of** *temp* **by** $\Delta x = j_1 - x_{0,1}$;
for $1 < i \leq \ell$ **do**
    temp $\leftarrow$ **stretching of** $\mathcal{A}_{i-1}$ **by** $\Delta x = j_i - x_{i-1,i}$ **at** j_{i-1};
    $\mathcal{A}_i \leftarrow$ **stretching of** *temp* **by** $\Delta z = j_i - z_{i-1,i}$ **at** j_{i-1};
end
$\mathcal{B}_\mathcal{S} \leftarrow \mathcal{A}_\ell$

Note that, by the definition of the stretching operation, blocks in the ceiling class $\mathcal{A}_{i-1}(z_{i-1,k})$, for $1 \leq k < i$, remain unchanged after the operation in line 6 is performed.

3.3. Small examples

When $n = 1$, we set $\mathcal{B}_1$ to be a single block. For $n = 2$, we get two blocks which intersect (see Figure 3). For $n = 3$, the construction yields a cycle on

9 blocks (see Figure 4 where the slanted line means these two blocks do not intersect). When $n = 4$, the construction has 72088 blocks so a figure is not possible in this case.

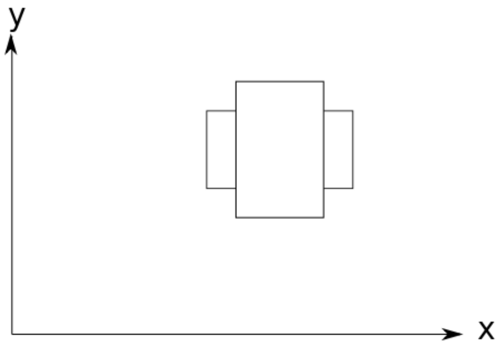

Figure 3. $\mathcal{B}_2$

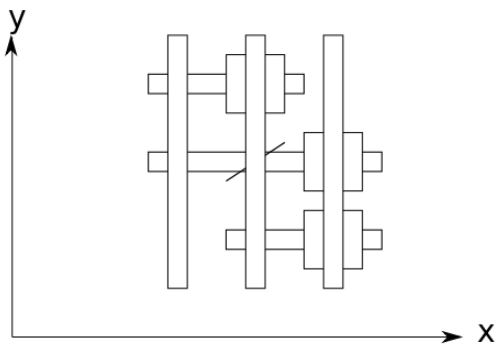

Figure 4. $\mathcal{B}_3$

3.4. Analysis of the construction

We first claim that, for any ℓ-subset $\mathcal{S} \in \mathcal{T}$, the projection of $\mathcal{B}_\mathcal{S}$ onto the y-axis falls inside the interval $[\mathrm{idx}(\mathcal{S}), \mathrm{idx}(\mathcal{S}) + 1)$. A brief scan through Algorithm 1 reveals that no stretching is performed on the y-axis in the process of obtaining $\mathcal{B}_\mathcal{S}$ from $\mathcal{B}_{n-1}$. Therefore, the projection of $\mathcal{B}_\mathcal{S}$ onto the y-axis is only a translation of the projection of $\mathcal{B}_{n-1}$. This translation corresponds precisely to the transformation performed in line 1. Under the assumption that $\min_{B\in\mathcal{B}_{n-1}} \min Y_B = 0$, the execution of line 1 implies that $\min_{B\in\mathcal{B}_{\mathcal{S}}} \min Y_B = \mathrm{idx}(\mathcal{S})$. Since $\mathcal{B}_{n-1}$ fits in a cube of side less than 1, the claim follows.

The discussion in the above paragraph leads to the conclusion that, for distinct ℓ-subsets $\mathcal{S} \neq \mathcal{S}' \in \mathcal{T}$, any two blocks $B \in \mathcal{B}_{\mathcal{S}}$ and $B' \in \mathcal{B}_{\mathcal{S}'}$ do not intersect. Now fix an arbitrary ℓ-subset $\mathcal{S} \subseteq \mathcal{T}$. Assume $\mathcal{S} = \{T_{j_1}, \dots, T_{j_\ell}\}$. It remains to show that a block $T \in \mathcal{T}$ and a block $B \in \mathcal{B}_{\mathcal{S}}$ intersect (and intersect at boundary points) if and only if $T = T_{j_k}$ for some k, and B belongs to the k-th smallest ceiling class of $\mathcal{B}_{\mathcal{S}}$. We consider the following list of invariants after line 6 of Algorithm 1.

(a) $z_{i,k} = j_k$ for all $1 \le k \le i \le \ell$, and $z_{i,k} < j_i + 1$ for all $1 \le i < k \le \ell$;
(b) $x_{i,k} = j_k$ for all $1 \le k \le i \le \ell$;
(c) $j_i + \varepsilon \le \min X_B$ for all $B \in \mathcal{A}_i(z_{i,k})$ with $i < k$;
(d) $\min X_B < j_i - \varepsilon$ for all $B \in \mathcal{A}_i(z_{i,k})$ with $k \le i$.

For conciseness, we omit their proofs. The reader, however, may use induction on i if they wish to verify (a)–(d). These invariants imply that $\mathcal{B}_{\mathcal{S}}$ (which equals $\mathcal{A}_\ell$) is a valid set of blocks with the desired intersections.

Finally, the definition of $\mathcal{T}$ together with properties (c) and (d) ensure that $\mathcal{B}_n$ is graded. In fact, one can list the ceiling labels of $\mathcal{B}_n$ as

$$1 < 1 + \frac{1}{2} < 2 < 2 + \frac{1}{2} < \cdots < t < t + \frac{1}{2}.$$

A certificate of gradation for $\mathcal{B}_n$ is the sequence

$$1 - \frac{3\varepsilon}{4} < 1 < 2 - \frac{3\varepsilon}{4} < 2 < \cdots < t - \frac{3\varepsilon}{4} < t,$$

with slack $\frac{\varepsilon}{16}$. This concludes the argument that $\mathcal{B}_n$ is a graded set of blocks.

3.5. The chromatic number

In view of the preceding sections, note that $G(\mathcal{B}_n)$ was constructed precisely in the way described by Lemma 2. In particular, for each ℓ-subset $\mathcal{S} \in \mathcal{T}$, there exists a copy of $G(\mathcal{B}_{n-1})$ where each one of its vertices has exactly one neighbor in $\mathcal{S}$. Therefore, the chromatic number of $G(\mathcal{B}_n)$ is n. This completes the induction step of the proof, and the theorem follows. ■

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References

[1] J.P. Burling, On coloring problems of families of prototypes, Ph.D. Thesis - University of Colorado, 1, (1965).

[2] T.R. Jensen and B. Toft, Graph coloring problems, Wiley-Interscience Series in Discrete Mathematics and Optimization (John Wiley & Sons Inc., New York, 1995). A Wiley-Interscience Publication.

[3] A.V. Kostochka and J. Nešetřil, *Properties of Descartes' construction of triangle-free graphs with high chromatic number*, Combin. Probab. Comput. **8** (1999) 467–472.

[4] B. Reed and D. Allwright, *Painting the office*, MICS Journal **1** (2008) 1–8.

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