Discussiones Mathematicae
Graph Theory 31 (2011) 143–159

CLOSURE FOR SPANNING TREES AND DISTANT AREA

Jun Fujisawa

Department of Applied Science
Kochi University
2–5–1 Akebono-cho, Kochi 780–8520, Japan
e-mail: fujisawa@is.kochi-u.ac.jp

Akira Saito*

Department of Computer Science
Nihon University
Sakurajosui 3–25–40, Setagaya-Ku, Tokyo 156–8550, Japan
e-mail: asaito@chs.nihon-u.ac.jp

and

Ingo Schiermeyer

Institut für Diskrete Mathematik und Algebra
Technische Universität
Bergakademie Freiberg, D–09596 Freiberg, Germany
e-mail: schierme@mailserver.tu-freiberg.de

Abstract

A k-ended tree is a tree with at most k endvertices. Broersma and Tuinstra [3] have proved that for $k \geq 2$ and for a pair of nonadjacent vertices u, v in a graph G of order n with $\deg_G u + \deg_G v \geq n-1$, G has a spanning k-ended tree if and only if $G + uv$ has a spanning k-ended tree. The distant area for u and v is the subgraph induced by the set of vertices that are not adjacent with u or v. We investigate the relationship between the condition on $\deg_G u + \deg_G v$ and the

*Partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C), 19500017, 2009.

structure of the distant area for u and v. We prove that if the distant area contains K_r, we can relax the lower bound of $\deg_G u + \deg_G v$ from $n-1$ to $n-r$. And if the distant area itself is a complete graph and G is 2-connected, we can entirely remove the degree sum condition.

Keywords: spanning tree, k-ended tree, closure.

2010 Mathematics Subject Classification: 05C05, 05C45.

1. Introduction

A closure operation is a useful operation in the study of a hamiltonian cycle and related topics. It was first introduced by Bondy and Chvátal [1]

Theorem A ([1])**.** *Let G be a graph of order n and let u and v be a pair of nonadjacent vertices in G.*

(1) *Suppose* $\deg_G u + \deg_G v \geq n$. *Then G has a hamiltonian cycle if and only if $G + u$ has a hamiltonian cycle.*

(2) *Suppose* $\deg_G u + \deg_G v \geq n - 1$. *Then G has a hamiltonian path if and only if $G + uv$ has a hamiltonian path.*

An endvertex is a vertex of degree one. For a positive integer k, a tree of order at least two is said to be a k-ended tree if it has at most k endvertices. A hamiltonian path is a spanning 2-ended tree. Thus, we can interpret the second statement of Theorem A as a closure theorem for a spanning 2-ended tree. Broersma and Tuinstra [3] extended this line of research, and proved the following theorem.

Theorem B ([3])**.** *Let k be an integer with $k \geq 2$. Let G be a graph of order n and let u and v be a pair of nonadjacent vertices in G with $\deg_G u + \deg_G v \geq n - 1$. Then G has a spanning k-ended tree if and only if $G + uv$ has a spanning k-ended tree.*

One peculiar point of Theorem B is that the requirement for $\deg_G u + \deg_G v$ does not depend on k. Though the existence of a spanning k-ended tree becomes stronger as k grows, Theorem B always requires $n-1$ for the lower bound of $\deg_G u + \deg_G v$. Though it looks strange, Broersma and Tuinstra proved that this requirement is sharp. In other words, they constructed a graph G with a pair of nonadjacent vertices u and v satisfying $\deg_G u +$

$\deg_G v = |V(G)| - 2$ such that $G + uv$ has a spanning k-ended tree, but G has no spanning k-ended tree.

For a pair of nonadjacent vertices u and v in G, we call the set of vertices that are not adjacent with u or v the *distant set* for u and v, and denote it by W_{uv}:

$$W_{uv} = V(G) - \big(\{u, v\} \cup N_G(u) \cup N_G(v)\big),$$

where $N_G(x)$ denotes the neighborhood of a vertex x in G. Also we call the graph induced by W_{uv} the *distant area* for u and v. The sharpness example given by Broersma and Tuinstra [3] has only one vertex in the distant area. This motivates us to study the relationship between the structure of the distant area and the degree sum condition. In particular, we focus on a clique in the distant area. Our results claim that if the distant area contains a clique, we can relax the degree sum condition by the proportion of its order.

For graph-theoretic terminology not explained in this paper, we refer the reader to [4]. For a set of vertices S in G, we denote by $G[S]$ the subgraph of G induced by S. An endvertex is a vertex of degree one. We denote by $\mathrm{End}(G)$ the set of endvertices of G. For A, $B \subset V(G)$ with $A \cap B = \emptyset$, we denote by $E_G(A, B)$ the set of edges which have one endvertex in A and the other in B. Given a tree T of order at least two, we often orient the edges of T so that T becomes a rooted tree. For $v \in V(T)$, we denote by $N_T^+(v)$ the out-neighborhood of v, i.e., the set of children of v. If v is not the root of T, let v^- denote the parent of v. Moreover, if X is a set of vertices of T and X does not contain the root, we let $X^- = \{x^- : x \in X\}$. A vertex with no child is called a leaf of T, and we denote by $L(T)$ the set of leaves of T. Note that for a tree T of order at least two and an orientation of $E(T)$ which makes T a rooted tree, $\mathrm{End}(T)$ and $L(T)$ can be different: If the root r of T has degree one, $r \in \mathrm{End}(T) - L(T)$. But $L(T) \subset \mathrm{End}(T) \subset L(T) \cup \{r\}$ always holds.

In the next section, we prove that if $G[W_{uv}]$ contains a complete graph, we can relax the degree sum condition for the Broersma-Tuinstra closure. In Section 3, we consider the case in which $G[W_{uv}]$ itself is a complete graph, and prove that in this case we no longer need the degree sum condition, if G is 2-connected. In Section 4, we consider a closure for a hamiltonian cycle and study its relationship with the structure of the distant area. And in Section 5, we give a conclusion and some remarks.

2. Clique in the Distant Area

As we have noted in the introduction, the degree condition for the Broersma-Tuinstra closure depends on the order of a clique in the distant area.

Theorem 1. *Let n, k and r be integers with $n > k \geq 2$ and $r \geq 1$. Let G be a graph of order n and let u and v be a pair of nonadjacent vertices in G. Suppose $G[W_{uv}]$ contains a complete graph of order r and $\deg_G u + \deg_G v \geq n - r$. Then G has a spanning k-ended tree if and only if $G + uv$ has a spanning k-ended tree.*

Proof. Since the necessity of the theorem is trivial, we only prove the sufficiency. Assume, to the contrary, that $G + uv$ has a spanning k-ended tree but G does not have a spanning k-ended tree. Let W be a subset of W_{uv} with $G[W] = K_r$. We prove a series of claims to obtain a contradiction.

Claim 1. There exist a pair of trees T_1 and T_2 such that

(1) $V(T_1) \cup V(T_2) = V(G)$, $V(T_1) \cap V(T_2) = \emptyset$,

(2) $u \in V(T_1)$ and $v \in V(T_2)$,

(3) $|V(T_1)| \geq 2$, $|V(T_2)| \geq 2$, and

(4) if we orient the edges of T_1 and T_2 so that T_1 and T_2 are rooted trees with roots u and v, respectively, then $|L(T_1)| + |L(T_2)| \leq k$.

Proof. Let T be a spanning k-ended tree of $G + uv$. Since G does not have a spanning k-ended tree, $uv \in E(T)$. Let T_1 and T_2 be the components of $T - uv$ with $u \in V(T_1)$ and $v \in V(T_2)$, and orient the edges of T_1 and T_2 so that they become rooted trees with roots u and v, respectively.

If neither u nor v is an endvertex of T, then T_1 and T_2 satisfy the conditions (1), (2) and (3). Moreover, since $\mathrm{End}(T) = L(T_1) \cup L(T_2)$, we have $|L(T_1)| + |L(T_2)| = |\mathrm{End}(T)| \leq k$.

Suppose u or v is an endvertex of T. By symmetry, we may assume $u \in \mathrm{End}(T)$. Since $n \geq 3$, $|V(T_1)| = 1$ and $|V(T_2)| \geq 2$. Since $|W_{uv}| \geq |W| = r$ and $N_G(v) \cap \big(W_{uv} \cup \{u, v\}\big) = \emptyset$, $\deg_G v \leq n - r - 2$. Then since $\deg_G u + \deg_G v \geq n - r$, we have $\deg_G u \geq 2$. Let $y \in N^+_{T_2}(v)$ and let $x \in N_G(u) - \{y\}$. Let $T' = T - \{uv, xx^-\} + ux$ and let T'_1 and T'_2 be the components of T'. Orient the edges of T'_1 and T'_2 so that T'_1 and T'_2 become rooted trees with roots u and v, respectively. Then T'_1 and T'_2 satisfy the conditions (1) and (2). Furthermore, since $x \in V(T'_1)$ and $y \in V(T'_2)$, they satisfy the condition (3). By the construction and the orientation of T, $L(T_1') \cup L(T_2') \subset \big(\mathrm{End}(T) - \{u\}\big) \cup \{x^-\}$, and hence $|L(T_1')| + |L(T_2')| \le k$. $\square$

In the subsequent arguments, when we have two trees satisfying the conditions of Claim 1, we always assume that their edges are oriented so that they become rooted trees with roots u and v.

Now choose two trees T_1 and T_2 satisfying the conditions of Claim 1 so that

(C1) $\sum_{w \in W \cap V(T_1)} d_{T_1}(u, w) + \sum_{w \in W \cap V(T_2)} d_{T_2}(v, w)$ is as large as possible, and

(C2) $\sum_{z \in V(T_1) - W} d_{T_1}(u, z) + \sum_{z \in V(T_2) - W} d_{T_2}(v, z)$ is as small as possible, subject to the condition (C1).

Let $T = (T_1 \cup T_2) + uv$. Note that T is a spanning k-ended tree of $G + uv$. Let

$$\begin{aligned} X_{1,1} &= \big(N_G(u) \cap V(T_1)\big)^-, & X_{1,2} &= N_G(v) \cap V(T_1), \\ X_{2,1} &= N_G(u) \cap V(T_2), & X_{2,2} &= \big(N_G(v) \cap V(T_2)\big)^-, \\ X_1 &= X_{1,1} \cup X_{1,2} \cup L(T_1), & X_2 &= X_{2,1} \cup X_{2,2} \cup L(T_2) \end{aligned}$$

and $X = X_1 \cup X_2$.

Claim 2.
(1) $X_{1,1}$, $X_{1,2}$ and $L(T_1)$ are mutually disjoint, and
(2) $X_{2,1}$, $X_{2,2}$ and $L(T_2)$ are mutually disjoint.

Proof. Since (1) and (2) are symmetric, we only prove (1). By definition, $X_{1,1} \cap L(T_1) = \emptyset$. Assume $X_{1,2} \cap L(T_1) \ne \emptyset$ and let $x \in X_{1,2} \cap L(T_1)$. Let $T' = (T_1 \cup T_2) + xv$. Then T' is a spanning tree of G. Moreover, since $|V(T_2)| \ge 2$, $v \notin \mathrm{End}(T')$, and hence $\mathrm{End}(T') \subset \big(\mathrm{End}(T) - \{x\}\big) \cup \{u\}$. Thus, T' is a k-ended tree. This contradicts the assumption.

Next assume $X_{1,1} \cap X_{1,2} \ne \emptyset$. Let $x \in X_{1,1} \cap X_{1,2}$. Then $N^+_{T_1}(x) \cap N_G(u) \ne \emptyset$. Let $y \in N^+_{T_1}(x) \cap N_G(u)$. Let $T' = (T_1 \cup T_2) - \{xy\} + \{vx, uy\}$. Then T' is a spanning tree of G. Moreover, since $\deg_T a = \deg_{T'} a$ for every $a \in V(G)$, T' is a k-ended tree. This is a contradiction. $\square$

Claim 3.
(1) $|X_{1,1}| \ge |N_G(u) \cap V(T_1)| - |L(T_1)| + 1$,

(2) $|X_{2,2}| \geq |N_G(v) \cap V(T_2)| - |L(T_2)| + 1$.

Proof. Since (1) and (2) are symmetric, we only prove (1).

$$\begin{aligned}
|X_{1,1}| &= \sum_{x \in X_{1,1}} (1 - |N^+_{T_1}(x) \cap N_G(u)| + |N^+_{T_1}(x) \cap N_G(u)|) \\
&= \sum_{x \in X_{1,1}} |N^+_{T_1}(x) \cap N_G(u)| - \sum_{x \in X_{1,1}} (|N^+_{T_1}(x) \cap N_G(u)| - 1) \\
&= |N_G(u) \cap V(T_1)| - \sum_{x \in X_{1,1}} (|N^+_{T_1}(x) \cap N_G(u)| - 1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{x \in X_{1,1}} (|N^+_{T_1}(x) \cap N_G(u)| - 1) &\leq \sum_{x \in X_{1,1}} (|N^+_{T_1}(x)| - 1) \\
&\leq \sum_{x \in V(T_1) - L(T_1)} (|N^+_{T_1}(x)| - 1) = |L(T_1)| - 1.
\end{aligned}$$

Therefore, (1) follows. □

By Claim 2 and Claim 3, we have

$$\begin{aligned}
|X| &= |X_1| + |X_2| = |X_{1,1}| + |X_{1,2}| + |L(T_1)| + |X_{2,1}| + |X_{2,2}| + |L(T_2)| \\
&\geq |N_G(u) \cap V(T_1)| - |L(T_1)| + 1 + |N_G(v) \cap V(T_1)| + |L(T_1)| \\
&\quad + |N_G(u) \cap V(T_2)| + |N_G(v) \cap V(T_2)| - |L(T_2)| + 1 + |L(T_2)| \\
&= |N_G(u) \cap V(T_1)| + |N_G(v) \cap V(T_1)| + |N_G(u) \cap V(T_2)| \\
&\quad + |N_G(v) \cap V(T_2)| + 2 \\
&= \deg_G u + \deg_G v + 2 \geq n - r + 2.
\end{aligned}$$

Claim 4. *$W \cap X_1 = \emptyset$ or $W \cap X_2 = \emptyset$.*

Proof. Assume, to the contrary, $W \cap X_1 \neq \emptyset$ and $W \cap X_2 \neq \emptyset$. Let $w_1 \in W \cap X_1$ and $w_2 \in W \cap X_2$. Since $w_1 \in W \subset W_{uv}$, $w_1 \notin X_{1,2}$ and hence $w_1 \in X_{1,1} \cup L(T_1)$. Similarly, $w_2 \in X_{2,2} \cup L(T_2)$. Note $w_1w_2 \in E(G)$ since W induces a complete graph.

If $w_1 \in L(T_1)$ and $w_2 \in L(T_2)$, then let $T' = (T_1 \cup T_2) + w_1w_2$. Then T' is a spanning tree of G, and since $\mathrm{End}(T') \subset (\mathrm{End}(T) - \{w_1, w_2\}) \cup \{u, v\}$, T' is a k-ended tree. This contradicts the assumption.

If $w_1 \in L(T_1)$ and $w_2 \in X_{2,2}$, then $N^+_{T_2}(w_2) \cap N_G(v) \neq \emptyset$. Let $x \in N^+_{T_2}(w_2) \cap N_G(v)$. Let $T' = (T_1 \cup T_2) - w_2x + \{vx, w_1w_2\}$. Then T' is a spanning tree of G. Moreover, since $\mathrm{End}(T') \subset (\mathrm{End}(T) - \{w_1\}) \cup \{u\}$, T' is a k-ended tree. This is a contradiction. By a similar argument, we obtain a a contradiction if $w_1 \in X_{1,1}$ and $w_2 \in L(T_2)$.

Finally, suppose $w_1 \in X_{1,1}$ and $w_2 \in X_{2,2}$. Let $x_1 \in N^+_{T_1}(w_1) \cap N_G(u)$ and $x_2 \in N^+_{T_2}(w_2) \cap N_G(v)$, and let $T' = T - \{x_1w_1, x_2w_2\} + \{ux_1, w_1w_2, vx_2\}$. Then T' is a spanning tree of G. Moreover, since $\deg_{T'}(a) = \deg_T(a)$ for each $a \in V(G)$, T' is a k-ended tree. This is a contradiction, and the claim follows. □

For $a \in V(T_1)$, let T_a be the subtree of T_1 induced by a and all of its descendants. Note that T_a is a rooted tree with root a. Similarly, for $b \in V(T_2)$, let T_b be the subtree of T_2 induced by b and all of its descendants.

Claim 5.

(1) For each $w \in W \cap V(T_1)$ and $x \in N^+_{T_1}(w^-) - \{w\}$, $W \cap V(T_x) = \emptyset$.

(2) For each $w \in W \cap V(T_2)$ and $x \in N^+_{T_2}(w^-) - \{w\}$, $W \cap V(T_x) = \emptyset$.

Proof. By symmetry, we have only to prove (1). Assume $W \cap V(T_x) \neq \emptyset$. Let $w' \in W \cap V(T_x)$. Then $ww' \in E(G)$. Let $T' = T_1 - ww^- + ww'$. Since $V(T'_1) = V(T_1)$, (T'_1, T_2) satisfies the condition (1), (2) and (3) of Claim 1. Note $\deg_{T'_1} w^- = \deg_{T_1} w^- - 1$, $\deg_{T'_1} w' = \deg_{T_1} w' + 1$ and $\deg_{T'_1} a = \deg_{T_1} a$ for each $a \in V(T_1) - \{w^-, w'\}$. Since $x \in N^+_{T'_1}(w^-)$, $w^- \notin L(T'_1)$, which implies $L(T'_1) \subset L(T_1)$ and $|L(T'_1)| + |L(T_2)| \leq k$. Therefore, (T'_1, T_2) satisfies all the conditions of Claim 1. Let $a \in V(T_1)$. If $a \notin V(T_w)$, then uT_1a is also a path in T'_1, and hence $d_{T'}(u, a) = d_{T_1}(u, a)$. On the other hand, if $a \in V(T_w)$, then $w^-w \in E(uT_1a)$. In this case, $uT_1w'wT_1a$ is a ua-path in T'_1, and $d_{T'_1}(u, a) > d_{T_1}(u, a)$. In particular, $d_{T'_1}(u, w) > d_{T_1}(u, w)$. Therefore,

$$\sum_{w \in W \cap V(T'_1)} d_{T'_1}(u, w) + \sum_{w \in W \cap V(T_2)} d_{T_2}(v, w)$$

$$> \sum_{w \in W \cap V(T_1)} d_{T_1}(u, w) + \sum_{w \in W \cap V(T_2)} d_{T_2}(v, w).$$

This contradicts the choice of (T_1, T_2). □

By Claim 5, we have the following.

Claim 6. For each $w_1,\ w_2 \in W$ with $w_1 \neq w_2$, we have $w_1^- \neq w_2^-$. In particular, $|W^-| = |W|$.

Claim 7. $W^- \cap X_1 \subset X_{1,2}$ and $W^- \cap X_2 \subset X_{2,1}$.

Proof. By symmetry, we have only to prove $W^- \cap X_1 \subset X_{1,2}$. Let $x \in W^- \cap X_1$ and assume $x \notin X_{1,2}$. Since x is the parent of a vertex in W, $x \notin L(T_1)$. Thus, $x \in X_{1,1}$. Then $N_{T_1}^+(x) \cap N_G(u) \neq \emptyset$. Let $y \in N_{T_1}^+(x) \cap N_G(u)$. Note that since $y \in N_G(u)$, $y \notin W$. On the other hand, since $x \in W^-$, $N_{T_1}^+(x) \cap W \neq \emptyset$. Let $w \in N_{T_1}^+(x) \cap W$. Note $y \neq w$. Note also that since $xw \in E(G)$, $u \neq x$.

By Claim 5, $V(T_y) \cap W = \emptyset$. Let $T_1' = T_1 - xy + uy$. Then (T_1', T_2) satisfies the conditions (1), (2) and (3) of Claim 1. Note $\deg_{T_1'} u = \deg_{T_1} u + 1$, $\deg_{T_1'} x = \deg_{T_1} x - 1$ and $\deg_{T_1'} a = \deg_{T_1} a$ for each $a \in V(T_1) - \{u, x\}$. Moreover, since $w \in N_{T_1'}^+(x)$, $x \notin L(T_1')$. Hence $|L(T_1')| = |L(T_1)|$ and $|L(T_1')| + |L(T_2)| \leq k$. Therefore, (T_1', T_2) satisfies all the conditions of Claim 1.

Let $a \in V(T_1)$. If $a \notin V(T_y)$, then uT_1a is also a unique ua-path in T_1' and $d_{T_1'}(u,a) = d_{T_1}(u,a)$. On the other hand, if $a \in V(T_y)$, then uyT_1a is a unique ua-path in T_1'. Since $u \neq x$, we have $d_{T_1'}(u,a) < d_{T_1}(u,a)$. In particular, $d_{T_1'}(u,y) < d_{T_1}(u,y)$. Since $V(T_y) \cap W = \emptyset$, we have

$$\sum_{w \in V(T_1') \cap W} d_{T_1'}(u,w) + \sum_{w \in V(T_2) \cap W} d_{T_2}(v,w) = \sum_{w \in V(T_1) \cap W} d_{T_1}(u,w) + \sum_{w \in V(T_2) \cap W} d_{T_2}(v,w)$$

and

$$\sum_{x \in V(T_1') - W} d_{T_1'}(u,x) + \sum_{x \in V(T_2) - W} d_{T_2}(v,x) < \sum_{x \in V(T_1) - W} d_{T_1}(u,x) + \sum_{x \in V(T_2) - W} d_{T_2}(v,x).$$

This contradicts the choice of (T_1, T_2), and the claim follows. □

Since $|X| \geq n - r + 2$ and $|W^-| = |W| = r$, we have $|W^-| + |X| \geq n + 2$, which implies $W^- \cap X \neq \emptyset$. By symmetry, we may assume $W^- \cap X_1 \neq \emptyset$.

Also by symmetry, if $W^- \cap X_2 \neq \emptyset$, we may assume $\max\{d_{T_1}(u,a)\colon a \in W^- \cap X_1\} \geq \max\{d_{T_2}(v,a)\colon a \in W^- \cap X_2\}$. Choose $z_1 \in W^- \cap X_1$ so that $d_{T_1}(u,z_1)$ is as large as possible. Let $d_0 = d_{T_1}(u,z_1)$. By Claim 7, $z_1 \in N_G(v)$. By Claim 6, $|N^+_{T_1}(z_1) \cap W| = 1$. Let $N^+_{T_1}(z_1) \cap W = \{w_1\}$.

Let $W_0 = W \cap V(T_{w_1})$. Note $w_1 \in W_0$ and hence $z_1 \in X \cap W_0^-$. On the other hand, if $X \cap W_0^- \neq \{z_1\}$, we can take $z \in X \cap W_0^- - \{z_1\}$. Then $z \in V(T_{w_1})$, and hence $d_{T_1}(u,z) \geq d_{T_1}(u,w_1) = d_{T_1}(u,z_1)+1 \geq d_0+1$. This contradicts the maximality of $d_{T_1}(u,z_1)$. Therefore, $X \cap W_0^- = \{z_1\}$.

Since $W_0 \subset V(T_{w_1})$ and $(W-W_0)\cap V(T_{w_1}) = \emptyset$, $(W-W_0)\cap W_0^- \subset \{z_1\}$. However, since $z_1 \in N_G(v)$, $z_1 \notin W$. Thus, we have $(W-W_0)\cap W_0^- = \emptyset$.

Let $Z = (W-W_0) \cup W_0^-$. By Claim 6, $|W_0^-| = |W_0|$, and since $(W - W_0) \cap W_0^- = \emptyset$, we have $|Z| = |W - W_0| + |W_0^-| = |W| = r$. Then $|Z| + |X| \geq n+2$, which implies $|Z \cap X| \geq 2$. Since $|X \cap W_0^-| = 1$, $X \cap (W - W_0) \neq \emptyset$. Let $w_2 \in X \cap (W - W_0)$.

First, suppose $w_2 \in V(T_1)$. Then $w_2 \in X_1 \cap W$. This implies $X_2 \cap W = \emptyset$ by Claim 4. Since $w_2 \in W$, $w_2 \in X_{1,1} \cup L(T_1)$. Note $w_1w_2 \in E(G)$.

If $w_2 \in X_{1,1}$, then $N^+_{T_1}(w_2) \cap N_G(u) \neq \emptyset$. Let $z_2 \in N^+_{T_1}(w_2) \cap N_G(u)$. Since $w_2 \notin W_0$, $z_2 \notin V(T_{w_1})$ and z_1 is the successor of w_1 in $w_1T_1w_2$. Let $P = z_1T_1z_2$. If $z_1 \in V(T_{z_2})$, then $w_2 \notin V(P)$. On the other hand, if $z_1 \notin V(T_{z_2})$, then w_2 is the predecessor of z_2 in P. In either case, let $T' = (T_1 \cup T_2) - \{w_1z_1, w_2z_2\} + \{vz_1, w_1w_2, z_2u\}$. Then since $w_1z_1 \in V(w_1T_1w_2)$, T' is a spanning tree of G. Furthermore, since $\deg_{T'} a = \deg_T a$ for every $a \in V(G)$, T' is a k-ended tree. This is a contradiction.

Suppose $w_2 \in L(T_1)$. Since $w_2 \notin V(T_{w_1})$, z_1 is the successor of w_1 in $w_1T_1w_2$. Let $T' = (T_1 \cup T_2) - w_1z_1 + \{vz_1, w_1w_2\}$. Then T' is a spanning tree of G. Moreover, $\deg_{T'} u = \deg_T u - 1$, $\deg_{T'} w_2 = \deg_{T'} w_2 + 1$ and $\deg_{T'} a = \deg_{T'} a$ for every $a \in V(G) - \{u, w_2\}$. Hence $\mathrm{End}(T') \subset (\mathrm{End}(T) - \{w_2\}) \cup \{u\}$ and T' is a k-ended tree. This is a contradiction.

Next, consider the case $w_2 \in V(T_2)$. Then $W \cap X_2 \neq \emptyset$, which implies $W \cap X_1 = \emptyset$ by Claim 4. Then $L(T_1) \cap W = \emptyset$, and hence $L(T_1) \subset N_G(u) \cup N_G(v)$. If $L(T_1) \cap N_G(v) \neq \emptyset$, let $x \in L(T_1) \cap N_G(v)$ and $T' = (T_1 \cup T_2) + vx$. Then T' is a spanning tree of G, and since $\mathrm{End}(T') \subset (\mathrm{End}(T) - \{x\}) \cup \{u\}$, T' is a k-ended tree, a contradiction. Therefore, we have $L(T_1) \subset N_G(u)$.

Let $x \in \mathrm{End}(T_{w_1})$. Since $w_2 \notin N_G(u)$, $w_2 \in X_{2,2} \cup L(T_2)$. If $w_2 \in X_{2,2}$, $N^+_{T_2}(w_2) \cap N_G(v) \neq \emptyset$. Let $z_2 \in N^+_{T_2}(w_2) \cap N_G(v)$, and let $T' = (T_1 \cup T_2) - \{w_1z_1, w_2z_2\} + \{xu, vz_2, w_1w_2\}$. Then T' is a spanning tree of G. We also have $\deg_{T'} x = \deg_T x + 1$, $\deg_{T'} z_1 = \deg_T z_1 - 1$ and $\deg_{T'} a = \deg_T a$ for every $a \in V(G) - \{x, z_1\}$. Hence $\mathrm{End}(T') \subset (\mathrm{End}(T) - \{x\}) \cup \{z_1\}$ and T' is a k-ended tree, a contradiction

Finally, suppose $w_2 \in L(T_2)$. Let $T' = (T_1 \cup T_2) - z_1w_1 + \{vz_1, w_1w_2\}$. Then T' is a spanning tree of G. Moreover, $\deg_{T'} u = \deg_T u - 1$, $\deg_{T'} w_2 = \deg_T w_2 + 1$ and $\deg_{T'} a = \deg_T a$ for every $a \in V(G) - \{u, w_2\}$. These imply $\mathrm{End}(T') = (\mathrm{End}(T) - \{w_2\}) \cup \{u\}$ and hence T' is a k-ended tree. This is a final contradiction, and the theorem follows. ∎

Theorem 1 is best-possible in the sense that we cannot replace the degree sum condition from $\deg_G u + \deg_G v \geq n - r$ to $\deg_G u + \deg_G v \geq n - r - 1$ if $1 \leq r \leq n - k - 2$. Let G_1 and G_2 be copies of $K_{n-k-r-1}$ and K_r, respectively. Let T_3 be a copy of $K_{1,k-1}$. Introduce a new vertex x, and join x to every vertex of G_1, every vertex of G_2 and the center of T_3. Let G be the resulting graph. Let u be the center of T_3 and let v be a vertex of G_1.

The order of G is n, and $\deg_G u + \deg_G v = n - r - 1$. The distant area for u and v is G_2 and it is a complete graph of order r. Let T be a spanning tree of G. All the endvertices of T_3 are also endvertices of T. Furthermore, since $G - x$ has G_1 and G_2 as its components, T has endvertices both in G_1 and G_2. Thus, $|\mathrm{End}(T)| \geq k + 1$ and G has no spanning k-ended tree. On the other hand, $G + uv$ has a spanning k-ended tree with no endvertex in G_1.

3. The Distant Area Forming a Complete Graph

As we have seen in Section 2, if the distant area contains a clique, we can relax the degree sum condition of the Broersma-Tuinstra closure. One extreme case is the one in which $G[W_{uv}]$ itself is a complete graph. Theorem 1 requires $G[W_{uv}]$ to have K_r for the condition $\deg_G u + \deg_G v \geq n - r$ to work. Therefore, if $|W_{uv}| < r$, we cannot apply Theorem 1 even if $G[W_{uv}]$ is a complete graph. However, in this section, we prove that if $G[W_{uv}]$ is complete and G is 2-connected, we no longer need any degree sum condition. Actually, we prove a stronger statement under a weaker assumption that G is just connected, which gives structural information when the closure does not work.

Theorem 2. *Let k be an integer with $k \geq 2$. Let G be a connected graph and let u and v be a pair of nonadjacent vertices in G. Suppose W_{uv} induces a complete graph. If $G + uv$ has a spanning k-ended tree but G has no spanning k-ended tree, then one of the following holds.*

(1) *There exists a vertex* $x \in W_{uv}$ *such that* x *is a cutvertex of* G *and* $W_{uv} - \{x\}$ *is a component of* $G - x$.

(2) *There exists a vertex* $x \in V(G) - W_{uv}$ *such that* x *is a cutvertex of* G *and* W_{uv} *is a component of* $G - x$.

Corollary 3. *Let* $k \geq 2$. *Let* G *be a* 2*-connected graph and let* u *and* v *be a pair of nonadjacent vertices in* G. *Suppose* W_{uv} *induces a complete graph in* G. *Then* G *has a spanning* k*-ended tree if and only if* $G+uv$ *has a spanning* k*-ended tree.*

***Proof of Theorem* 2.** If $N_G(u) = N_G(v) = \{x\}$ for some $x \in V(G)$, then $W_{uv} = V(G) - \{u, v, x\}$ and the second statement of the theorem follows. Therefore, we may assume $N_G(u) \neq N_G(v)$ or $|N_G(v)| \geq 2$.

Claim 1. There exist a pair of trees T_1 and T_2 such that

(1) $V(T_1) \cup V(T_2) = V(G)$, $V(T_1) \cap V(T_2) = \emptyset$,

(2) $u \in V(T_1)$ and $v \in V(T_2)$,

(3) $|V(T_1)| \geq 2$ and $|V(T_2)| \geq 2$, and

(4) if we orient the edges of T_1 and T_2 so that T_1 and T_2 become rooted trees with roots u and v, respectively, then $|L(T_1)| + |L(T_2)| \leq k$.

Proof. Let T be a spanning k-ended tree of $G + uv$. Since G does not have a spanning k-ended tree, $uv \in E(T)$. Let T_1 and T_2 be the components of $T - uv$ with $\in V(T_1)$ and $V(T_2)$, and orient the edges of T_1 and T_2 so that T_1 and T_2 become rooted trees with roots u and v, respectively. If $\{u, v\} \cap \mathrm{End}(T) = \emptyset$, then T_1 and T_2 satisfy the conditions (1)–(3) of the claim. Moreover, since $L(T_1) \cup L(T_2) = \mathrm{End}(T)$, we also have $|L(T_1)| + |L(T_2)| \leq k$. Therefore, by symmetry, we may assume $u \in \mathrm{End}(T)$, which implies $V(T_1) = \{u\}$. Let $y \in N^+_{T_2}(v)$. If $N_G(u) - \{y\} \neq \emptyset$, then by the same argument as in the proof of Claim 1 of Theorem 1, we can obtain two trees satisfying the conditions (1)–(4) of the claim. Therefore, we may assume $N_G(u) = \{y\}$. Since $N_G(u) \neq N_G(v)$ or $|N_G(v)| \geq 2$, $N_G(v) - \{y\} \neq \emptyset$. Let $z \in N_G(v) - \{y\}$ and $T' = T - \{uv, vy, zz^-\} + \{uy, vz\}$. Let T'_1 and T'_2 be the components of T' with $u \in V(T'_1)$ and $v \in V(T'_2)$, and orient the edges of T'_1 and T'_2 so that T'_1 and T'_2 become rooted trees with roots u and v, respectively. Then T'_1 and T'_2 satisfy the conditions (1) and (2). Moreover, since $y \in V(T'_1)$ and $z \in V(T'_2)$, we have $|V(T'_1)| \geq 2$ and $|V(T'_2)| \geq 2$.

By the construction of T_1' and T_2', $L(T_1') \cup L(T_2') \subset (L(T) - \{u\}) \cup \{z^-\}$, and hence $|L(T_1')| + |L(T_2')| \leq k$. $\square$

As in the proof of Theorem 1, in the subsequent arguments, when we have a pair of trees satisfying the conditions of Claim 1, we always assume that their edges are oriented so that they become rooted trees with roots u and v.

Now choose a pair of trees (T_1, T_2) satisfying the conditions (1)–(4) of Claim 1 so that $|(L(T_1) \cup L(T_2)) \cap W_{uv}|$ is as small as possible. Let $T = (T_1 \cup T_2) + uv$. Note that T is a spanning k-ended tree of $G + uv$.

Claim 2. $E_G(L(T_1) \cup \{u\}, L(T_2) \cup \{v\}) = \emptyset$.

Proof. Assume $E_G(L(T_1) \cup \{u\}, L(T_2) \cup \{v\}) \neq \emptyset$. Let $xy \in E(G)$ with $x \in L(T_1) \cup \{u\}$ and $y \in L(T_2) \cup \{v\}$. Let $T' = (T_1 \cup T_2) + xy$. Then T' is a spanning tree of G. If $x = u$, then $y \in L(T_2)$ and hence $\mathrm{End}(T') \subset (\mathrm{End}(T) - \{y\}) \cup \{v\}$. If $y = v$ and $x \in L(T_1)$, then $\mathrm{End}(T') \subset (\mathrm{End}(T) - \{x\}) \cup \{u\}$. And if $x \in L(T_1)$ and $y \in L(T_2)$, then $\mathrm{End}(T') \subset (\mathrm{End}(T) - \{x, y\}) \cup \{u, v\}$. Hence T' is a spanning k-ended tree of G in every case. This contradicts the assumption, and the claim follows. $\square$

Claim 3. $(L(T_1) \cup L(T_2)) \cap W_{uv} \neq \emptyset$.

Proof. Assume $(N_G(u) \cup N_G(v)) \cap W_{uv} = \emptyset$. Then by Claim 2, $L(T_1) \subset N_G(u)$ and $L(T_2) \subset N_G(v)$. Since G is connected, G has an edge xy with $x \in V(T_1)$ and $y \in V(T_2)$. Take $u' \in L(T_1)$ and $v' \in L(T_2)$ with $x \in V(uT_1u')$ and $y \in V(vT_2v')$. Note that $u' \in N_G(u)$ and $v' \in N_G(v)$, but possibly $uu' \in E(T_1)$ or $vv' \in E(T_2)$.

If $x \notin \{u, u'\}$ and $y \notin \{v, v'\}$, then $uu' \notin E(T_1)$ and $vv' \notin E(T_2)$. Let $T' = (T_1 \cup T_2) - \{xx^-, yy^-\} + \{xy, uu', vv'\}$. Then T' is a spanning tree of G, and since $L(T') \subset (L(T) - \{u', v'\}) \cup \{x^-, y^-\}$, T' is a k-ended tree. This contradicts the assumption. Suppose $x = u'$. By Claim 2, $y \notin \{v, v'\}$ and hence $vv' \notin E(T)$. Let $T' = (T_1 \cup T_2) - yy^- + \{u'y, vv'\}$. Then T' is a spanning tree of G, and since $L(T') \subset (L(T) - \{u', v'\}) \cup \{u, y^-\}$, T' is a k-ended tree, again a contradiction. By a similar argument, we reach a contradiction if $y = v'$.

Suppose $x = u$. Then again by Claim 2, $y \notin \{v, v'\}$ and $vv' \notin E(T)$. Let $T' = (T_1 \cup T_2) - yy^- + \{uy, vv'\}$. Then T' is a spanning tree of G, and since $L(T') \subset (L(T) - \{v'\}) \cup \{y^-\}$, T' is a k-ended tree, a contradiction. By a similar argument, we reach a contradiction if $y = v$. Therefore, the claim follows. $\square$

By symmetry, we may assume $L(T_1) \cap W_{uv} \neq \emptyset$. Let $w_0 \in L(T_1) \cap W_{uv}$.

Claim 4. $N_G(w_0) \subset V(T_1)$. *In particular,* $W_{uv} \subset V(T_1)$.

Proof. Assume $N_G(w_0) \not\subset V(T_1)$. Then $N_G(w_0) \cap V(T_2) \neq \emptyset$. Choose $z \in N_G(w_0) \cap V(T_2)$ so that $d_{T_2}(v, z)$ is as small as possible. By Claim 2, $z \neq v$.

If $z^- \in W_{uv}$, then $z^- \in N_G(w_0)$ since W_{uv} induces a complete graph in G. This contradicts the minimality of $d_{T_2}(v, z)$. Hence $z^- \notin W_{uv}$.

Let T_1' and T_2' be the two components of $(T_1 \cup T_2) - zz^- + w_0z$ with $u \in V(T_1')$ and $v \in V(T_2')$. Then (T_1', T_2') satisfies the conditions (1) and (2) of Claim 1. Moreover, since $V(T_1) \subset V(T_1')$, $|V(T_1')| \geq 2$.

Assume $|V(T_2')| \geq 2$. Since $L(T_1') \cup L(T_2') \subset (L(T_1) \cup L(T_2)) - \{w_0\}) \cup \{z^-\}$, $|L(T_1')| + |L(T_2')| \leq k$. Therefore, (T_1', T_2') satisfies all the conditions of Claim 1. On the other hand, since $w_0 \in W_{uv}$ and $z^- \notin W_{uv}$, $|(L(T_1') \cup L(T_2')) \cap W_{uv}| = |(L(T_1) \cup L(T_2)) \cap W_{uv}| - 1$. This contradicts the minimality of $|(L(T_1) \cup L(T_2)) \cap W_{uv}|$. Therefore, we have $|V(T_2')| = 1$, or $V(T_2') = \{v\}$. This implies $N^+_{T_2}(v) = \{z\}$. By Claim 2, $z \notin L(T_2)$. Let $v' \in L(T_2)$. Then $z \in V(vT_2v')$. If $v' \in W_{uv}$, then $w_0v' \in E(G)$ since $G[W_{uv}]$ is complete. However, this contradicts Claim 2. Hence $v' \notin W_{uv}$. Also by Claim 2, we have $v' \notin N_G(u)$. Therefore, $v' \in N_G(v)$. Let $T' = (T_1 \cup T_2) - \{vz\} + \{vv', w_0z\}$. Then since $z \in V(vT_2v')$, T' is a spanning tree of G. Moreover, since $\mathrm{End}(T') \subset (\mathrm{End}(T) - \{w_0, v'\}) \cup \{u, v\}$, T' is a k-ended tree. This contradicts the assumption, and we have $N_G(w_0) \subset V(T_1)$.

Since W_{uv} induces a complete graph in G and $w_0 \in W_{uv}$, we have $W_{uv} \subset V(T_1)$. □

Claim 5. $N_G(w_0) \subset V(uT_1w_0)$. *In particular,* $W_{uv} \subset V(uT_1w_0)$.

Proof. Assume $N_G(w_0) \not\subset V(uT_1w_0)$, and choose $z \in N_G(w_0) - V(uT_1w_0)$ so that $d_{T_1}(u, z)$ is as small as possible. Then either $z^- \notin N_G(w_0)$ or $z^- \in V(uT_1w_0)$. Let $T_1' = T_1 - zz^- + zw_0$. Then T_1' is a tree with $V(T_1') = V(T_1)$. Since $L(T_1') \subset (L(T_1) - \{w_0\}) \cup \{z^-\}$, $|L(T_1')| + |L(T_2)| \leq k$ and hence (T_1', T_2) satisfies the conditions (1)–(4) of Claim 1.

If $z^- \notin N_G(w_0)$, then since $G[W_{uv}]$ is complete and $w_0 \in W_{uv}$, we have $z^- \notin W_{uv}$. If $z^- \in N_G(w_0)$, then $z^- \in V(uT_1w_0)$. This implies that z^- has at least two children, z and the one in uT_1w_0, and hence $z^- \notin L(T_1')$. Therefore, in either case, we have $z^- \notin L(T_1') \cap W_{uv}$ and $|(L(T_1') \cup L(T_2)) \cap$

$W_{uv}| = |(L(T_1) \cup L(T_2)) \cap W_{uv}| - 1$. This contradicts the minimality of $|(L(T_1) \cup L(T_2)) \cap W_{uv}|$. $\square$

By Claim 5, $L(T_1) \cap W_{uv} = \{w_0\}$. Let w_1 be the vertex in W_{uv} that is closest from u in T_1.

Claim 6. *$W_{uv} = V(w_1 T_1 w_0)$.*

Proof. Assume $W_{uv} \neq V(w_1 T_1 w_0)$. Since $W_{uv} \subset V(w_1 T_1 w_0)$ by Claim 5, we can choose $x \in V(w_1 T_1 w_0) - W_{uv}$ so that x is as close to u as possible in T_1. Then $x^- \in W_{uv}$. Since W_{uv} induces a complete graph, $w_0 x^- \in E(G)$. Let $T_1' = T_1 - xx^- + w_0 x^-$. Then T_1' is a tree with $V(T_1') = V(T_1)$ and $L(T_1') \subset (L(T_1) - \{w_0\}) \cup \{x\}$, and since $x \notin W_{uv}$, we have $L(T_1') \cap W_{uv} = \emptyset$. This contradicts the minimality of $|(L(T_1) \cup L(T_2)) \cap W_{uv}|$. $\square$

Claim 7. *$N_G(w_0) \subset W_{uv} \cup \{w_1^-\}$.*

Proof. Assume the contrary. Then by Claim 5, $N_G(w_0) \cap V(u T_1 w_1^{--}) \neq \emptyset$. Let $x \in N_G(w_0) \cap V(u T_1 w_1^{--})$. Let $N_{T_1}^+(x) \cap V(u T_1 w_0) = \{y\}$. Then since $y \in V(u T_1 w_1^-)$, $y \notin W_{uv}$. Let $T_1' = T_1 - xy + xw_0$. Then T_1' is a tree with $V(T_1') = V(T_1)$ and $L(T_1') \subset (L(T_1) - \{w_0\}) \cup \{y\}$, and hence $L(T_1') \cap W_{uv} = \emptyset$. This contradicts the minimality of $|(L(T_1) \cup L(T_2)) \cap W_{uv}|$. $\square$

Let $w \in V(w_1^+ T_1 w_0) = W_{uv} - \{w_1\}$. Then $w^- \in V(w_1 T_1 w_0^-) \subset W_{uv}$ and hence $w^- w_0 \in E(G)$. Let $T^w = T_1 - ww^- + w^- w_0$. Then T^w is a tree with $V(T^w) = V(T_1)$ and $L(T^w) \subset (L(T_1) - \{w_0\}) \cup \{w\}$. By the minimality of $|(L(T_1) \cup L(T_2)) \cap W_{uv}|$, we have $L(T^w) \cap W_{uv} = \{w\}$. Then we can apply Claims 1–6 to (T^w, T_2) instead of (T_1, T_2), and obtain $N_G(w) \subset W_{uv} \cup \{w_1^-\}$ for each $w \in W_{uv} - \{w_1\}$.

If $N_G(w_1^-) \cap (W_{uv} - \{w_1\}) = \emptyset$, then $N_G(w) \subset W_{uv}$ for each $w \in W_{uv} - \{w_1\}$. Thus, w_1 is a cutvertex of G and $W_{uv} - \{w_1\}$ is a component of $G - w_1$.

Suppose $N_G(w_1^-) \cap (W_{uv} - \{w_1\}) \neq \emptyset$. Let $w_2 \in N_G(w_1^-) \cap (W_{uv} - \{w_1\})$. Then $w_2 \in V(w_1^+ T_1 w_0)$ and hence $w_2^- \in V(w_1 T_1 w_0) \subset W_{uv}$. Therefore, $w_0 w_2^- \in E(G)$. Let $T_1'' = T_1 - \{w_1^- w_1, w_2^- w_2\} + \{w_1^- w_2, w_2^- w_0\}$. Then T_1'' is a tree with $V(T_1) = V(T_1'')$ and $L(T_1'') = (L(T_1) - \{w_0\}) \cup \{w_1\}$, and we can apply Claims 1–6 to (T_1'', T_2) and obtain $N_G(w_1) \subset W_{uv} \cup \{w_1^-\}$. Now w_1^- is a cutvertex of G and W_{uv} is a component of $G - w_1^-$. ■

Two cases described in Theorem 2 can happen, and they are independent. Consider the sharpness example G for Theorem 1, which is given at the end of Section 2. In this example, G_2 is the distant area for u and v, and it is a component of $G - x$. On the other hand, no vertex in G_2 is a cutvertex of G.

For a graph satisfying the statement (1) of Theorem 2, again let G_1, G_2, T_3, x, u and v be as in the previous example, where $2 \leq r \leq n - k - 3$. Take one vertex y in $G_1 - v$ and one vertex z in G_2 and add edges ux, xy, xz and yz. Let G' be the resulting graph. Every spanning tree of G has $k - 1$ endvertices of T_3 as its endvertices. Moreover, it has at least one endvertex in $G_1 - y$ and another endvertex in $G_2 - z$. Thus, G' does not have a spanning k-ended tree. On the other hand, $G + uv$ has a spanning k-ended tree with no endvertices in G_1. In this graph, G_2 is the distant area for u and v, and $G_2 - z$ is a component of $G' - z$. On the other hand, G' does not have a cutvertex which satisfies the condition (1) of Theorem 2.

4. Closure for Hamiltonian Cycles and Distant Area

In Section 2, we have proved that if the distant area contains a clique, the Broersma-Tuinstra closure for a spanning k-ended tree can be applied under a weaker degree sum condition. We can ask a similar question for the Bondy-Chvátal closure for hamiltonian cycles, and give the following theorem.

Theorem 4. *Let n and r be positive integers. Let G be a 2-connected graph of order n and let u and v be a pair of nonadjacent vertices in G. Suppose $G[W_{uv}]$ contains a complete graph of order r and $\deg_G u + \deg_G v \geq n - r + 1$. Then G has a hamiltonian cycle if and only if $G + uv$ has a hamiltonian cycle.*

However, we can prove it as a corollary of the following theorem, which was proved by Broersma and Schiermeyer [2].

Theorem C ([2]). *Let u and v be a pair of nonadjacent vertices of a 2-connected graph G of order n with $|N_G(u) \cap N_G(v)| \geq 3$. If $|N_G(u) \cup N_G(v) \cup N_G(w)| \geq n - |N_G(u) \cap N_G(v)|$ holds for at least $|W_{uv}| + 2 - |N_G(u) \cap N_G(v)|$ vertices w in W_{uv}, then G has a hamiltonian cycle if and only if $G + uv$ has a hamiltonian cycle.*

***Proof of Theorem* 4.** Let W be a set of vertices in W_{uv} which induces K_r in G. Since $\deg_G u + \deg_G v \geq n - r + 1$, $|N_G(u) \cup N_G(v)| \geq n - r + 1 - |N_G(u) \cap$

$N_G(v)|$, and hence $|W_{uv}| = n-2-|N_G(u)\cup N_G(v)| \le r+|N_G(u)\cap N_G(v)|-3$. On the other hand, since $W \subset W_{uv}$, we have $|W_{uv}| \ge r$. Thus, we have $|N_G(u) \cap N_G(v)| \ge 3$.

Let $w \in W$. Then since $G[W] \simeq K_r$, $|N_G(w) \cap W_{uv}| \ge r-1$. Therefore,

$$\begin{aligned}|N_G(u) \cup N_G(v) \cup N_G(w)| &= |N_G(u) \cup N_G(v)| + |N_G(w) \cap W_{uv}| \\ &\ge n - r + 1 - |N_G(u) \cap N_G(v)| + r - 1 \\ &= n - |N_G(u) \cap N_G(v)|.\end{aligned}$$

Since $|W_{uv}| \le r + |N_G(u) \cap N_G(v)| - 3$, $|W_{uv}| + 2 - |N_G(u) \cap N_G(v)| \le r-1$. Since $|W| \ge r$, we see that $|N_G(u) \cup N_G(v) \cup N_G(w)| \ge n - |N_G(u) \cap N_G(v)|$ holds for at least $|W_{uv}| + 2 - |N_G(u) \cap N_G(v)|$ vertices w in W_{uv}. Thus, the theorem follows from Theorem C. ■

In Section 3, we have proved that if u and v are a pair of nonadjacent vertices in a 2-connected graph G and W_{uv} induces a complete graph, then G has a spanning k-ended tree if and only if $G + uv$ has a spanning k-ended tree (Corollary 3). An analogue for a hamiltonian cycle has been proved by Zhu, Tian and Deng [5].

Theorem D ([5]). *Let u and v be a pair of nonadjacent vertices in a 3-connected graph G. If $G[W_{uv}]$ is complete, then G has a hamiltonian cycle if and only if $G + uv$ has a hamiltonian cycle.*

5. Conclusion

In this paper, we have investigated the relationship between the degree sum condition of the Broersma-Tuinstra closure and the order of a clique in the distant area. We have proved that if the distant area contains K_r, then we can relax the degree sum condition of their closure theorem. Moreover, if the distant area is complete and the graph is 2-connected, we can entirely remove the degree sum condition. We have also considered a similar problem for the Bondy-Chvátal closure for hamiltonian cycles and studied the relationship with the previous work.

We have proved in Section 4 that Theorem C implies Theorem 4. We suspect that Theorem 1 admits a similar generalization to a closure based on the neighborhood union of independent triples. Also, in Section 3, we have given some structural information of the distant area when it induces a complete graph but the closure fails. We believe that Theorem D admits a similar extension.

References

[1] J.A. Bondy and V. Chvátal, *A method in graph theory*, Discrete Math. **15** (1976) 111–135.

[2] H.J. Broersma and I. Schiermeyer, *A closure concept based on neighborhood unions of independent triples*, Discrete Math. **124** (1994) 37–47.

[3] H. Broersma and H. Tuinstra, *Independence trees and Hamilton cycles*, J. Graph Theory **29** (1998) 227–237.

[4] G. Chartrand and L. Lesniak, Graphs & Digraphs (4th ed.), (Chapman and Hall/CRC, Boca Raton, Florida, U.S.A. 2005).

[5] Y.J. Zhu, F. Tian and X.T. Deng, *Further consideration on the Bondy-Chvátal closure theorems*, in: Graph Theory, Combinatorics, Algorithms, and Applications (San Francisco, CA, 1989), 518–524.

Received 29 September 2009
Accepted 19 April 2010