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THE INDEPENDENT DOMINATION NUMBER OF A RANDOM GRAPH

Lane Clark and Darin Johnson

Department of Mathematics
Southern Illinois University Carbondale
Carbondale, IL 62901–4408, USA

Abstract

We prove a two-point concentration for the independent domination number of the random graph $G_{n,p}$ provided $p^2 \ln(n) \geq 64\ln((\ln n)/p)$.

Keywords: random graph, two-point concentration, independent domination.

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1. Introduction

Let G be a graph with vertex set $[n]$ and let $S \subset [n]$. If for every vertex $u \notin S$ there is a vertex $v \in S$ such that u and v are adjacent then S is called a *dominating set*. If further for every $v, w \in S$ there is no edge between v and w then S is called an *independent dominating set*. The *domination number*, $\gamma(G)$ is the smallest integer s such that there exists a dominating set of cardinality s. The *independent domination number*, $i(G)$ is the smallest integer s such that there exists an independent dominating set of cardinality s. $\mathcal{G}(n,p)$ is the set of all graphs $G_{n,p}$ with vertex set $[n]$ and edges chosen independently with probability $0 \leq p = p(n) \leq 1$. Hence, for each $G_{n,p}$ $P(G_{n,p}) = p^{e(G_{n,p})}(1-p)^{\binom{n}{2}-e(G_{n,p})}$. For a graph property A we say A occurs *asymptotically almost surely* (a.a.s.) if $P(G_{n,p}$ has property $A) \to 1$ as $n \to \infty$. See Bollobás [2] for notation and terminology.

Weber [7] showed if $p = 1/2$ then a.a.s. $\gamma(G_{n,p})$ is either $\lfloor \log_2 n - \log_2(\log_2 n \ln n)\rfloor + 1$ or $\lfloor \log_2 n - \log_2(\log_2 n \ln n)\rfloor + 2$ and a.a.s. $i(G_{n,p})$

is $\lfloor \log_2 n - \log_2(\log_2 n \ln n) \rfloor + 2$ or $\lfloor \log_2 n - \log_2(\log_2 n \ln n) \rfloor + 3$. Godbole and Wieland [4] extended Weber's result showing if p is constant or $p = p(n) \to 0$ such that $p^2 \ln n \geq 40 \ln((\ln^2 n)/p)$ then a.a.s. $\gamma(G_{n,p})$ is either $\lfloor \log_b n - \log_b(\log_b n \ln n) \rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n \ln n) \rfloor + 2$, where $b = 1/(1-p)$. Very recently Bonato and Wang [3] showed that if p is constant then a.a.s. $\lfloor \log_b n - \log_b(\log_b n \ \ln n) \rfloor + 1 \leq i(G_{n,p}) \leq \lfloor \log_b n \rfloor$. In this paper we show that if p is constant or $p = p(n) \to 0$ such that $p^2 \ln(n) \geq 64 \ln(\ln(n)/p)$ then a.a.s. $i(G_{n,p})$ is either $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2$. This extends Weber's result (the case $p = 1/2$) and immediately implies Bonato and Wang's result (the case p is constant). We then empirically explore the number of independent dominating sets of size k ranging on $[n]$ and make a conjecture about the distribution.

2. TWO-POINT CONCENTRATION

Throughout this section we will use p as the probability an edge exists in $G = G_{n,p}$, $q = 1 - p$ the probability an edge does not exist in G and $b = \frac{1}{q}$. We will also make extensive use of two inequalities,

$$1 - x \leq \exp\{-x\}, x \in \mathbb{R}, \tag{1}$$

$$1 - x \geq \exp\left\{\frac{-x}{1-x}\right\}, \ x \in [0, 1). \tag{2}$$

We begin by defining the random variables X_k and Y_s as the number of independent dominating sets of cardinality k in G and the number of independent dominating sets of cardinality s or less in G respectively. Clearly $Y_s = \sum_{k=1}^{s} X_k$. It is now obvious that

$$E(X_k) = \binom{n}{k}(1 - q^k)^{n-k} q^{\binom{k}{2}}$$

and by linearity of expectation,

$$E(Y_s) = \sum_{k=1}^{s} E(X_k) = \sum_{k=1}^{s} \binom{n}{k}(1 - q^k)^{n-k} q^{\binom{k}{2}}.$$

We now state our first lemma.

Lemma 2.1. *Let* $s = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor$, *then* $E(Y_s) \to 0$ *if* p *fixed or if* $p \to 0$ *as* $n \to \infty$ *and* $p \geq \frac{e \ln^2 n}{n}$.

Proof. Lemma 2 of [4] states the expected number of dominating sets of size less than or equal to $r = \lfloor \log_b n - \log_b (\log_b n \ln n) \rfloor$ goes to 0 if $p \geq \frac{e \ln^2 n}{n}$. Since every independent dominating set is a dominating set it is clear $E(Y_r) \to 0$ as $n \to \infty$. It remains to show,

$$\sum_{k=r+1}^{s} E(X_k) \to 0.$$

Using Stirling's inequality, inequality (1),

$$\begin{aligned} E(X_k) &= \binom{n}{k} (1-q^k)^{n-k} q^{\binom{k}{2}} \\ &\leq \exp\left\{ k \ln n + 2k - k \ln k - nq^k + \frac{k^2}{2} \ln q - \frac{k}{2} \ln q \right\} \\ &:= \exp\{f(k)\}. \end{aligned}$$

Now,

$$f'(k) = \ln n + 1 - \ln k + nq^k \ln\left(\frac{1}{q}\right) - k \ln\left(\frac{1}{q}\right) - \frac{1}{2} \ln\left(\frac{1}{q}\right).$$

Note $f'(k)$ is decreasing for all positive value of k and $f'(\log_b n - \log_b (\log_b n \ \ln n) + \log_b 2) \geq 0$ for sufficiently large n. So for sufficiently large n, we have $f(k)$ increasing for all $k \leq \log_b n - \log_b(\log_b n \ \ln n) + \log_b 2$. Hence, setting $k = \log_b n - \log_b(\log_b n \ \ln n) + \log_b 2$ we have

$$\begin{aligned} E(Y_s) &\leq (k-r) \exp\{f(k)\} \\ &\leq (\log_b 2) \exp\left\{ k \ln n + 2k - k \ln k - nq^k + \frac{k^2}{2} \ln q - \frac{k}{2} \ln q \right\} \\ &\leq (\log_b 2) \exp\left\{ -k \ln k + 3k + \frac{k}{2} \ln\left(\frac{1}{q}\right) \right\} \\ &\to 0 \end{aligned}$$

since $k \ln k$ clearly dominates the other two terms in the exponent.

We now note that since $i(G)$ is always at least 1,

$$\log_b n - \log_b(\log_b n \ln n) + \log_b 2 \geq 1.$$

A condition satisfied if

$$p \geq \frac{e \ln^2 n}{2n}$$

which is easily seen after noting $p \leq \ln \frac{1}{q}$. However, the condition $p \geq \frac{e \ln^2 n}{n}$ used above is stronger so we must use it instead. ■

Lemma 2.2. *If p fixed or if $p \to 0$ and $\frac{p^2}{64} \geq \frac{\ln\left(\frac{\ln n}{p}\right)}{\ln n}$ then $E(X_s) \to \infty$ for $s = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2$.*

Proof. Using inequality (2), Stirling's Formula, and that for $k^2 = o(n)$ $(n)_k = (1 - o(1))n^k$

$$\begin{aligned}
&E(X_k) \\
&= \binom{n}{k}(1-q^k)^{n-k} q^{\binom{k}{2}} \\
&\geq \binom{n}{k}(1-q^k)^{n} q^{\frac{k^2}{2}} \\
(3)\quad &\geq \binom{n}{k} \exp\left\{\frac{-nq^k}{1-q^k} + \frac{k^2}{2}\ln q\right\} \\
&\geq (1-o(1))\frac{n^k}{k!} \exp\left\{\frac{-nq^k}{1-q^k} + \frac{k^2}{2}\ln q\right\} \quad (\text{if } k^2 = o(n)) \\
&\geq (1-o(1))\left(\frac{ne}{k}\right)^k (2\pi k)^{-\frac{1}{2}} \exp\left\{\frac{-nq^k}{1-q^k} + \frac{k^2}{2}\ln q\right\} \quad (\text{if } k \to \infty) \\
&\geq (1-o(1)) \exp\left\{k\ln n + k - k\ln k - \frac{1}{2}\ln(2\pi k) - \frac{nq^k}{1-q^k} + \frac{k^2}{2}\ln q\right\}.
\end{aligned}$$

The condition $k^2 = o(n)$ is satisfied if $p \gg \frac{\ln n}{n^{\frac{1}{2}}}$ and $k = \log_b n - \log_b (\log_b n \ln n) + \log_b 2 + \epsilon$, where $\epsilon > 0$. One can easily show $\frac{d}{dk}\left(k\ln n + k - k\ln k - \frac{1}{2}\ln(2\pi k) - \frac{nq^k}{1-q^k} + \frac{k^2}{2}\ln q\right) \geq 0$ as long as k is much smaller than nq^k, which is true for large n when assuming the just mentioned conditions. Substituting in (3) $k = s$ on the left and $k =$

$\log_b n - \log_b(\log_b n \ln n) + \log_b 2 + \frac{1}{2}$ on the right it is shown for sufficiently large n

$$\begin{aligned} E(X_s) \geq & \ (1-o(1))\exp\left\{\frac{1}{2}\log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{1-q^k}\right) + \log_b n \right. \\ & - \log_b(\log_b n \ln n)\ln(\log_b n \ln n) + \frac{1}{2}\ln(\log_b n \ln n) \\ & \left. - \log_b n \ln s - (1+\log_b 2)\ln s - \frac{1}{2}\ln 2\pi - \frac{1}{8}\ln\frac{1}{q}\right\} \\ \geq & \ (1-o(1))\exp\{A - B\}, \end{aligned}$$

where

$$\begin{aligned} A &= \frac{1}{2}\log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{1-q^s}\right) + \log_b n, \\ B &= \log_b(\log_b n \ln n)\ln(\log_b n \ln n) + \log_b n \ln(\log_b n), \\ &\quad + (1+\log_b 2)\ln(\log_b n) + \frac{1}{2}\ln(2\pi e^L) \end{aligned}$$

and L is any constant bounding $\frac{1}{8}\ln\frac{1}{q}$, which exists since $\ln\left(\frac{1}{q}\right)$ is constant or $\ln\left(\frac{1}{q}\right) \to 0$. Since $p \gg \frac{\ln n}{n^{\frac{1}{2}}}$ and $\log_b n \sim \frac{\ln n}{p}$ we have $p \gg \frac{\log_b n \ln n}{n}$. So for n sufficiently large,

$$\begin{aligned} A &= \frac{1}{2}\log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{(1-q^s)}\right) + \log_b n \\ &= \frac{1}{2}\log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{(1 - \frac{q^{\frac{1}{2}}\log_b n \ln n}{2n})}\right) + \log_b n \\ &\geq \frac{1}{2}\log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{(1 - \frac{pq^{\frac{1}{2}}}{2})}\right) + \log_b n \\ &= \frac{1}{2}\log_b n \ln n \left(\frac{1 - \frac{pq^{\frac{1}{2}}}{2} - q^{\frac{1}{2}}}{1 - \frac{pq^{\frac{1}{2}}}{2}}\right) + \log_b n. \end{aligned}$$

Using the inequality, $\frac{x}{2} \leq 1 - (1-x)^{\frac{1}{2}}$, we obtain

$$\begin{aligned} A &\geq \frac{p}{4} \log_{\mathrm{b}} n \ln n \left(\frac{1-(1-p)^{\frac{1}{2}}}{1-\frac{pq^{\frac{1}{2}}}{2}} \right) + \log_{\mathrm{b}} n \\ &\geq \frac{p}{4} \log_{\mathrm{b}} n \ln n \left(\frac{\frac{p}{2}}{1-\frac{pq^{\frac{1}{2}}}{2}} \right) + \log_{\mathrm{b}} n \\ &\geq \frac{p^2}{8} \log_{\mathrm{b}} n \ln n + \log_{\mathrm{b}} n. \end{aligned}$$

Define C as:

$$C = \frac{p^2 \log_{\mathrm{b}} n \ln n}{8} + \log_{\mathrm{b}} n. \tag{4}$$

We will now find p such that for n sufficiently large $\frac{C}{8}$ is larger than all terms in B. Hence

$$\begin{aligned} (1-o(1)) \exp\{A-B\} &\geq (1-o(1)) \exp\{C-B\} \\ &\geq (1-o(1)) \exp\{C/2\} \\ &\to \infty. \end{aligned} \tag{5}$$

It is obvious that the third and fourth terms of B are dominated by the first so we will only compare the first and second terms to $C/8$. Comparing the first term,

$$C/8 \geq \frac{1}{2} \log_{\mathrm{b}} (\log_{\mathrm{b}} n \ln n) \ln (\log_{\mathrm{b}} n \ln n)$$

if for sufficiently large n

$$\frac{p}{8} \geq \frac{\ln\left(\frac{\ln^2 n}{p}\right)}{\sqrt{2} \ln n}. \tag{6}$$

Comparing the second term,

$$C/8 \geq \log_{\mathrm{b}} n \ln (\log_{\mathrm{b}} n)$$

if for sufficiently large n

$$\frac{p^2}{64} \geq \frac{\ln\left(\frac{\ln n}{p}\right)}{\ln n}. \tag{7}$$

Clearly (7) implies (6) and the condition $p \gg \frac{\ln n}{n^{\frac{1}{2}}}$ and the lemma is proven. ■

Lemma 2.3. *If p fixed or if $p \to 0$ and $\frac{p^2}{64} \geq \frac{\ln\left(\frac{\ln n}{p}\right)}{\ln n}$ then $\frac{Var X_s}{E^2(X_s)} \to 0$ for* $s = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2$.

Proof. Following the proof of Lemma 3 in [4] it is easily derived that

$$\begin{aligned} Var(X_s) \leq\ & E(X_s) - E^2(X_s) \\ & + \binom{n}{s} \sum_{m=0}^{s-1} \binom{s}{m} \binom{n-s}{s-m} \left(1 - 2q^s + q^{2s-m}\right)^{n-2s+m} q^{2\binom{s}{2} - \binom{m}{2}}. \end{aligned}$$

We write $s = \log_b n - \log_b(\log_b n \ln n) + \log_b 2 + \epsilon$ where $\epsilon = \epsilon(n) = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2 - \log_b n + \log_b(\log_b n \ln n) - \log_b 2$ and observe that $1 \leq \epsilon \leq 2$.

It is immediately obvious for any s such that $E(X_s) \to \infty$,

$$E(X_s) = o(E^2(X_s)).$$

We will now show

$$\binom{n}{s}\binom{s}{0}\binom{n-s}{s}\left(1 - 2q^s + q^{2s}\right)^{n-2s} q^{2\binom{s}{2}} - E^2(X_s) = o(E^2(X_s)) \tag{8}$$

and

$$\begin{aligned} & \binom{n}{s} \sum_{m=1}^{s-1} \binom{s}{m}\binom{n-s}{s-m}\left(1 - 2q^s + q^{2s-m}\right)^{n-2s+m} q^{2\binom{s}{2} - \binom{m}{2}} \\ & = o(E^2(X_s)). \end{aligned} \tag{9}$$

To show (8) note,

$$\begin{aligned} & \binom{n}{s}\binom{s}{0}\binom{n-s}{s}\left(1 - 2q^s + q^{2s}\right)^{n-2s} q^{2\binom{s}{2}} - E^2(X_s) \\ & \leq E^2(X_s)\left((1-q^s)^{-2s} - 1\right) \\ & \leq E^2(X_s)\left(\exp\left\{\frac{2sq^s}{1-q^s}\right\} - 1\right) \quad \text{(by (2))}. \end{aligned}$$

Since $p \gg \frac{\ln^{\frac{3}{2}} n}{n^{\frac{1}{2}}}$, we know that $\frac{2sq^s}{1-q^s} \geq 0$ and approaches 0 as $n \to \infty$. Thus,

$$\left(\exp\left\{\frac{2sq^s}{1-q^s}\right\} - 1\right) \to 0.$$

To show (9) let

$$f(m) = \binom{s}{m}\binom{n-s}{s-m}\left(1 - 2q^s + q^{2s-m}\right)^{n-2s+m} q^{2\binom{s}{2}-\binom{m}{2}}$$

and note for sufficiently large n

$$\begin{aligned} f(m) &\leq \binom{s}{m}\frac{n^{s-m}}{(s-m)!}\left(1 - 2q^s + q^{2s-m}\right)^{n-2s+m} q^{2\binom{s}{2}-\binom{m}{2}} \\ &\leq 2\binom{s}{m}\frac{n^{s-m}}{(s-m)!}\left(1 - 2q^s + q^{2s-m}\right)^{n} q^{2\binom{s}{2}-\binom{m}{2}} \\ &\leq 2\binom{s}{m}\frac{n^{s-m}}{(s-m)!}\exp\left(n(-2q^s + q^{2s-m})\right) q^{2\binom{s}{2}-\binom{m}{2}} \quad \text{(by (1))} \end{aligned}$$

where the second inequality holds for $p \gg \frac{\ln \frac{3}{2} n}{n^{\frac{1}{2}}}$. Define

$$g(m) := 2\binom{s}{m}\frac{n^{s-m}}{(s-m)!}\exp\left(n(-2q^s + q^{2s-m})\right) q^{2\binom{s}{2}-\binom{m}{2}}$$

and consider the the ratio of consecutive terms of $g(m)$.

$$h(m) := \frac{g(m+1)}{g(m)} = \frac{(s-m)^2}{nq^m(m+1)}\exp\left\{npq^{2s-m-1}\right\}. \tag{10}$$

We will show $h(m) \geq 1$ iff $m \geq m_0$ for some $m_0(n) \to \infty$, hence g is first decreasing and then increasing. Further we will show $g(1) \geq g(s-1)$, which implies $\sum_{m=1}^{s-1} f(m) \leq sg(1)$. Observe for sufficiently large n,

$$\begin{aligned} h(1) &= \frac{(s-1)^2}{2nq}\exp\left\{\frac{np}{q^2}q^{2s}\right\} \\ &\leq \frac{{\log_b}^2 n}{2nq}\exp\left\{\frac{(\log_b n \ln n)^2 p}{4nq^{2-2\epsilon}}\right\} \end{aligned}$$

$$\leq \frac{\ln^2 n}{2np^2q} \exp\left\{\frac{\ln^4 n}{4npq^{2-2\epsilon}}\right\} \quad \text{(by (1))}$$

$$\to 0$$

since $p \gg \frac{\ln n}{n^{\frac{1}{2}}}$ and

$$h(s-1) = \frac{1}{nsq^{s-1}} \exp\{npq^s\}$$

$$\geq \frac{2q^{1-\epsilon}}{\log_{\mathrm{b}}{}^2 n \ln n} \exp\left\{\frac{pq^\epsilon \log_{\mathrm{b}} n \ln n}{2}\right\}$$

$$= \frac{2q^{1-\epsilon} \ln^2 \frac{1}{q}}{\ln^3 n} \exp\left\{\frac{pq^\epsilon \ln^2 n}{-2\ln q}\right\}$$

$$\geq \frac{2q^{1-\epsilon}p^2}{\ln^3 n} \exp\left\{\frac{q^{1+\epsilon}\ln^2 n}{2}\right\} \quad \text{(by (1), (2))}$$

$$\geq 1$$

provided $p \neq 1 - o(1)$. Also,

$$h(m) = \frac{(s-m)^2}{n(m+1)q^m} \exp\left\{npq^{2s-m-1}\right\} \geq 1$$

iff

$$npq^{2s-m-1} \geq \ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)$$

iff

$$m \geq \log_{\mathrm{b}}\left(\frac{\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)}{npq^{2s-1}}\right)$$

iff

$$m \geq \log_{\mathrm{b}}\left(\frac{4n\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)}{p\log_{\mathrm{b}}{}^2(n)\ln^2(n)}\right) + 2\epsilon - 1$$

iff

$$m \geq \log_{\mathrm{b}}\left(\frac{4n}{p}\right) + \log_{\mathrm{b}}\left(\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)\right) - 2\log_{\mathrm{b}}(\ln n \log_{\mathrm{b}} n) + 2\epsilon - 1.$$

Define

$$x(m) = \log_b\left(\frac{4n}{p}\right) + \log_b\left(\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)\right) - 2\log_b(\ln n \log_b n) + 2\epsilon - 1.$$

Now,

$$\frac{d}{dm}x(m) = \frac{\left(m^2 - \left(s - 1 - \frac{1}{\ln\left(\frac{1}{q}\right)}\right)m - \left(1 - \frac{1}{\ln\left(\frac{1}{q}\right)}\right)s + \frac{2}{\ln\left(\frac{1}{q}\right)}\right)}{(m+1)(s-m)\left(\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)\right)}$$

and the roots of the numerator are:

$$\frac{s - 1 - \frac{1}{\ln\left(\frac{1}{q}\right)} \pm \sqrt{\left(s - 1 - \frac{1}{\ln\left(\frac{1}{q}\right)}\right)^2 + 4s\left(1 - \frac{1}{\ln\left(\frac{1}{q}\right)}\right) - \frac{8}{\ln\left(\frac{1}{q}\right)}}}{2}$$

$$= \frac{s - 1 - \frac{1}{\ln\left(\frac{1}{q}\right)} \pm (s+1)\sqrt{\left(1 - \frac{3}{(s+1)\ln\left(\frac{1}{q}\right)}\right)^2 - \frac{8}{(s+1)\ln^2\left(\frac{1}{q}\right)}}}{2}.$$

Using Taylor Series with remainder about 0, one can show if $0 \le z \le 3 - 2\sqrt{2}$ then for any y such that $|y| \le z$

$$1 - 3y - \frac{8z^2}{(1 - 6z + z^2)^{\frac{3}{2}}} \le \sqrt{(1-3y)^2 - 8y^2} \le 1 - 3y + \frac{8z^2}{(1 - 6z + z^2)^{\frac{3}{2}}}.$$

Letting $y = z = \frac{1}{(s-1)\ln\left(\frac{1}{q}\right)}$, we show

$$\frac{d}{dm}x(m) = \frac{\left(m + 1 - \frac{1}{\ln\left(\frac{1}{q}\right)} - \delta\right)\left(m - s + \frac{2}{\ln\left(\frac{1}{q}\right)} + \delta\right)}{(m+1)(s-m)\left(\ln\left(\frac{n(m+1)q^m}{(s-m)^2}\right)\right)}$$

$$\text{where } |\delta| \le \frac{8}{(s+1)\ln^2\left(\frac{1}{q}\right)\left(1 - \frac{6}{(s+1)\ln\left(\frac{1}{q}\right)} + \frac{1}{(s+1)^2\ln^2\left(\frac{1}{q}\right)}\right)^{\frac{3}{2}}}.$$

Thus $\delta = \Theta\left(\frac{1}{p \ln n}\right) \to 0$ as $n \to \infty$ since $p \gg \frac{1}{\ln(n)}$.

So on $(-\infty, -1)$ and $\left(\ln^{-1}\left(\frac{1}{q}\right) - 1 + \delta, s - 2\ln^{-1}\left(\frac{1}{q}\right) - \delta\right)$ $x(m)$ is decreasing and on $\left(-1, \ln^{-1}\left(\frac{1}{q}\right) - 1 + \delta\right)$ and $\left(s - 2\ln^{-1}\left(\frac{1}{q}\right) - \delta, s\right)$ $x(m)$ is increasing. Thus $m_1 = \ln^{-1}\left(\frac{1}{q}\right) - 1 + \delta$ is a relative maximum and $m_2 = s - 2\ln^{-1}\left(\frac{1}{q}\right) - \delta$ is a relative minimum of $x(m)$.

Note $m_1 \in [1, s-1]$ iff $p \le 1 - e^{-\frac{1}{2-\delta}}$ and $m_2 \in [1, s-1]$ iff $p \le 1 - e^{-\frac{2}{1-\delta}}$. Also for n sufficiently large, $x(m)$ is continuous on $[1, s-1]$, for every $m \in [1, s-1]$ $x(m) \in [1, s-1]$, and $s - 1 > x(1) > x(s-1) > 1$.

If $p > 1 - e^{-\frac{2}{1-\delta}}$, on $[1, s-1]$ $x(m)$ has an absolute maximum at 1 and an absolute minimum at $s-1$. So by the above information and the intermediate value theorem there exists a unique $m_0 \in [1, s-1]$ such that $m_0 = x(m_0)$ and $x(m_0) > x(s-1)$.

If $1 - e^{-\frac{1}{2-\delta}} < p \le 1 - e^{-\frac{2}{1-\delta}}$, on $[1, s-1]$ $x(m)$ has an absolute maximum at 1 and an absolute minimum at m_2. So by the above information and the intermediate value theorem there exists a unique $m_0 \in [1, s-1]$ such that $m_0 = x(m_0)$. Further, one can show by iteration that $x(m_0) \ge x(s-1)$.

If $p \le 1 - e^{-\frac{1}{2-\delta}}$ or $p \to 0$, on $[1, s-1]$ $x(m)$ has an absolute maximum at m_1 and an absolute minimum at m_2. So by the above information and the intermediate value theorem there exists a unique $m_0 \in [1, s-1]$ such that $m_0 = x(m_0)$. Further, one can show by iteration that $x(m_0) \ge x(s-1)$.

Thus, in any of the three cases there exists a unique $m_0 \in [1, s-1]$ such that $\forall m \ge m_0 = x(m_0) \ge x(m)$.

Now, for n sufficiently large

$$\ln\left(\frac{n(m_0+1)q^{m_0}}{(s-m_0)^2}\right) \ge \ln\left(nsq^{s-1}\right) \ge \ln\left(\frac{\log_b(n)\ln(n)s}{4q^{1-\epsilon}}\right)$$

which goes to infinity as n goes to infinity. Also, $\log_b\left(\frac{4n}{p}\right) \gg 2\log_b$ $(\log_b(n)\ln(n))$ and $2\epsilon - 1$ is bounded, thus $m_0 \to \infty$. Therefore, $h(m) \ge 1$ iff $m \ge m_0 \to \infty$ as $n \to \infty$.

Also, $g(1) \ge g(s-1)$ iff

$$\frac{n^{s-1}}{(s-1)!}\exp\left\{nq^{2s-1}\right\} \ge n\exp\left\{nq^{s+1}\right\}q^{-\binom{s-1}{2}}$$

iff

$$\frac{n^s}{s!}\exp\left\{-n(q^{s+1} - q^{2s-1})\right\}q^{\binom{s-1}{2}} \ge \frac{n^2}{s}$$

which is true since

$$\frac{n^s}{s!}\exp\left\{-n(q^{s+1}-q^{2s-1})\right\}q^{\binom{s-1}{2}} \geq E(X_s) \geq (1-o(1))\exp\{C/2\}$$

where $C = \frac{p^2 \log_b n \ln n}{8} + \log_b n$ and $(1-o(1))\exp\{C/2\} \geq n^2/s$ if $p \geq \frac{24}{\ln n}$, a condition clearly satisfied by our hypothesis. Hence we have shown,

$$\binom{n}{s}\sum_{m=1}^{s-1}\binom{s}{m}\binom{n-s}{s-m}\left(1-2q^s-q^{2s-m}\right)^{n-2s+m}q^{2\binom{s}{2}-\binom{m}{2}} \leq s\binom{n}{s}g(1).$$

Finally, we show $s\binom{n}{s}g(1) = o(E^2(X_s))$,

$$\begin{aligned}
\frac{s\binom{n}{s}g(1)}{E^2(X_s)} &= \frac{2s^2n^{s-1}\exp\{n(q^{2s-1}-2q^s)\}}{\binom{n}{s}(1-q^s)^{2(n-s)}(s-1)!} \\
&\leq \frac{2s^3\exp\{n(q^{2s-1}-2q^s)\}}{(1-o(1))n(1-q^s)^{2n}} \; (s^2 = o(n)) \\
&\leq \frac{2s^3}{(1-o(1))n}\exp\left\{n\left(q^{2s-1}-2q^s+\frac{2q^s}{1-q^s}\right)\right\} \; (\text{by } (2)) \\
&\leq \frac{2s^3}{(1-o(1))n}\exp\left\{\frac{nq^{2s-1}(1+2q)}{1-q^s}\right\} \\
&\leq \frac{2\log_b{}^3 n}{(1-o(1))n}\exp\left\{\frac{3\log_b{}^2 n\ln^2 n}{4n(1-q^s)}q^{2\epsilon-1}\right\} \\
&\to 0
\end{aligned}$$

since $p \gg \frac{\ln n}{n^{\frac{1}{3}}}$.

We have thus shown if $s = \log_b n - \log_b(\log_b n \ln n) + \log_b 2 + \epsilon = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2\rfloor + 2$ then $Var(X_s) = o(E^2(X_s))$ provided $\frac{p^2}{64} \geq \frac{\ln\left(\frac{\ln n}{p}\right)}{\ln n}$. ∎

We now can state our main result.

Theorem 2.4. *Let p be fixed or $\frac{p^2}{64} \geq \frac{\ln\left(\frac{\ln n}{p}\right)}{\ln n}$ then $i(G)$ is equal to $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2\rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2\rfloor + 2$ a.a.s.*

Proof. By Markov's Inequality and Lemma 2.1 if $s = \lfloor \log_b n - \log_b (\log_b n \ln n) + \log_b 2 \rfloor$ then

$$P(i(G) \leq s) = P(Y_s \geq 1) \leq E(Y_s) \to 0$$

and by Chebyshev's Inequality, Lemma 2.2, and Lemma 2.3 if $s = \lfloor \log_b n - \log_b (\log_b n \ln n) + \log_b 2 \rfloor + 2$ then

$$P(i(G) > s) \leq P(X_s = 0) \leq P(|X_s - E(X_s)| \geq E(X_s)) \leq \frac{Var(X_s)}{E^2(X_s)} \to 0.$$

■

3. Empirical Data

In this section we used a standard random number generator which we verified to return a nearly uniform distribution for samples of size 10^6 lending some credibility to the empirical results.

We generated $N = 1000$ graphs G of order n, by randomly assigning edges with probability p, using a standard uniform random number generator. We then ran an algorithm of our design to count the number of graphs with an independent domination number of size k for each $k \in [n]$. The results are presented in the chart below for $n = 16, 32, 64,$ and 128 with probabilities $p = .5$ and .75.

n, p	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
16, .5	0	522	476	2	0
32, .5	0	36	962	2	0
64, .5	0	0	693	307	0
64, .75	0	998	2	0	0
128, .75	0	419	561	0	0

We generated $N = 1000$ graphs G of order n, by randomly assigning edges with probability p, using a standard uniform random number generator. We then ran an algorithm of our design to calculate the average number of independent dominating sets of size k contained in all G for each $k \in [n]$. The results are presented in the chart below for $n = 12, 16, 20, 24$ and $p = .5$.

n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
12	0.1	1.96	7.87	4	0.56	.11	0	0
16	0	1.12	13.02	12.98	3.91	.14	0	0
20	0	0.55	13.08	26.33	8.83	1.39	0	0
24	0	0.17	15.81	45.94	21.6	2.66	0.05	0.01

This data appears to imply the following conjecture:

Conjecture 3.1. $Y = X_1 + X_2 + \cdots + X_n$ suitably normalized converges weakly to $N(0,1)$.

We have no proof to date. The random variables $\{X_k\}$ are highly dependent posing difficulty in verifying the conditions needed in many central limit theorems for dependent sums.

References

[1] N. Alon and J. Spencer, The Probabilistic Method (John Wiley, New York, 1992).

[2] B. Bollobás, Random Graphs (Second Edition, Cambridge University Press, New York, 2001).

[3] A. Bonato and C. Wang, *A note on domination parameters in random graphs*, Discuss. Math. Graph Theory **28** (2008) 307–322.

[4] A. Godbole and B. Wieland, *On the domination number of a Random graph*, Electronic J. Combin. **8** (2001) 1–13.

[5] T. Haynes, S. Hedetniemi and P. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).

[6] T. Haynes, S. Hedetniemi and P. Slater, Domination in Graphs: Advanced Topics (Marcel Dekker, Inc., New York, 1998).

[7] K. Weber, *Domination number for almost every graph*, Rostocker Matematisches Kolloquium **16** (1981) 31–43.

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