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A MAGICAL APPROACH TO SOME LABELING CONJECTURES

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Dedicated to our mentor and friend Professor Arthur T. White

Abstract

In this paper, a complete characterization of the (super) edge-magic linear forests with two components is provided. In the process of establishing this characterization, the super edge-magic, harmonious, sequential and felicitous properties of certain 2-regular graphs are investigated, and several results on super edge-magic and felicitous labelings of unions of cycles and paths are presented. These labelings resolve one conjecture on harmonious graphs as a corollary, and make headway towards the resolution of others. They also provide the basis for some new conjectures (and a weaker form of an old one) on labelings of 2-regular graphs.

Keywords: edge-magic labelling, edge-magic total labelling, felicitous labelling, harmonious labelling, sequential labelling.

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1. Introduction

For most of the graph theory terminology and notation utilized throughout this paper, we will follow Chartrand and Lesniak [2]. In particular, we will consider finite and simple graphs, that is, there are no loops and multiple edges.

The thrust of this paper is towards giving a complete characterization of the (super) edge-magic linear forest with two components: $P_m \cup P_n$. However, as the authors assembled the necessary results for this characterization, they were pleased to realize that a conjecture on harmonious labelings was settled and progress towards the resolution of other labeling conjectures was made.

As a road map to this paper, we now delineate a general strategy of attack.

(1) Find super edge-magic labelings of $C_m \cup C_n$ for certain values of m and n;

(2) Find super edge-magic labelings of $C_m \cup P_n$ for certain values of m and n;

(3) Obtain super edge-magic labelings of $P_m \cup P_n$ for certain values of m and n from the labelings found in steps 1 and 2 by removing edges;

(4) For each of those cases not handled by step 3, either find a labeling or show none is possible.

To do these things, we next introduce the necessary definitions and some fundamental results.

In 1970, Kotzig and Rosa [18] initiated the study of magic valuations. These labelings are currently referred to as either edge-magic labelings or edge-magic total labelings; these terms were coined by Ringel and Lladó [23], and Wallis [25], respectively. In this paper, we will use the former for the sake of brevity. A graph G of order p and size q is called *edge-magic* if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ such that $f(u) + f(v) + f(uv)$ is a constant (called the *valence* or *magic number*) for any edge $uv \in E(G)$. Such a function is called an *edge-magic labeling.* In 1998, Enomoto *et al.* [3] defined an edge-magic labeling f to be a *super edge-magic labeling* if it has the additional property that $f(V(G)) = \{1, 2, \ldots, p\}$ (an alternative term exists for this kind of labeling, namely, strongly edge-magic labeling; see Wallis [25]). Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling.

The following result found in [4] allows us to exhibit only the vertex labels of a super edge-magic labeling of a graph as it explains how the edge labels will be induced by them.

Lemma 1. *A graph G of order p and size q is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set $S = \{f(u) + f(v)|uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = p+q+s$, where $s = \min(S)$ and*

$$\begin{aligned} S &= \{f(u) + f(v)\,|uv \in E(G)\} \\ &= \{k - (p + 1), k - (p + 2), \ldots, k - (p + q)\}\,. \end{aligned}$$

In [4], Figueroa-Centeno *et al.* established the following necessary condition for an r-regular graph to be super edge-magic.

Lemma 2. *If G is a super edge-magic r-regular graph of order p and size q, where $r \geq 1$, then q is odd and the valence of any super edge-magic labeling of G is $(4p + q + 3)/2$.*

It is worthwhile to mention that Kotzig and Rosa [18] proved that a 1-regular graph, that is, the linear forest nP_2 is super edge-magic if and only if n is odd. Moreover, it was shown in [4] that an r-regular graph is super edge-magic only when $0 \leq r \leq 3$. Therefore, as super edge-magic 0 and 1-regular graphs are completely characterized, the study of super edge-magic 2 and 3-regular graphs is of interest. In this paper, we significantly add to what is known about super edge-magic 2-regular graphs.

In [3], Enomoto *et al.* showed that all cycles of odd length are super edge-magic. Subsequently, Figueroa-Centeno *et al.* extended in [5] their result as follows.

Theorem 1.1. *The* 2*-regular graph* mC_n *is super edge-magic if and only if* m *and* n *are odd.*

We now consider some kinds of graph labelings that are somehow related to super edge-magic labelings.

In 1980, Graham and Sloane [14] introduced the notion of harmonious labelings. A graph G of order p and size q with $q \geq p$ is called *harmonious* if there exists an injective function $f : V(G) \to \mathbb{Z}_q$ such that each $uv \in E(G)$ is labeled $f(u)+f(v) \pmod q$ and the resulting edge labels are distinct. Such a function is called a *harmonious labeling.* If G is a tree (so that $q = p-1$) exactly two vertices are labeled the same; otherwise, the definition is the same.

The definition of sequential labelings was introduced by Grace [13], who was inspired by the above definition of harmonious labelings. For a graph G of size q, a *sequential labeling* is defined to be the injective function $f : V(G) \to \{0, 1, \ldots, q-1\}$ (with the label q allowed if G is a tree) such that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting edge labels are $\{m, m+1, \ldots, m+q-1\}$ for some positive integer m. Moreover, G is called *sequential* if such a labeling exists.

In [21], Shee defined the notion of felicitous labelings as a generalization of harmonious labelings. A graph G of size q is *felicitous* if there exists an injective function $f : V(G) \to \mathbb{Z}_{q+1}$ such that each $uv \in E(G)$ is labeled $f(u) + f(v) \pmod q$ and the resulting edge labels are distinct. Such a function is called a *felicitous labeling.*

The following result established in [4] shows the relationship between a super edge-magic graph and a graph admitting a harmonious labeling.

Lemma 3. *If* G *is a super edge-magic graph of order* p *and size* q*, then* G *is harmonious and sequential whenever it is a tree or satisfies* $q \geq p$*.*

Lemma 3 together with the fact that every harmonious graph of order p and size q with $q \geq p$ is felicitous yields that if G is a super edge-magic graph of order p and size q with $q \geq p$, then G is felicitous. This result extends easily to graphs of order p and size q with $q \geq p-1$; hence, we state the following lemma.

Lemma 4. *If G is a super edge-magic graph of order p and size q with $q \geq p-1$, then G is felicitous.*

In [24], Rosa introduced the notion of β-valuations, which were subsequently named graceful labelings by Golomb [12]. A graph G of size q is called *graceful* if there exists an injective function $f : V(G) \to \{0, 1, \ldots, q\}$ such that each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Such a function is called a *graceful labeling*. In [24], Rosa also defined an *α-valuation* of a graph G as a graceful labeling f with the additional property that there exists an integer λ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$.

In [6], Figueroa-Centeno *et al.* recently introduced a particular type of felicitous labelings, namely, strongly felicitous labelings. A felicitous labeling f of a graph G of size q is *strongly felicitous* if there exists an integer λ so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. Thus, a *strongly felicitous graph* is a graph that admits a strongly felicitous labeling.

The following result found in [6] shows the relationship between a strongly felicitous graph and a graph admitting an α-valuation.

Lemma 5. *A graph G of order p and size q with $q \geq p-1$ is strongly felicitous if and only if G admits an α-valuation.*

2. Results on 2-Regular Graphs with two Components

In this section, we study the super edge-magic properties of 2-regular graphs with two components. These are interesting, since Kotzig and Rosa [18] showed that every cycle is edge-magic and then posed a problem, which is still open: characterize the 2-regular graphs, which are edge-magic. The interested reader is directed to [4, 5, 7, 9] for some recent advances towards the resolution of this problem.

Now, notice that Gray and MacDougall proved in [15] that $C_3 \cup C_{2n}$ and $C_4 \cup C_{2n-1}$ ($n > 2$) admit *strong vertex-magic total labelings*, and that it can be shown that all such graphs are also super edge-magic. Thus, Theorems 2.1 and 2.2 below are corollaries to their results; however, notice that our proofs will not only establish these theorems, but also they provide constructions that will allow us to prove Theorems 3.1 and 3.2.

We start with the following result.

Theorem 2.1. *The* 2*-regular graph* $G \cong C_3 \cup C_n$ *is super edge-magic if and only if* $n \geq 6$ *and* n *is even.*

Proof. In [20], it was shown that the 2-regular graph $C_3 \cup C_4$ is not harmonious; hence, by Lemma 3, it is not super edge-magic either. Thus, by Lemma 2, if the 2-regular graph $G \cong C_3 \cup C_n$ is super edge-magic, then $n \geq 6$ and n is even.

For the converse, assume that $n \geq 6$ and n is even, and let

$$V(G) = \{u_1, u_2, u_3\} \cup \{v_i | 1 \leq i \leq n\}$$

and

$$E(G) = \{u_1u_2, u_2u_3, u_1u_3\} \cup \{v_1v_n\} \cup \{v_iv_{i+1} | 1 \leq i \leq n-1\}.$$

Then consider the following cases for the vertex labeling $f : V(G) \to \{1, 2, \ldots, n+3\}$.

Case 1. For $n = 8k - 2$, where k is a positive integer, let $f(u_1) = 1$; $f(u_2) = 4k + 2$; $f(u_3) = 4k + 3$;

$$f(v_l) = \begin{cases} i+1, & \text{if } l = 2i-1 \text{ and } 1 \leq i \leq 2k; \\ 4k+i+3, & \text{if } l = 2i \text{ and } 1 \leq i \leq 2k; \\ 2k+3, & \text{if } l = 4k+1; \\ 2k+2i, & \text{if } l = 4k+4i-2 \text{ and } 1 \leq i \leq k; \\ 6k+2i+3, & \text{if } l = 4k+4i-1 \text{ and } 1 \leq i \leq k-1; \\ 2k+2i+3, & \text{if } l = 4k+4i \text{ and } 1 \leq i \leq k-1; \\ 6k+2i+2, & \text{if } l = 4k+4i+1 \text{ and } 1 \leq i \leq k-1. \end{cases}$$

Case 2. For $n = 8k + 2$, where k is a positive integer, let $f(u_1) = 1$; $f(u_2) = 4k + 4$; $f(u_3) = 4k + 5$;

$$f(v_l) = \begin{cases} i+1, & \text{if } l = 2i-1 \text{ and } 1 \leq i \leq 2k+1; \\ 4k+i+5, & \text{if } l = 2i \text{ and } 1 \leq i \leq 2k+2; \\ 2k+2i+2, & \text{if } l = 4k+4i+2 \text{ and } 1 \leq i \leq k; \\ 6k+2i+7, & \text{if } l = 4k+4i+3 \text{ and } 1 \leq i \leq k-1; \\ 2k+2i+5, & \text{if } l = 4k+4i+4 \text{ and } 1 \leq i \leq k-1; \\ 6k+2i+6, & \text{if } l = 4k+4i+5 \text{ and } 1 \leq i \leq k-1; \end{cases}$$

$f(v_{4k+3}) = 2k + 5$; $f(v_{4k+5}) = 2k + 3$.

Case 3. For $n = 12k - 4$, where k is a positive integer, let $f(u_1) = 3k$; $f(u_2) = 9k - 1$; $f(u_3) = 9k$;

$$f(v_l) = \begin{cases} 6k + i - 1, & \text{if } l = 2i - 1 \text{ and } 1 \leq i \leq 3k - 1; \\ i, & \text{if } l = 2i \text{ and } 1 \leq i \leq 3k - 1; \\ 3k + 3i - 2, & \text{if } l = 6k + 6i - 6 \text{ and } 1 \leq i \leq k - 1; \\ 9k + 3i - 2, & \text{if } l = 6k + 6i - 5 \text{ and } 1 \leq i \leq k; \\ 3k + 3i, & \text{if } l = 6k + 6i - 4 \text{ and } 1 \leq i \leq k - 1; \\ 9k + 3i, & \text{if } l = 6k + 6i - 3 \text{ and } 1 \leq i \leq k - 1; \\ 3k + 3i - 1, & \text{if } l = 6k + 6i - 2 \text{ and } 1 \leq i \leq k - 1; \\ 9k + 3i + 2, & \text{if } l = 6k + 6i - 1 \text{ and } 1 \leq i \leq k - 1; \end{cases}$$

$f(v_{6k-1}) = 9k + 2$; $f(v_{12k-6}) = 6k - 1$; $f(v_{12k-4}) = 6k - 2$.

Case 4. For $n = 12k$, where k is a positive integer, let $f(u_1) = 3k + 1$; $f(u_2) = 9k + 2$; $f(u_3) = 9k + 3$;

$$f(v_l) = \begin{cases} 6k + i + 1, & \text{if } l = 2i - 1 \text{ and } 1 \leq i \leq 3k; \\ i, & \text{if } l = 2i \text{ and } 1 \leq i \leq 3k; \\ 9k + 3i + 2, & \text{if } l = 6k + 6i - 5 \text{ and } 1 \leq i \leq k; \\ 3k + 3i - 1, & \text{if } l = 6k + 6i - 4 \text{ and } 1 \leq i \leq k; \\ 9k + 3i + 1, & \text{if } l = 6k + 6i - 3 \text{ and } 1 \leq i \leq k; \\ 3k + 3i + 1, & \text{if } l = 6k + 6i - 2 \text{ and } 1 \leq i \leq k; \\ 9k + 3i + 3, & \text{if } l = 6k + 6i - 1 \text{ and } 1 \leq i \leq k; \\ 3k + 3i, & \text{if } l = 6k + 6i \text{ and } 1 \leq i \leq k. \end{cases}$$

Case 5. For $n = 12k+4$, where k is a positive integer, let $f(u_1) = 3k+2$; $f(u_2) = 9k + 5$; $f(u_3) = 9k + 6$;

$$f(v_l) = \begin{cases} 6k + i + 3, & \text{if } l = 2i - 1 \text{ and } 1 \leq i \leq 3k + 1; \\ i, & \text{if } l = 2i \text{ and } 1 \leq i \leq 3k + 1; \\ 9k + 3i + 5, & \text{if } l = 6k + 6i - 3 \text{ and } 1 \leq i \leq k; \\ 3k + 3i, & \text{if } l = 6k + 6i - 2 \text{ and } 1 \leq i \leq k - 1; \\ 9k + 3i + 4, & \text{if } l = 6k + 6i - 1 \text{ and } 1 \leq i \leq k; \\ 3k + 3i + 2, & \text{if } l = 6k + 6i \text{ and } 1 \leq i \leq k - 1; \\ 9k + 3i + 6, & \text{if } l = 6k + 6i + 1 \text{ and } 1 \leq i \leq k - 1; \\ 3k + 3i + 1, & \text{if } l = 6k + 6i + 2 \text{ and } 1 \leq i \leq k - 1; \end{cases}$$

$f(v_{12k-2}) = 6k + 1$; $f(v_{12k}) = 6k$; $f(v_{12k+1}) = 12k + 7$; $f(v_{12k+2}) = 6k + 3$; $f(v_{12k+3}) = 12k + 6$; $f(v_{12k+4}) = 6k + 2$.

Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $5n/2+9$ in all five cases. ■

As a consequence of Lemma 3 and Theorem 2.1, we have the following result, which was a conjecture posed by Seoud *et al.* [20].

Corollary 2.1. *The 2-regular graph $C_3 \cup C_n$ is harmonious if and only if $n \geq 6$ and n is even.*

Now, another result on the super edge-magicness of 2-regular graphs is presented.

Theorem 2.2. *The 2-regular graph $G \cong C_4 \cup C_n$ is super edge-magic if and only if $n \geq 5$ and n is odd.*

Proof. First, assume that $n \geq 5$ and n is odd, and let

$$V(G) = \{u_1, u_2, u_3, u_4\} \cup \{v_i | 1 \leq i \leq n\}$$

and

$$E(G) = \{u_1u_2, u_2u_3, u_3u_4, u_1u_4\} \cup \{v_1v_n\} \cup \{v_iv_{i+1} | 1 \leq i \leq n-1\}.$$

Then consider two cases for the vertex labeling $f : V(G) \to \{1, 2, \dots, n+4\}$.

Case 1. For $n = 4k+1$, where k is a positive integer, let $f(u_1) = 1$; $f(u_2) = 2k+3$; $f(u_3) = 2$; $f(u_4) = 2k+5$;

$$f(v_l) = \begin{cases} 2k+2i+2, & \text{if } l = 4i-3 \text{ and } 1 \leq i \leq k+1; \\ 2i+2, & \text{if } l = 4i-2 \text{ and } 1 \leq i \leq k; \\ 2k+2i+5, & \text{if } l = 4i-1 \text{ and } 1 \leq i \leq k; \\ 2i+1, & \text{if } l = 4i \text{ and } 1 \leq i \leq k. \end{cases}$$

Case 2. For $n = 4k+3$, where k is a positive integer, let $f(u_1) = 1$; $f(u_2) = 2k+4$; $f(u_3) = 2$; $f(u_4) = 2k+6$;

$$f(v_l) = \begin{cases} 4i-1, & \text{if } l = 4i \text{ and } 1 \leq i \leq k; \\ 2k+2i+7, & \text{if } l = 4i+1 \text{ and } 1 \leq i \leq k; \\ 2k+2i+6, & \text{if } l = 4i+3 \text{ and } 1 \leq i \leq k; \\ 2i+4, & \text{if } l = 4i+6 \text{ and } 1 \leq i \leq k-1; \end{cases}$$

$f(v_1) = 2k+5$; $f(v_2) = 4$; $f(v_3) = 2k+7$; $f(v_6) = 5$.

Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $(5n+23)/2$.

Finally, the converse follows from Lemma 2 and the fact mentioned in the proof of Theorem 2.1 that $C_3 \cup C_4$ is not super edge-magic. ■

We next present the following result.

Theorem 2.3. *The 2-regular graph $G \cong C_5 \cup C_n$ is super edge-magic if and only if $n \geq 4$ and n is even.*

Proof. Assume that $n \geq 4$ and n is even, and let

$$V(G) = \{u_1, u_2, u_3, u_4, u_5\} \cup \{v_i | 1 \leq i \leq n\}$$

and

$$E(G) = \{u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_1u_5\} \cup \{v_1v_n\} \cup \{v_iv_{i+1} | 1 \leq i \leq n-1\}.$$

First, suppose the 2-regular graph $C_5 \cup C_{10}$. Then label the vertices of C_5 with $4-13-5-15-6-4$, and the ones of C_{10} with $1-10-2-11-3-12-7-9-14-8-1$ to obtain a super edge-magic labeling of $C_5 \cup C_{10}$ with valence 39.

Next, consider the following cases for the vertex labeling $f : V(G) \to \{1, 2, \ldots, n+5\}$.

Case 1. For $n = 8k-2$, where k is a positive integer, let $f(u_1) = 1$; $f(u_2) = 4k+2$; $f(u_3) = 2$; $f(u_4) = 4k+5$; $f(u_5) = 4k+4$;

$$f(v_l) = \begin{cases} i+2, & \text{if } l = 2i-1 \text{ and } 1 \leq i \leq 2k+1; \\ 4k+i+5, & \text{if } l = 2i \text{ and } 1 \leq i \leq 2k; \\ 2k+2i+3, & \text{if } l = 4k+4i-2 \text{ and } 1 \leq i \leq k; \\ 8k+2i+1, & \text{if } l = 4k+4i-1 \text{ and } 1 \leq i \leq k-1; \\ 2k+2i+2, & \text{if } l = 4k+4i \text{ and } 1 \leq i \leq k-1; \\ 8k+2i, & \text{if } l = 4k+4i+1 \text{ and } 1 \leq i \leq k-1. \end{cases}$$

Case 2. For $n = 8k+2$, where k is an integer with $k \geq 2$, let $f(u_1) = 1$; $f(u_2) = 4k+4$; $f(u_3) = 2$; $f(u_4) = 4k+7$; $f(u_5) = 4k+6$;

$$f(v_l) = \begin{cases} i+2, & \text{if } l = 2i-1 \text{ and } 1 \leq i \leq 2k+2; \\ 4k+i+7, & \text{if } l = 2i \text{ and } 1 \leq i \leq 2k+1; \\ 2k+2i+3, & \text{if } l = 4k+4i+2 \text{ and } 1 \leq i \leq k-1; \\ 6k+2i+9, & \text{if } l = 4k+4i+3 \text{ and } 1 \leq i \leq k-1; \\ 2k+2i+6, & \text{if } l = 4k+4i+4 \text{ and } 1 \leq i \leq k-2; \\ 6k+2i+8, & \text{if } l = 4k+4i+5 \text{ and } 1 \leq i \leq k-1; \end{cases}$$

$f(v_{4k+4}) = 2k+6$; $f(v_{4k+5}) = 6k+9$; $f(v_{8k}) = 4k+3$; $f(v_{8k+2}) = 4k+5$.

Case 3. For $n = 12k-8$, where k is a positive integer, let $f(u_1) = 9k-1$; $f(u_2) = 3k$; $f(u_3) = 9k$; $f(u_4) = 3k+1$; $f(u_5) = 9k-3$;

$$f(v_l) = \begin{cases} i, & \text{if } l = 2i-1 \text{ and } 1 \le i \le 3k-2; \\ 6k+i-2, & \text{if } l = 2i \text{ and } 1 \le i \le 3k-2; \\ 3k+3i-4, & \text{if } l = 6k+6i-9 \text{ and } 1 \le i \le k; \\ 9k+3i-5, & \text{if } l = 6k+6i-8 \text{ and } 1 \le i \le k; \\ 3k+3i+1, & \text{if } l = 6k+6i-7 \text{ and } 1 \le i \le k-1; \\ 9k+3i, & \text{if } l = 6k+6i-6 \text{ and } 1 \le i \le k-1; \\ 3k+3i, & \text{if } l = 6k+6i-5 \text{ and } 1 \le i \le k-1; \\ 9k+3i-1, & \text{if } l = 6k+6i-4 \text{ and } 1 \le i \le k-1. \end{cases}$$

Case 4. For $n = 12k-4$, where k is a positive integer, let $f(u_1) = 3k$; $f(u_2) = 9k+1$; $f(u_3) = 3k+1$; $f(u_4) = 9k+3$; $f(u_5) = 3k+2$;

$$f(v_l) = \begin{cases} i, & \text{if } l = 2i-1 \text{ and } 1 \le i \le 3k-1; \\ 6k+i+1, & \text{if } l = 2i \text{ and } 1 \le i \le 3k-1; \\ 3k+3i, & \text{if } l = 6k+6i-7 \text{ and } 1 \le i \le k; \\ 9k+3i+3, & \text{if } l = 6k+6i-6 \text{ and } 1 \le i \le k-1; \\ 3k+3i+2, & \text{if } l = 6k+6i-5 \text{ and } 1 \le i \le k-1; \\ 9k+3i-1, & \text{if } l = 6k+6i-4 \text{ and } 1 \le i \le k-1; \\ 3k+3i+1, & \text{if } l = 6k+6i-3 \text{ and } 1 \le i \le k-1; \\ 9k+3i+1, & \text{if } l = 6k+6i-2 \text{ and } 1 \le i \le k-1; \end{cases}$$

$f(v_{12k-6}) = 12k+1$; $f(v_{12k-5}) = 6k+1$; $f(v_{12k-4}) = 12k-1$.

Case 5. For $n = 12k$, where k is a positive integer, let $f(u_1) = 3k+1$; $f(u_2) = 9k+4$; $f(u_3) = 3k+2$; $f(u_4) = 9k+5$; $f(u_5) = 3k+3$;

$$f(v_l) = \begin{cases} i, & \text{if } l = 2i-1 \text{ and } 1 \le i \le 3k; \\ 6k+i+3, & \text{if } l = 2i \text{ and } 1 \le i \le 3k; \\ 3k+3i+3, & \text{if } l = 6k+6i-5 \text{ and } 1 \le i \le k; \\ 9k+3i+5, & \text{if } l = 6k+6i-4 \text{ and } 1 \le i \le k; \\ 3k+3i+2, & \text{if } l = 6k+6i-3 \text{ and } 1 \le i \le k; \\ 9k+3i+4, & \text{if } l = 6k+6i-2 \text{ and } 1 \le i \le k; \\ 3k+3i+1, & \text{if } l = 6k+6i-1 \text{ and } 1 \le i \le k; \\ 9k+3i+3, & \text{if } l = 6k+6i \text{ and } 1 \le i \le k. \end{cases}$$

Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $5n/2 + 15$ in all five cases.

Finally, the converse is an immediate consequence of Lemma 2. ■

The following result is a partial generalization of Theorem 2.2.

Theorem 2.4. *If m is even with $m \geq 4$ and n is odd with $n \geq m/2 + 2$, then the 2-regular graph $G \cong C_m \cup C_n$ is super edge-magic.*

Proof. In light of Theorem 2.2, assume that m is even with $m \geq 6$ and n is odd with $n \geq m/2 + 2$. Then define the 2-regular graph $G \cong C_m \cup C_n$ with

$$V(G) = \{u_i | 1 \leq i \leq m\} \cup \{v_i | 1 \leq i \leq n\}$$

and

$$E(G) = \{u_1 u_m, v_1 v_n\} \cup \{u_i u_{i+1} | 1 \leq i \leq m-1\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\}.$$

Now, consider the following cases for the vertex labeling $f : V(G) \rightarrow \{1, 2, \ldots, m+n\}$.

Case 1. For $m = 4k + 2$ and $n = 2k + 6l - 3$, where k and l are positive integers, let

$$f(u_j) = \begin{cases} 6k + 6l - 3i + 2, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k+1; \\ 6l + 3i - 6, & \text{if } j = 2i - 1 \text{ and } k+2 \leq i \leq 2k+1; \\ 3k + 3l - 3i + 1, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3l - 3k + 3i - 4, & \text{if } j = 2i \text{ and } k+1 \leq i \leq 2k+1; \end{cases}$$

$$f(v_j) = \begin{cases} 6k + 6l - 3i + 1, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k; \\ 3k + 3l - 3i, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3k + 6l - 3i, & \text{if } j = 2k + 6i - 5 \text{ and } 1 \leq i \leq l; \\ 3l - 3i + 1, & \text{if } j = 2k + 6i - 4 \text{ and } 1 \leq i \leq l; \\ 3k + 6l - 3i + 1, & \text{if } j = 2k + 6i - 3 \text{ and } 1 \leq i \leq l; \\ 3l - 3i - 1, & \text{if } j = 2k + 6i - 2 \text{ and } 1 \leq i \leq l-1; \\ 3k + 6l - 3i - 1, & \text{if } j = 2k + 6i - 1 \text{ and } 1 \leq i \leq l-1; \\ 3l - 3i, & \text{if } j = 2k + 6i \text{ and } 1 \leq i \leq l-1. \end{cases}$$

Case 2. For $m = 4k + 2$ and $n = 2k + 6l - 1$, where k and l are positive integers, there are two subcases to pursue.

Subcase 2.1. For $k \geq 1$ and $l = 1$, let

$$f(u_j) = \begin{cases} 6k - 3i + 10, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k + 1; \\ 3i + 2, & \text{if } j = 2i - 1 \text{ and } k + 2 \leq i \leq 2k + 1; \\ 3k - 3i + 6, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3i - 3k + 1, & \text{if } j = 2i \text{ and } k + 1 \leq i \leq 2k + 1; \end{cases}$$

$$f(v_j) = \begin{cases} 3k - 3i + 5, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k; \\ 6k - 3i + 9, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ i, & \text{if } j = 2k + 2i - 1 \text{ and } 1 \leq i \leq 3; \\ 3k + i + 4, & \text{if } j = 2k + 2i \text{ and } 1 \leq i \leq 2. \end{cases}$$

Subcase 2.2. For $k \geq 1$ and $l \geq 2$, let

$$f(u_j) = \begin{cases} 6k + 6l - 3i + 4, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k + 1; \\ 6l + 3i - 4, & \text{if } j = 2i - 1 \text{ and } k + 2 \leq i \leq 2k + 1; \\ 3k + 3l - 3i + 2, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3l - 3k + 3i - 3, & \text{if } j = 2i \text{ and } k + 1 \leq i \leq 2k + 1; \end{cases}$$

$$f(v_j) = \begin{cases} 6k + 6l - 3i + 3, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k; \\ 3k + 3l - 3i + 1, & \text{if } j = 2i \text{ and } 1 \leq i \leq k + 1; \\ 3l - 3i + 2, & \text{if } j = 2k + 6i - 2 \text{ and } 1 \leq i \leq l; \\ 3k + 6l - 3i, & \text{if } j = 2k + 6i - 1 \text{ and } 1 \leq i \leq l - 1; \\ 3l - 3i, & \text{if } j = 2k + 6i \text{ and } 1 \leq i \leq l - 1; \\ 3k + 6l - 3i + 1, & \text{if } j = 2k + 6i + 1 \text{ and } 1 \leq i \leq l - 1; \\ 3l - 3i - 2, & \text{if } j = 2k + 6i + 2 \text{ and } 1 \leq i \leq l - 1; \\ 3k + 6l - 3i - 1, & \text{if } j = 2k + 6i + 3 \text{ and } 1 \leq i \leq l - 2; \end{cases}$$

$f(v_{2k+1}) = 3k + 3l - 1$; $f(v_{2k+3}) = 3k + 3l$; $f(v_{2k+6l-3}) = 3k + 3l + 1$; $f(v_{2k+6l-1}) = 3k + 3l + 2$.

Case 3. For $m = 4k + 2$ and $n = 2k + 6l + 1$, where k and l are positive integers, let

$$f(u_j) = \begin{cases} 6k + 6l - 3i + 6, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k + 1; \\ 6l + 3i - 1, & \text{if } j = 2i - 1 \text{ and } k + 2 \leq i \leq 2k + 1; \\ 3k + 3l - 3i + 3, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3l - 3k + 3i - 1, & \text{if } j = 2i \text{ and } k + 1 \leq i \leq 2k + 1; \end{cases}$$

$$f(v_j) = \begin{cases} 3k+3l-3i+4, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k; \\ 6k+6l-3i+4, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3l-3i+3, & \text{if } j = 2k+6i-5 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+5, & \text{if } j = 2k+6i-4 \text{ and } 1 \leq i \leq l; \\ 3l-3i+4, & \text{if } j = 2k+6i-3 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i-2 \text{ and } 1 \leq i \leq l; \\ 3l-3i+2, & \text{if } j = 2k+6i-1 \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i+4, & \text{if } j = 2k+6i \text{ and } 1 \leq i \leq l; \end{cases}$$

$f(v_{2k+6l-1}) = 1$; $f(v_{2k+6l+1}) = 2$.

Case 4. For $m = 4k+4$ and $n = 2k+6l-1$, where k and l are positive integers, let

$$f(u_j) = \begin{cases} 6k+6l-3i+6, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6l+3i-4, & \text{if } j = 2i-1 \text{ and } k+2 \leq i \leq 2k+2; \\ 3k+3l-3i+3, & \text{if } j = 2i \text{ and } 1 \leq i \leq k+1; \\ 3l-3k+3i-4, & \text{if } j = 2i \text{ and } k+2 \leq i \leq 2k+2; \end{cases}$$

$$f(v_j) = \begin{cases} 3k+3l-3i+4, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6k+6l-3i+4, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i-4 \text{ and } 1 \leq i \leq l-1; \\ 3l-3i+2, & \text{if } j = 2k+6i-3 \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i+4, & \text{if } j = 2k+6i-2 \text{ and } 1 \leq i \leq l-1; \\ 3l-3i, & \text{if } j = 2k+6i-1 \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i+2, & \text{if } j = 2k+6i \text{ and } 1 \leq i \leq l-1; \\ 3l-3i+1, & \text{if } j = 2k+6i+1 \text{ and } 1 \leq i \leq l-1; \end{cases}$$

$f(v_{2k+6l-4}) = 3k+3l+3$; $f(v_{2k+6l-3}) = 1$; $f(v_{2k+6l-2}) = 3k+3l+4$; $f(v_{2k+6l-1}) = 2$.

Case 5. For $m = 4k+4$ and $n = 2k+6l+1$, where k and l are positive integers, let

$$f(u_j) = \begin{cases} 6k+6l-3i+8, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6l+3i-3, & \text{if } j = 2i-1 \text{ and } k+2 \leq i \leq 2k+2; \\ 3k+3l-3i+4, & \text{if } j = 2i \text{ and } 1 \leq i \leq k+1; \\ 3l-3k+3i-4, & \text{if } j = 2i \text{ and } k+2 \leq i \leq 2k+2; \end{cases}$$

$$f(v_j) = \begin{cases} 6k+6l-3i+7, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k+1; \\ 3k+3l-3i+3, & \text{if } j = 2i \text{ and } 1 \le i \le k; \\ 3l-3i+2, & \text{if } j = 2k+6i-4 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+5, & \text{if } j = 2k+6i-3 \text{ and } 1 \le i \le l; \\ 3l-3i+3, & \text{if } j = 2k+6i-2 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i-1 \text{ and } 1 \le i \le l; \\ 3l-3i+1, & \text{if } j = 2k+6i \text{ and } 1 \le i \le l; \\ 3k+6l-3i+4, & \text{if } j = 2k+6i+1 \text{ and } 1 \le i \le l. \end{cases}$$

Case 6. For $m = 4k+4$ and $n = 2k+6l+3$, where k and l are positive integers, let

$$f(u_j) = \begin{cases} 6k+6l-3i+10, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k+1; \\ 6l+3i-1, & \text{if } j = 2i-1 \text{ and } k+2 \le i \le 2k+2; \\ 3k+3l-3i+5, & \text{if } j = 2i \text{ and } 1 \le i \le k+1; \\ 3l-3k+3i-3, & \text{if } j = 2i \text{ and } k+2 \le i \le 2k+2; \end{cases}$$

$$f(v_j) = \begin{cases} 6k+6l-3i+9, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k+2; \\ 3k+3l-3i+4, & \text{if } j = 2i \text{ and } 1 \le i \le k; \\ 3k+6l-3i+7, & \text{if } j = 2k+6i-1 \text{ and } 1 \le i \le l; \\ 3l-3i+1, & \text{if } j = 2k+6i \text{ and } 1 \le i \le l; \\ 3k+6l-3i+5, & \text{if } j = 2k+6i+1 \text{ and } 1 \le i \le l-1; \\ 3l-3i+2, & \text{if } j = 2k+6i+2 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i+3 \text{ and } 1 \le i \le l-1; \\ 3l-3i, & \text{if } j = 2k+6i+4 \text{ and } 1 \le i \le l-1; \end{cases}$$

$f(v_{2k+2}) = 3l$; $f(v_{2k+4}) = 3l+1$; $f(v_{2k+6l+1}) = 3k+3l+4$; $f(v_{2k+6l+3}) = 3k+3l+5$.

Therefore, by Lemma1, f extends to a super edge-magic labeling of G with valence $5(m+n-1)/2+4$. ∎

In light of Lemma 2 and Theorems 2.1, 2.3 and 2.4, we now obtain the following result.

Corollary 2.2. *For $m = 6, 8$ or 10, the 2-regular graph $C_m \cup C_n$ is super edge-magic if and only if $n \ge 3$ and n is odd.*

The preceding results lead us to the following conjecture.

Conjecture 1. The 2-regular graph $C_m \cup C_n$ is super edge-magic if and only if $m+n \ge 9$ and $m+n$ is odd.

Notice that Holden *et al.* conjectured in [16] that with the exception of $C_3 \cup C_4$, $3C_3 \cup C_4$ and $2C_3 \cup C_5$, all odd order 2-regular graphs possess strong vertex-magic total labelings, which is a stronger conjecture than Conjecture 1.

The remainder of this section exhibits the felicitous properties of some 2-regular graphs. These provide us some affirmative answers to an open problem posed by Lee *et al.* [19], namely, which are the pairs of integers m and n such that the 2-regular graph mC_n is felicitous?

Before presenting our next result, we first note that the cycle C_n is felicitous if and only if $n \equiv 0, 1$ or $3 \pmod 4$, and the 2-regular graph mC_n is not felicitous when $mn \equiv 2 \pmod 4$(see [19]).

We are now able to present the following result.

Theorem 2.5. *The 2-regular graph $G \cong 2C_n$ is strongly felicitous if and only if $n \geq 4$ and n is even.*

Proof. In [17], Kotzig proved that the 2-regular graph $G \cong 2C_n$ admits an α-valuation when $n \geq 4$ and n is even. Thus, by Lemma 5, G is strongly felicitous.

The converse has already been demonstrated by Lee *et al.* [19]. ■

As a consequence of Theorem 1.1 and Lemma 4, we have the following result.

Corollary 2.3. *If m and n are odd with $m \geq 1$ and $n \geq 3$, then the 2-regular graph mC_n is felicitous.*

With the aid of Corollary 2.3, we now obtain the following result.

Corollary 2.4. *The 2-regular graph $G \cong 3C_n$ is felicitous if and only if $n \not\equiv 2 \pmod 4$.*

Proof. First, consider the following felicitous labeling of $3C_4$: $0-7-2-8-0$, $3-10-4-12-3$, and $1-5-6-11-1$. Moreover, notice that Kotzig [17] showed that the 2-regular graph $G \cong 3C_n$ admits an α-valuation for $n \geq 8$ and $n \equiv 0 \pmod 4$. Thus, G is felicitous for $n \equiv 0 \pmod 4$. Now, as an immediate consequence of Lemma 5 and Corollary 2.3, we have that G is also felicitous for $n \equiv 1$ or $3 \pmod 4$.

The converse has already been proved in [19]. ■

Theorem 2.5 and Corollaries 2.3 and 2.4 motivate us to conjecture the following.

Conjecture 2. The 2-regular graph mC_n is felicitous if and only if $mn \not\equiv 2 \pmod 4$.

It is now worthwhile to mention that Abrham and Kotzig [1] proved that the 2-regular graph $C_m \cup C_n$ admits an α-valuation if and only if m and n are even and $m + n \equiv 0 \pmod 4$. Thus, by Lemma 5 and the results in the section above, we suspect the following conjecture to be true.

Conjecture 3. The 2-regular graph $C_m \cup C_n$ is felicitous if and only if $m + n \not\equiv 2 \pmod 4$.

3. Results on Unions of Cycles and Paths

With the knowledge in the previous section in hand, we present several results on super edge-magic labelings of the unions of cycles and paths, which in light of Lemma 4 advance the conjecture of Shee [21] that the graph $C_m \cup P_n$ is felicitous for every two integers $m \geq 3$ and $n \geq 2$. This conjecture is only known to be true when $m = 3$ and n is any positive integer, and m is odd and $n = 2$ or 3 (see [22] and [19], respectively).

With the aid of Theorem 2.1, we now present the following result.

Theorem 3.1. *For every integer n with $n \geq 6$, the 2-regular graph $H \cong C_3 \cup P_n$ is super edge-magic.*

Proof. Let n be an integer with $n \geq 6$, and define the 2-regular graph $G \cong C_3 \cup C_n$ as in Theorem 2.1. Also, consider the super edge-magic labeling f of G provided in the proof of the same theorem. Then proceed with four cases.

Case 1. For $n \equiv 2$ or $6 \pmod 8$, remove the edge v_1v_n from G to obtain a super edge-magic labeling of H.

Case 2. For $n \equiv 0$ or $4 \pmod 8$, remove the edge v_1v_2 from G to obtain a super edge-magic labeling of H.

Case 3. For $n \equiv 1$, 5 or $9 \pmod{12}$, consider the graph H obtained from G as follows: let $V(H) = V(G) \cup \{v\}$ and $E(H) = (E(G) - \{v_1v_2\}) \cup \{uv_1\}$. Now, define the vertex labeling $g : V(H) \to \{1, 2, \ldots, n+4\}$ to be such that $g(u) = n+4$ and $g(v) = f(v)$ for each $v \in V(G)$. Then g extends to a super edge-magic labeling of H.

Case 4. For $n \equiv 3, 7$ or $11 \pmod{12}$, remove the vertex v_2 from G, and define the vertex labeling $g : V((G - v_2)) \to \{1, 2, \ldots, n+3\}$ such that $g(x) = f(x) - 1$ for each $x \in V(G - v_2)$. Then g extends to a super edge-magic labeling of H.

Therefore, by Lemma 1, we obtain super edge-magic labelings of H with valence $5n/2 + 9$ for $1, 5$ or $9 \pmod{12}$ and $5n/2 + 8$; otherwise. ■

Now, with the aid of Theorem 2.2, we have the following result.

Theorem 3.2. *The graph $H \cong C_4 \cup P_n$ is super edge-magic if and only if $n \neq 3$.*

Proof. By Table 1, it suffices to show that the graph $H \cong C_4 \cup P_n$ is super edge-magic for every integer $n \geq 4$.

Now, let n be odd with $n \geq 5$, and define the 2-regular graph $G \cong C_4 \cup C_n$ as in Theorem 2.2, and consider the super edge-magic labeling of G given in the proof of the same result. Then remove the edge $v_1 v_n$ from G to obtain a super edge-magic labeling of H with valence $(5n + 21)/2$ for $n \equiv 1 \pmod 4$ and $(5n + 23)/2$ for $n \equiv 3 \pmod 4$.

Next, let n be even with $n \geq 4$, and define the graph $H \cong C_4 \cup P_n$ with $V(H) = V(G)$ and $E(H) = E(G) - \{v_1 v_n\}$. Now, consider two cases for the vertex labeling $g : V(H) \to \{1, 2, \ldots, n+4\}$.

Case 1. Let $n = 4k$, where k is a positive integer, and let $g(u_1) = 1$; $g(u_2) = 2k + 3$; $g(u_3) = 2$; $g(u_4) = 2k + 5$;

$$g(v_l) = \begin{cases} 2k + 2i + 2, & \text{if } l = 4i - 3 \text{ and } 1 \leq i \leq k; \\ 2i + 2, & \text{if } l = 4i - 2 \text{ and } 1 \leq i \leq k; \\ 2k + 2i + 5, & \text{if } l = 4i - 1 \text{ and } 1 \leq i \leq k - 1; \\ 2i + 1, & \text{if } l = 4i \text{ and } 1 \leq i \leq k; \\ 4k + 4, & \text{if } l = 4k - 1. \end{cases}$$

Case 2. Let $n = 4k + 2$, where k is a positive integer. For $k = 1$, the result follows from Table 1; hence, without loss of generality, assume that

$k \geq 2$, and let $g(u_1) = 1$; $g(u_2) = 2k + 4$; $g(u_3) = 2$; $g(u_4) = 2k + 6$;

$$g(v_l) = \begin{cases} 2k + 2i + 3, & \text{if } l = 2i - 1 \text{ and } 1 \leq i \leq 2; \\ i + 3, & \text{if } l = 4i - 2 \text{ and } 1 \leq i \leq 2; \\ 3, & \text{if } l = 4; \\ 2k + 2i + 7, & \text{if } l = 4i + 1 \text{ and } 1 \leq i \leq k - 1; \\ 2k + 2i + 6, & \text{if } l = 4i + 3 \text{ and } 1 \leq i \leq k - 1; \\ 2i + 5, & \text{if } l = 4i + 4 \text{ and } 1 \leq i \leq k - 1; \\ 2i + 4, & \text{if } l = 4i + 6 \text{ and } 1 \leq i \leq k - 1; \\ 4k + 6, & \text{if } l = 4k + 1. \end{cases}$$

Therefore, by Lemma 1, g extends to a super edge-magic labeling of H with valence $5n/2 + 11$. For the converse, note that by exhaustive computer search, one can verify that $C_4 \cup P_3$ is not super edge-magic. ■

Table 1. Super Edge-Magic Labelings of $C_m \cup P_n$ for small m and n.

m	n	C_m	P_n	k
4	1	$(1,3,2,5,1)$	(4)	13
	2	$(2,3,5,4,2)$	$(1,6)$	16
	6	$(2,5,9,6,2)$	$(1,8,4,7,3,10)$	26
5	4	$(6,4,9,3,8,6)$	$(1,7,2,5)$	24
	6	$(2,5,11,4,10,2)$	$(9,1,7,6,3,8)$	28
	7	$(1,8,2,6,12,1)$	$(4,7,10,5,11,3,9)$	31
	10	$(4,13,5,15,6,4)$	$(1,10,2,11,3,12,7,9,14,8)$	39
	11	$(1,10,2,8,16,1)$	$(5,13,3,11,4,9,14,6,15,7,12)$	41
6	2	$(1,5,2,8,4,7,1)$	$(3,6)$	21
	3	$(1,6,7,4,2,8,1)$	$(5,3,9)$	23
	4	$(1,7,3,10,4,8,1)$	$(5,2,9,6)$	26
8	1	$(1,5,2,6,3,8,4,9,1)$	(7)	23
	2	$(1,6,2,7,5,10,4,9,1)$	$(3,8)$	26
	3	$(1,7,2,8,6,11,5,10,1)$	$(3,9,4)$	29
	4	$(1,7,2,8,5,12,6,10,1)$	$(9,3,11,4)$	31
	5	$(1,8,2,9,6,13,7,11,1)$	$(3,10,4,12,5)$	34
10	2	$(1,10,8,6,2,7,3,12,5,11,1)$	$(4,9)$	31
	3	$(1,7,2,8,4,13,6,12,3,10,1)$	$(9,5,11)$	33
	4	$(1,8,3,11,6,14,7,12,4,9,1)$	$(2,10,5,13)$	36
	5	$(1,8,3,12,4,14,7,15,5,9,1)$	$(10,2,11,6,13)$	38
	6	$(1,9,4,15,7,16,8,13,2,10,1)$	$(3,11,5,12,6,14)$	41

With the aid of Theorem 2.3, we now present the following result.

Theorem 3.3. *For every integer $n \geq 4$, the graph $H \cong C_5 \cup P_n$ is super edge-magic.*

Proof. First, assume that $n \geq 4$ and n is even, and define the 2-regular graph $G \cong C_5 \cup C_n$ as in Theorem 2.3. Moreover, consider the super edge-magic labeling of G given in the proof of the same result. Then, by Table 1, the result is true for $n = 4$, 6 and 10. Now, consider next three cases in which each of the labelings has valence $5n/2 + 14$.

Case 1. For $n \equiv 2$ or $6 \pmod 8$, where $n \geq 14$, remove the edge $v_{n-1}v_n$ from G to obtain a super edge-magic labeling of H.

Case 2. For $n \equiv 0$ or $4 \pmod{12}$, where $n \geq 12$, remove the edge $v_{n-5}v_{n-4}$ from G to obtain a super edge-magic labeling of H.

Case 3. For $n \equiv 8 \pmod{12}$, remove the edge $v_{n-2}v_{n-1}$ from G to obtain a super edge-magic labeling of H.

Next, assume that $n \geq 5$ and n is odd, and define the graph $H \cong C_5 \cup P_n$ with $V(H) = V(G)$ and $E(H) = E(G) - \{v_1v_n\}$. Then consider four cases for the vertex labeling $g : V(H) \to \{1, 2, \ldots, n+5\}$.

Case 4. For $n = 8k - 3$, where k is a positive integer, let $g(u_1) = 1$; $g(u_2) = 4k + 3$; $g(u_3) = 2$; $g(u_4) = 4k + 1$; $g(u_5) = 8k + 2$;

$$g(v_l) = \begin{cases} 6k + i + 1, & \text{if } l = 4i - 3 \text{ and } 1 \leq i \leq k; \\ 2k + 2i + 2, & \text{if } l = 4i - 2 \text{ and } 1 \leq i \leq 2k - 1; \\ 6k + 2i + 3, & \text{if } l = 4i - 1 \text{ and } 1 \leq i \leq k - 1; \\ 2k + 2i + 1, & \text{if } l = 4i \text{ and } 1 \leq i \leq k - 1; \\ 2i + 2, & \text{if } l = 4k + 4i - 5 \text{ and } 1 \leq i \leq k; \\ 4k + 2i + 3, & \text{if } l = 4k + 4i - 4 \text{ and } 1 \leq i \leq k; \\ 2i + 1, & \text{if } l = 4k + 4i - 3 \text{ and } 1 \leq i \leq k. \end{cases}$$

Case 5. For $n = 8k - 1$, where k is a positive integer, by Table 1, the result is true for $k = 1$. Hence, assume that $k \geq 2$, and let $g(u_1) = 1$; $g(u_2) = 4k + 4$; $g(u_3) = 2$; $g(u_4) = 4k + 2$; $g(u_5) = 8k + 4$;

$$g(v_l) = \begin{cases} 2k+2i+1, & \text{if } l = 4i-3 \text{ and } 1 \le i \le k+1; \\ 6k+2i+3, & \text{if } l = 4i-2 \text{ and } 1 \le i \le k; \\ 2k+2i, & \text{if } l = 4i-1 \text{ and } 1 \le i \le k; \\ 6k+2i+2, & \text{if } l = 4i \text{ and } 1 \le i \le k; \\ 4k+2i+5, & \text{if } l = 4k+4i+1 \text{ and } 1 \le i \le k-1; \\ 2i+5, & \text{if } l = 4k+4i+4 \text{ and } 1 \le i \le k-2; \\ 2i+4, & \text{if } l = 4k+4i+6 \text{ and } 1 \le i \le k-2; \\ 4k+2i+6, & \text{if } l = 4k+4i+7 \text{ and } 1 \le i \le k-2; \end{cases}$$

$g(v_{4k+2}) = 4$; $g(v_{4k+3}) = 4k+5$; $g(v_{4k+4}) = 3$; $g(v_{4k+6}) = 5$; $g(v_{4k+7}) = 4k+6$.

Case 6. For $n = 8k+1$, where k is a positive integer, let $g(u_1) = 1$; $g(u_2) = 4k+5$; $g(u_3) = 2$; $g(u_4) = 4k+3$; $g(u_5) = 8k+6$;

$$g(v_l) = \begin{cases} 6k+2i+3, & \text{if } l = 4i-3 \text{ and } 1 \le i \le k+1; \\ 2k+2i-1, & \text{if } l = 4i-2 \text{ and } 1 \le i \le k+1; \\ 6k+2i+2, & \text{if } l = 4i-1 \text{ and } 1 \le i \le k+1; \\ 2k+2i+2, & \text{if } l = 4i \text{ and } 1 \le i \le 2k; \\ 2i+2, & \text{if } l = 4k+4i+1 \text{ and } 1 \le i \le k; \\ 4k+2i+5, & \text{if } l = 4k+4i+2 \text{ and } 1 \le i \le k-1; \\ 2i+1, & \text{if } l = 4k+4i+3 \text{ and } 1 \le i \le k-1. \end{cases}$$

Case 7. For $n = 8k+3$, where k is a positive integer, by Table 1, the result is true for $k = 1$; so assume that $k \ge 2$, and let $g(u_1) = 1$; $g(u_2) = 4k+6$; $g(u_3) = 2$; $g(u_4) = 4k+4$; $g(u_5) = 8k+8$;

$$g(v_l) = \begin{cases} 2k+2i, & \text{if } l = 4i-3 \text{ and } 1 \le i \le k+1; \\ 6k+2i+4, & \text{if } l = 4i-2 \text{ and } 1 \le i \le k+1; \\ 2k+2i+3, & \text{if } l = 4i-1 \text{ and } 1 \le i \le k+1; \\ 6k+2i+7, & \text{if } l = 4i \text{ and } 1 \le i \le k; \\ 4k+2i+7, & \text{if } l = 4k+4i+3 \text{ and } 1 \le i \le k; \\ 4k+2i+6, & \text{if } l = 4k+4i+5 \text{ and } 1 \le i \le k-1; \\ 2i+5, & \text{if } l = 4k+4i+6 \text{ and } 1 \le i \le k-1; \\ 2i+4, & \text{if } l = 4k+4i+8 \text{ and } 1 \le i \le k-2; \end{cases}$$

$g(v_{4k+4}) = 4$; $g(v_{4k+5}) = 4k+7$; $g(v_{4k+6}) = 3$; $g(v_{4k+8}) = 5$.

Therefore, by Lemma 1, g extends to a super edge-magic labeling of H with valence $(5n+27)/2$. ■

If we label the vertices of C_5 with $0-2-4-1-3-0$, and label the vertex of P_1 with 5, we obtain a felicitous labeling of $C_5 \cup P_1$. Also, note that $C_5 \cup P_n$ is felicitous for $n = 2$ and 3, since Lee *et al.* [19] showed that the graph $C_{2m+1} \cup P_n$ is felicitous for every positive integer m and $n = 2$ or 3. Thus, by Theorem 3.3 and Lemma 4, we have the following corollary.

Corollary 3.1. *For every positive integer n, the graph $C_5 \cup P_n$ is felicitous.*

The following results make inroads towards solving the conjecture by Frucht and Salinas [10] that the graph $C_m \cup P_n$ is graceful for $m + n \geq 7$, since most of the bipartite graphs in this section can be shown to admit an α-valuation (see [4] for the relationship between certain super edge-magic bipartite graphs and ones admitting α-valuations).

Theorem 3.4. *If m is even with $m \geq 4$ and $n \geq m/2 + 2$, then the graph $H \cong C_m \cup P_n$ is super edge-magic.*

Proof. In light of Theorem 3.2, assume that m is even with $m \geq 6$ and n is odd with $n \geq m/2 + 2$. Then define the 2-regular graph $G \cong C_m \cup C_n$ as in Theorem 2.4 with the super edge-magic labeling of G given in the proof of the same result. Now, consider the following cases in which each of the labelings has valence $5(m + n + 1)/2 + 4$.

Case 1. For $m = 4k + 2$ and $n = 2k + 6l - 3$, where $k \geq 1$ and $l \geq 1$, remove the edge $v_{n-2}v_{n-1}$ from G to obtain a super edge-magic labeling of H.

Case 2. For $m = 4k+2$ and $n = 2k+6l-1$, where $k \geq 1$ and $l \geq 1$, there are two possibilities. If $k \geq 1$ and $l = 1$, remove the edge $v_{n-4}v_{n-3}$ from G to obtain a super edge-magic labeling of H, and if $k \geq 1$ and $l \geq 2$, remove the edge $v_{n-3}v_{n-2}$ from G to obtain a super edge-magic labeling of H.

Case 3. For $m = 4k + 2$ and $n = 2k + 6l + 1$, where $k \geq 1$ and $l \geq 1$, remove the edge $v_{n-3}v_{n-2}$ from G to obtain a super edge-magic labeling of H.

Case 4. For $m = 4k + 4$ and $n = 2k + 6l - 1$, where $k \geq 1$ and $l \geq 1$, remove the edge v_1v_n from G to obtain a super edge-magic labeling of H.

Case 5. For $m = 4k + 4$ and $n = 2k + 6l + 1$, where $k \geq 1$ and $l \geq 1$, remove the edge $v_{n-2}v_{n-1}$ from G to obtain a super edge-magic labeling of H.

Case 6. For $m = 4k + 4$ and $n = 2k + 6l + 3$, where $k \geq 1$ and $l \geq 1$, remove the edge $v_{n-3}v_{n-2}$ from G to obtain a super edge-magic labeling of H.

Next, assume that $m \geq 6$ and n is even with $n \geq m/2 + 2$. Then define the graph $H \cong C_m \cup P_n$ with $V(H) = V(G)$ and $E(H) = E(G) - \{v_1 v_n\}$. Now, consider the following cases for the vertex labeling $g : V(H) \to \{1, 2, \ldots, m + n\}$.

Case 7. For $m = 4k + 2$ and $n = 2k + 6l - 2$, where $k \geq 1$ and $l \geq 1$, let

$$g(u_j) = \begin{cases} 6k + 6l - 3i + 3, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k + 1; \\ 6l + 3i - 5, & \text{if } j = 2i - 1 \text{ and } k + 2 \leq i \leq 2k + 1; \\ 3k + 3l - 3i + 2, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3l - 3k + 3i - 3, & \text{if } j = 2i \text{ and } k + 1 \leq i \leq 2k + 1; \end{cases}$$

$$g(v_j) = \begin{cases} 6k + 6l - 3i + 2, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k; \\ 3k + 3l - 3i + 1, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3k + 6l - 3i + 1, & \text{if } j = 2k + 6i - 5 \text{ and } 1 \leq i \leq l; \\ 3l - 3i + 2, & \text{if } j = 2k + 6i - 4 \text{ and } 1 \leq i \leq l - 1; \\ 3k + 6l - 3i + 2, & \text{if } j = 2k + 6i - 3 \text{ and } 1 \leq i \leq l; \\ 3l - 3i, & \text{if } j = 2k + 6i - 2 \text{ and } 1 \leq i \leq l - 1; \\ 3k + 6l - 3i, & \text{if } j = 2k + 6i - 1 \text{ and } 1 \leq i \leq l - 1; \\ 3k + 6l + 1, & \text{if } j = 2k + 6i \text{ and } 1 \leq i \leq l - 1; \end{cases}$$

$g(v_{2k+6l-4}) = 1$; $g(v_{2k+6l-2}) = 2$.

Case 8. For $m = 4k + 2$ and $n = 2k + 6l$, where $k \geq 1$ and $l \geq 1$, let

$$g(u_j) = \begin{cases} 6k + 6l - 3i + 5, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k + 1; \\ 6l + 3i - 2, & \text{if } j = 2i - 1 \text{ and } k + 2 \leq i \leq 2k + 1; \\ 3k + 3l - 3i + 2, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3l - 3k + 3i - 2, & \text{if } j = 2i \text{ and } k + 1 \leq i \leq 2k + 1; \end{cases}$$

$$g(v_j) = \begin{cases} 3k + 3l - 3i + 3, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k; \\ 6k + 6l - 3i + 3, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3l - 3i + 2, & \text{if } j = 2k + 6i - 5 \text{ and } 1 \leq i \leq l; \\ 3k + 6l - 3i + 4, & \text{if } j = 2k + 6i - 4 \text{ and } 1 \leq i \leq l; \\ 3l - 3i + 3, & \text{if } j = 2k + 6i - 3 \text{ and } 1 \leq i \leq l; \\ 3k + 6l - 3i + 2, & \text{if } j = 2k + 6i - 2 \text{ and } 1 \leq i \leq l; \\ 3l - 3i + 1, & \text{if } j = 2k + 6i - 1 \text{ and } 1 \leq i \leq l; \\ 3k + 6l - 3i + 3, & \text{if } j = 2k + 6i \text{ and } 1 \leq i \leq l. \end{cases}$$

Case 9. For $m = 4k+2$ and $n = 2k+6l+2$, where $k \geq 1$ and $l \geq 1$, let

$$g(u_j) = \begin{cases} 6k+6l-3i+7, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6l+3i, & \text{if } j = 2i-1 \text{ and } k+2 \leq i \leq 2k+1; \\ 3k+3l-3i+3, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3l-3k+3i-1, & \text{if } j = 2i \text{ and } k+1 \leq i \leq 2k+1; \end{cases}$$

$$g(v_j) = \begin{cases} 3k+3l-3i+4, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k; \\ 6k+6l-3i+5, & \text{if } j = 2i \text{ and } 1 \leq i \leq k+1; \\ 3k+6l-3i+6, & \text{if } j = 2k+6i-2 \text{ and } 1 \leq i \leq l; \\ 3l-3i+1, & \text{if } j = 2k+6i-1 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+4, & \text{if } j = 2k+6i \text{ and } 1 \leq i \leq l; \\ 3l-3i+2, & \text{if } j = 2k+6i+1 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+2, & \text{if } j = 2k+6i+2 \text{ and } 1 \leq i \leq l; \\ 3l-3i, & \text{if } j = 2k+6i+3 \text{ and } 1 \leq i \leq l; \end{cases}$$

$g(v_{2k+1}) = 3l$; $g(v_{2k+3}) = 3l+1$; $g(v_{2k+6l}) = 3k+3l+3$; $g(v_{2k+6l+2}) = 3k+3l+4$.

Case 10. For $m = 4k+4$ and $n = 2k+6l-2$, where $k \geq 1$ and $l \geq 1$, let

$$g(u_j) = \begin{cases} 6k+6l-3i+5, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6l+3i-5, & \text{if } j = 2i-1 \text{ and } k+2 \leq i \leq 2k+2; \\ 3k+3l-3i+2, & \text{if } j = 2i \text{ and } 1 \leq i \leq k+1; \\ 3l-3k+3i-5, & \text{if } j = 2i \text{ and } k+2 \leq i \leq 2k+2; \end{cases}$$

$$g(v_j) = \begin{cases} 3k+3l-3i+3, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6k+6l-3i+3, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3k+6l-3i+2, & \text{if } j = 2k+6i-4 \text{ and } 1 \leq i \leq l; \\ 3l-3i+1, & \text{if } j = 2k+6i-3 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i-2 \text{ and } 1 \leq i \leq l; \\ 3l-3i-1, & \text{if } j = 2k+6i-1 \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i+1, & \text{if } j = 2k+6i \text{ and } 1 \leq i \leq l-1; \\ 3l-3i, & \text{if } j = 2k+6i+1 \text{ and } 1 \leq i \leq l-1. \end{cases}$$

Case 11. For $m = 4k+4$ and $n = 2k+6l$, where $k \geq 1$ and $l \geq 1$, let

$$g(u_j) = \begin{cases} 3i-2, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6k-3i+9, & \text{if } j = 2i-1 \text{ and } k+2 \leq i \leq 2k+2; \\ 3k+3l+3i+2, & \text{if } j = 2i \text{ and } 1 \leq i \leq k+1; \\ 9k+3l-3i+10, & \text{if } j = 2i \text{ and } k+2 \leq i \leq 2k+2; \end{cases}$$

$$g(v_j) = \begin{cases} 3k+3l+2, & \text{if } j=1; \\ 3k+3l+3i, & \text{if } j=2i-1 \text{ and } 2 \le i \le k+1; \\ 3i-1, & \text{if } j=2i \text{ and } 1 \le i \le k+1; \\ 6k+3l+3i+4, & \text{if } j=2k+6i-3 \text{ and } 1 \le i \le l; \\ 3k+3i+1, & \text{if } j=2k+6i-2 \text{ and } 1 \le i \le l; \\ 6k+3l+3i+3, & \text{if } j=2k+6i-1 \text{ and } 1 \le i \le l; \\ 3k+3i+3, & \text{if } j=2k+6i \text{ and } 1 \le i \le l; \\ 6k+3l+3i+5, & \text{if } j=2k+6i+1 \text{ and } 1 \le i \le l-1; \\ 3k+3i+2, & \text{if } j=2k+6i+2 \text{ and } 1 \le i \le l-1. \end{cases}$$

Case 12. For $m = 4k+4$ and $n = 2k+6l+2$, where $k \ge 1$ and $l \ge 1$, let

$$g(u_j) = \begin{cases} 6k+6l-3i+9, & \text{if } j=2i-1 \text{ and } 1 \le i \le k+1; \\ 6l+3i-2, & \text{if } j=2i-1 \text{ and } k+2 \le i \le 2k+2; \\ 3k+3l-3i+5, & \text{if } j=2i \text{ and } 1 \le i \le k+1; \\ 3l-3k+3i-3, & \text{if } j=2i \text{ and } k+2 \le i \le 2k+2; \end{cases}$$

$$g(v_j) = \begin{cases} 6k+6l-3i+8, & \text{if } j=2i-1 \text{ and } 1 \le i \le k+1; \\ 3k+3l-3i+4, & \text{if } j=2i \text{ and } 1 \le i \le k; \\ 3l-3i+3, & \text{if } j=2k+6i-4 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+6, & \text{if } j=2k+6i-3 \text{ and } 1 \le i \le l; \\ 3l-3i+4, & \text{if } j=2k+6i-2 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+4, & \text{if } j=2k+6i-1 \text{ and } 1 \le i \le l; \\ 3l-3i+2, & \text{if } j=2k+6i \text{ and } 1 \le i \le l-1; \\ 3k+6l-3i+5, & \text{if } j=2k+6i+1 \text{ and } 1 \le i \le l; \end{cases}$$

$g(v_{2k+6l}) = 1$; $g(v_{2k+6l+2}) = 2$.

Therefore, by Lemma 1, g extends to a super edge-magic labeling of H with valence $5(m+n)/2+1$. ∎

As we mentioned in the proof of Theorem 3.2, $C_4 \cup P_3$ is not super edge-magic. Also, note that in [8], the graph $C_{4n+2} \cup nK_1$ has shown to be not super edge-magic for any positive integer n, which implies that $C_6 \cup P_1$ and $C_{10} \cup P_1$ is not super edge-magic. Next, note that the graph $C_m \cup P_n$ is super edge-magic for $m \in \{4, 6, 8, 10\}$ and the small values of n shown in Table 1.

Thus, by Table 1 and Theorems 3.2, 2.3 and 3.4, we obtain the following result.

Corollary 3.2. *For every positive integer n, the graph $C_m \cup P_n$ is super edge-magic when $m = 4$, 5, 6, 8 or 10, unless $(m,n) = (4,3), (6,1), (10,1)$.*

Theorem 3.4 and Lemma 4 imply the following result.

Corollary 3.3. *If m is even with $m \geq 4$ and $n \geq m/2 + 2$, then the graph $C_m \cup P_n$ is felicitous.*

By Lemma 4 and Corollary 3.2, we obtain the following result.

Corollary 3.4. *For every positive integer n, the graph $C_m \cup P_n$ is felicitous when $m = 4$, 5, 6, 8 or 10, unless $(m,n) = (4,3), (6,1), (10,1)$.*

4. Results on Linear Forests with two Components

In this section, we completely characterize the classes of (super) edge-magic linear forests with two components. These extend the following result shown in [5].

Theorem 4.1. *For every integer $n \geq 3$, the linear forest $P_2 \cup P_n$ is super edge-magic.*

In [7], the forest $K_{1,m} \cup P_n$ is shown to be super edge-magic for every two integers $m \geq 1$ and $n \geq 4$. Hence, we have the following result.

Theorem 4.2. *For every integer $n \geq 4$, the linear forest $P_3 \cup P_n$ is super edge-magic.*

We are now able to present the following result.

Theorem 4.3. *The linear forest $F \cong P_m \cup P_n$ is super edge-magic if and only if $(m,n) \neq (2,2)$ or $(3,3)$.*

Proof. First, note that Kotzig and Rosa [18] proved that the linear forest nP_2 is (super) edge-magic if and only if n is odd and thus $2P_2$ is not super edge-magic. Also, one can verify by an exhaustive computer search that $2P_3$ is not super edge-magic either.

For the converse, assume that $(m,n) \neq (2,2)$ or $(3,3)$. Observe then that $P_1 \cup P_n$ is super edge-magic as it was shown in [23] that all paths are super edge-magic. Now, by Theorems 4.1 and 4.2, it is sufficient to show that the linear forest $P_m \cup P_n$ is super edge-magic for every pair of integers m and n with $n \geq m \geq 4$. Thus, consider the following cases.

Case 1. If $m = 4$ and $n \geq 4$, then consider the super edge-magic labeling of $H \cong C_4 \cup P_n$ found in Theorem 3.2. Now, remove the edge u_1u_4 from H to obtain a super edge-magic labeling of $P_4 \cup P_n$ with valence $5n/2 + 10$ if n is even, $(5n + 19)/2$ if $n \equiv 1 \pmod 4$ and $(5n + 21)/2$ if $n \equiv 3 \pmod 4$.

Case 2. If m is even with $m \geq 6$ and n is odd with $n \geq 7$, then consider the super edge-magic labeling of $G \cong C_m \cup C_n$ found in Theorem 3.4, and remove the edges u_1u_m and v_1v_n from G to obtain a super edge-magic labeling of F with valence $5(m + n - 1)/2 + 2$.

Case 3. If m and n are even with $m \geq 6$ and $n \geq 6$, then there are two subcases to pursue; so consider the super edge-magic labeling of $H \cong C_m \cup P_n$ found in Theorem 2.4.

Subcase 3.1. For $m = 4k + 2$ and $n = 2k + 6l + t$, where $t = -2$, 0 or 2, $k \geq 1$ and $l \geq \lceil (2k - t + 2)/6 \rceil$, or $m = 4k + 4$ and $n = 2k + 6l + t$, where $t = -2$ or 2, $k \geq 1$ and $l \geq \lceil (2k - t + 4)/6 \rceil$, remove the edge u_1u_m from G to obtain a super edge-magic labeling of F.

Subcase 3.1. For $m = 4k + 4$ and $n = 2k + 6l$, where $k \geq 1$ and $l \geq \lceil (k + 2)/3 \rceil$, define the linear forest F with $V(F) = V(G)$ and $E(F) = E(G) - \{u_1u_m, v_1v_n\}$. Now, let $h : V(F) \to \{1, 2, \ldots, 6k + 6l + 4\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 6k + 6l - 3i + 6, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k + 1; \\ 6l + 3i - 4, & \text{if } j = 2i - 1 \text{ and } k + 2 \leq i \leq 2k + 2; \\ 3k + 3l - 3i + 3, & \text{if } j = 2i \text{ and } 1 \leq i \leq k + 1; \\ 3l - 3k + 3i - 4, & \text{if } j = 2i \text{ and } k + 2 \leq i \leq 2k + 2; \end{cases}$$

$$h(v_j) = \begin{cases} 6k + 6l - 3i + 7, & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k + 1; \\ 3k + 3l - 3i + 4, & \text{if } j = 2i \text{ and } 1 \leq i \leq k + 1; \\ 3k + 6l - 3i + 3, & \text{if } j = 2k + 6i - 3 \text{ and } 1 \leq i \leq l; \\ 3l - 3i + 2, & \text{if } j = 2k + 6i - 2 \text{ and } 1 \leq i \leq l - 1; \\ 3k + 6l - 3i + 4, & \text{if } j = 2k + 6i - 1 \text{ and } 1 \leq i \leq l; \\ 3l - 3i, & \text{if } j = 2k + 6i \text{ and } 1 \leq i \leq l - 1; \\ 3k + 6l - 3i + 2, & \text{if } j = 2k + 6i + 1 \text{ and } 1 \leq i \leq l - 1; \\ 3l - 3i + 1, & \text{if } j = 2k + 6i + 2 \text{ and } 1 \leq i \leq l - 1; \\ i, & \text{if } j = 2k + 6l + 2i - 4 \text{ and } 1 \leq i \leq 2. \end{cases}$$

Thus, by Lemma 1, h extends to a super edge-magic labeling of F with valence $5(m+n)/2$.

Case 4. If m and n are odd with $m \geq 5$ and $n \geq 5$, then there are six subcases to pursue; so define the linear forest $F \cong P_m \cup P_n$ as given in Subcase 3.2.

Subcase 4.1. For $m = 4k+1$ and $n = 2k+6l-3$, where $k \geq 1$ and $l \geq \lceil (k+2)/3 \rceil$, let $h : V(F) \to \{1, 2, \ldots, 6k+6l-2\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 3k+3l-3i+2, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 3l-3k+3i-5, & \text{if } j = 2i-1 \text{ and } k+2 \leq i \leq 2k+1; \\ 6k+6l-3i, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 6l+3i-4, & \text{if } j = 2i \text{ and } k+1 \leq i \leq 2k; \end{cases}$$

$$h(v_j) = \begin{cases} 6k+6l-3i+1, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k; \\ 3k+3l-3i, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3k+6l-3i, & \text{if } j = 2k+6i-5 \text{ and } 1 \leq i \leq l; \\ 3l-3i+1, & \text{if } j = 2k+6i-4 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+1, & \text{if } j = 2k+6i-3 \text{ and } 1 \leq i \leq l; \\ 3l-3i-1, & \text{if } j = 2k+6i-2 \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i-1, & \text{if } j = 2k+6i-1 \text{ and } 1 \leq i \leq l-1; \\ 3l-3i, & \text{if } j = 2k+6i \text{ and } 1 \leq i \leq l-1. \end{cases}$$

Subcase 4.2. For $m = 4k+1$ and $n = 2k+6l-1$, where $k \geq 1$ and $l \geq \lceil (k+1)/3 \rceil$, consider the following possibilities for the pair of integers k and l.

For $(k,l) = (1,1)$, label the vertices of P_5 with $6-2-8-1-12$, and label the ones of P_7 with $4-7-10-5-11-3-9$ to obtain a super edge-magic labeling of $P_5 \cup P_7$ with valence 30.

For $(k,l) = (2,1)$, $F \cong 2P_9$, which is super edge-magic, since it was shown in [7] that the linear forest $2P_n$ is super edge-magic if and only if $n \neq 2$ or 3.

For $(k,l) \neq (1,1)$ or $(2,1)$, let $h : V(F) \to \{1, 2, \ldots, 6k+6l\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 3k+3l-3i+3, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 3l-3k+3i-4, & \text{if } j = 2i-1 \text{ and } k+2 \leq i \leq 2k+1; \\ 6k+6l-3i+2, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 6l+3i-2, & \text{if } j = 2i \text{ and } k+1 \leq i \leq 2k; \end{cases}$$

$$h(v_j) = \begin{cases} 6k+6l-3i+3, & \text{if } j=2i-1 \text{ and } 1\le i\le k;\\ 3k+3l-3i+1, & \text{if } j=2i \text{ and } 1\le i\le k+1;\\ 3l-3i+2, & \text{if } j=2k+6i-2 \text{ and } 1\le i\le l;\\ 3k+6l-3i, & \text{if } j=2k+6i-1 \text{ and } 1\le i\le l-1;\\ 3l-3i, & \text{if } j=2k+6i \text{ and } 1\le i\le l-1;\\ 3k+6l-3i+1, & \text{if } j=2k+6i+1 \text{ and } 1\le i\le l-1;\\ 3l-3i-2, & \text{if } j=2k+6i+2 \text{ and } 1\le i\le l-1;\\ 3k+6l-3i-1, & \text{if } j=2k+6i+3 \text{ and } 1\le i\le l-2; \end{cases}$$

$h(v_{2k+1}) = 3k+6l-1$; $h(v_{2k+3}) = 3k+6l$; $h(v_{2k+6l-3}) = 3k+3l+1$; $h(v_{2k+6l-1}) = 3k+3l+2$.

Subcase 4.3. For $m = 4k+1$ and $n = 2k+6l+1$, where $k \ge 1$ and $l \ge \lceil k/3 \rceil$, let $h : V(F) \to \{1,2,\ldots,6k+6l+2\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 6k+6l-3i+5, & \text{if } j=2i-1 \text{ and } 1\le i\le k+1;\\ 6l+3i-2, & \text{if } j=2i-1 \text{ and } k+2\le i\le 2k+1;\\ 3k+3l-3i+3, & \text{if } j=2i \text{ and } 1\le i\le k;\\ 3l-3k+3i-1, & \text{if } j=2i \text{ and } k+1\le i\le 2k; \end{cases}$$

$$h(v_j) = \begin{cases} 3k+3l-3i+4, & \text{if } j=2i-1 \text{ and } 1\le i\le k;\\ 6k+6l-3i+3, & \text{if } j=2i \text{ and } 1\le i\le k;\\ 3l-3i+3, & \text{if } j=2k+6i-5 \text{ and } 1\le i\le l;\\ 3k+6l-3i+4, & \text{if } j=2k+6i-4 \text{ and } 1\le i\le l;\\ 3l-3i+4, & \text{if } j=2k+6i-3 \text{ and } 1\le i\le l;\\ 3k+6l-3i+2, & \text{if } j=2k+6i-2 \text{ and } 1\le i\le l;\\ 3l-3i+2, & \text{if } j=2k+6i-1 \text{ and } 1\le i\le l-1;\\ 3k+6l-3i+3, & \text{if } j=2k+6i \text{ and } 1\le i\le l; \end{cases}$$

$h(v_{2k+6l-1}) = 1$; $h(v_{2k+6l+1}) = 2$.

Subcase 4.4. For $m = 4k+3$ and $n = 2k+6l-1$, where $k \ge 1$ and $l \ge \lceil (k+2)/3 \rceil$, let $h : V(F) \to \{1,2,\ldots,6k+6l+2\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 6k+6l-3i+5, & \text{if } j=2i-1 \text{ and } 1\le i\le k+1;\\ 6l+3i-5, & \text{if } j=2i-1 \text{ and } k+2\le i\le 2k+2;\\ 3k+3l-3i+3, & \text{if } j=2i \text{ and } 1\le i\le k+1;\\ 3l-3k+3i-4, & \text{if } j=2i \text{ and } k+2\le i\le 2k+1; \end{cases}$$

$$h(v_j) = \begin{cases} 3k+3l-3i+4, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k+1; \\ 6k+6l-3i+3, & \text{if } j = 2i \text{ and } 1 \le i \le k; \\ 3k+6l-3i+2, & \text{if } j = 2k+6i-4 \text{ and } 1 \le i \le l; \\ 3l-3i+2, & \text{if } j = 2k+6i-3 \text{ and } 1 \le i \le l-1; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i-2 \text{ and } 1 \le i \le l; \\ 3l-3i, & \text{if } j = 2k+6i-1 \text{ and } 1 \le i \le l-1; \\ 3k+6l-3i+1, & \text{if } j = 2k+6i \text{ and } 1 \le i \le l-1; \\ 3l-3i+1, & \text{if } j = 2k+6i+1 \text{ and } 1 \le i \le l-1; \end{cases}$$

$h(v_{2k+6l-3}) = 1$; $h(v_{2k+6l-1}) = 2$.

Subcase 4.5. For $m = 4k+3$ and $n = 2k+6l+1$, where $k \ge 1$ and $l \ge \lceil (k+1)/3 \rceil$, let $h : V(F) \to \{1, 2, \ldots, 6k+6l+4\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 3k+3l-3i+5, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k+1; \\ 3l-3k+3i-5, & \text{if } j = 2i-1 \text{ and } k+2 \le i \le 2k+2; \\ 6k+6l-3i+6, & \text{if } j = 2i \text{ and } 1 \le i \le k+1; \\ 6l+3i-1, & \text{if } j = 2i \text{ and } k+2 \le i \le 2k+1; \end{cases}$$

$$h(v_j) = \begin{cases} 6k+6l-3i+7, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k+1; \\ 3k+3l-3i+3, & \text{if } j = 2i \text{ and } 1 \le i \le k; \\ 3l-3i+2, & \text{if } j = 2k+6i-4 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+5, & \text{if } j = 2k+6i-3 \text{ and } 1 \le i \le l; \\ 3l-3i+3, & \text{if } j = 2k+6i-2 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i-1 \text{ and } 1 \le i \le l; \\ 3l-3i+1, & \text{if } j = 2k+6i \text{ and } 1 \le i \le l; \\ 3k+6l-3i+4, & \text{if } j = 2k+6i+1 \text{ and } 1 \le i \le l. \end{cases}$$

Subcase 4.6. For $m = 4k+3$ and $n = 2k+6l+3$, where $k \ge 1$ and $l \ge \lceil k/3 \rceil$, let $h : V(F) \to \{1, 2, \ldots, 6k+6l+6\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 3k+3l-3i+6, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k+1; \\ 3l-3k+3i-4, & \text{if } j = 2i-1 \text{ and } k+2 \le i \le 2k+2; \\ 6k+6l-3i+8, & \text{if } j = 2i \text{ and } 1 \le i \le k+1; \\ 6l+3i+1, & \text{if } j = 2i \text{ and } k+2 \le i \le 2k+1; \end{cases}$$

$$h(v_j) = \begin{cases} 6k+6l-3i+9, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+2; \\ 3k+3l-3i+4, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3k+6l-3i+7, & \text{if } j = 2k+6i-1 \text{ and } 1 \leq i \leq l; \\ 3l-3i+1, & \text{if } j = 2k+6i \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+5, & \text{if } j = 2k+6i+1 \text{ and } 1 \leq i \leq l-1; \\ 3l-3i+2, & \text{if } j = 2k+6i+2 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i+3 \text{ and } 1 \leq i \leq l-1; \\ 3l-3i, & \text{if } j = 2k+6i+4 \text{ and } 1 \leq i \leq l-1; \end{cases}$$

$h(v_{2k+2}) = 3l$; $h(v_{2k+4}) = 3l+1$; $h(v_{2k+6l+1}) = 3k+3l+4$; $h(v_{2k+6l+3}) = 3k+3l+5$.

Thus, by Lemma 1, h extends to a super edge-magic labeling of F with valence $5(m+n)/2$.

Case 5. If m is odd with $m \geq 5$ and n is even with $n \geq 6$, then there are six subcases to pursue; so define the linear forest $F \cong P_m \cup P_n$ as given in Subcase 3.2.

Subcase 5.1. For $m = 4k+1$ and $n = 2k+6l-2$, where $k \geq 1$ and $l \geq \lceil (k+2)/3 \rceil$, let $h : V(F) \to \{1, 2, \ldots, 6k+6l-1\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 3k+3l-3i+3, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 3l-3k+3i-4, & \text{if } j = 2i-1 \text{ and } k+2 \leq i \leq 2k+1; \\ 6k+6l-3i+1, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 6l+3i-3, & \text{if } j = 2i \text{ and } k+1 \leq i \leq 2k; \end{cases}$$

$$h(v_j) = \begin{cases} 6k+6l-3i+2, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k; \\ 3k+3l-3i+1, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3k+6l-3i+1, & \text{if } j = 2k+6i-5 \text{ and } 1 \leq i \leq l; \\ 3l-3i+2, & \text{if } j = 2k+6i-4 \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i+2, & \text{if } j = 2k+6i-3 \text{ and } 1 \leq i \leq l; \\ 3l-3i, & \text{if } j = 2k+6i-2 \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i, & \text{if } j = 2k+6i-1 \text{ and } 1 \leq i \leq l-1; \\ 3l-3i+1, & \text{if } j = 2k+6i \text{ and } 1 \leq i \leq l-1; \end{cases}$$

$h(v_{2k+6l-4}) = 1$; $h(v_{2k+6l-2}) = 2$.

Subcase 5.2. For $m = 4k+1$ and $n = 2k+6l$, where $k \geq 1$ and $l \geq \lceil (k+1)/3 \rceil$, let $h : V(F) \to \{1, 2, \ldots, 6k+6l+1\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 6k+6l-3i+4, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k+1; \\ 6l+3i-3, & \text{if } j = 2i-1 \text{ and } k+2 \le i \le 2k+1; \\ 3k+3l-3i+2, & \text{if } j = 2i \text{ and } 1 \le i \le k; \\ 3l-3k+3i-2, & \text{if } j = 2i \text{ and } k+1 \le i \le 2k; \end{cases}$$

$$h(v_j) = \begin{cases} 3k+3l-3i+3, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k; \\ 6k+6l-3i+2, & \text{if } j = 2i \text{ and } 1 \le i \le k; \\ 3l-3i+2, & \text{if } j = 2k+6i-5 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i-4 \text{ and } 1 \le i \le l; \\ 3l-3i+3, & \text{if } j = 2k+6i-3 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+1, & \text{if } j = 2k+6i-2 \text{ and } 1 \le i \le l; \\ 3l-3i+1, & \text{if } j = 2k+6i-1 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+2, & \text{if } j = 2k+6i \text{ and } 1 \le i \le l. \end{cases}$$

Subcase 5.3. For $m = 4k+1$ and $n = 2k+6l+2$, where $k \ge 1$ and $l \ge \lceil k/3 \rceil$, let $h : V(F) \to \{1, 2, \ldots, 6k+6l+3\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 6k+6l-3i+6, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k+1; \\ 6l+3i-1, & \text{if } j = 2i-1 \text{ and } k+2 \le i \le 2k+1; \\ 3k+3l-3i+3, & \text{if } j = 2i \text{ and } 1 \le i \le k; \\ 3l-3k+3i-1, & \text{if } j = 2i \text{ and } k+1 \le i \le 2k; \end{cases}$$

$$h(v_j) = \begin{cases} 3k+3l-3i+4, & \text{if } j = 2i-1 \text{ and } 1 \le i \le k; \\ 6k+6l-3i+4, & \text{if } j = 2i \text{ and } 1 \le i \le k+1; \\ 3k+6l-3i+5, & \text{if } j = 2k+6i-2 \text{ and } 1 \le i \le l; \\ 3l-3i+1, & \text{if } j = 2k+6i-1 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+3, & \text{if } j = 2k+6i \text{ and } 1 \le i \le l-1; \\ 3l-3i+2, & \text{if } j = 2k+6i+1 \text{ and } 1 \le i \le l; \\ 3k+6l-3i+1, & \text{if } j = 2k+6i+2 \text{ and } 1 \le i \le l-1; \\ 3l-3i, & \text{if } j = 2k+6i+3 \text{ and } 1 \le i \le l-1; \end{cases}$$

$h(v_{2k+1}) = 3l$; $h(v_{2k+3}) = 3l+1$; $h(v_{2k+6l}) = 3k+3l+2$; $h(v_{2k+6l+2}) = 3k+3l+3$.

Subcase 5.4. For $m = 4k+3$ and $n = 2k+6l$, where $k \ge 1$ and $l \ge \lceil (k+2)/3 \rceil$, consider the following possibilities for the integers k and l.

For $(k,l)=(1,1)$, label the vertices of P_7 with $13-6-10-4-11-7-14$, and label the ones of P_8 with $15-5-12-1-8-2-9-3$ to obtain a super edge-magic labeling of $P_7 \cup P_8$ with valence 37.

For $(k,l) \neq (1,1)$, let $h : V(F) \to \{1,2,\ldots,6k+6l+3\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 6k+6l-3i+6, & \text{if } j=2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6l+3i-4, & \text{if } j=2i-1 \text{ and } k+2 \leq i \leq 2k+2; \\ 3k+3l-3i+3, & \text{if } j=2i \text{ and } 1 \leq i \leq k+1; \\ 3l-3k+3i-4, & \text{if } j=2i \text{ and } k+2 \leq i \leq 2k+1; \end{cases}$$

$$h(v_j) = \begin{cases} 3k+3l-3i+4, & \text{if } j=2i-1 \text{ and } 1 \leq i \leq k+2; \\ 6k+6l-3i+4, & \text{if } j=2i \text{ and } 1 \leq i \leq k; \\ 3l-3i+2, & \text{if } j=2k+6i-1 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+1, & \text{if } j=2k+6i \text{ and } 1 \leq i \leq l-1; \\ 3l-3i, & \text{if } j=2k+6i+1 \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i+2, & \text{if } j=2k+6i+2 \text{ and } 1 \leq i \leq l-1; \\ 3l-3i-2, & \text{if } j=2k+6i+3 \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i, & \text{if } j=2k+6i+4 \text{ and } 1 \leq i \leq l-2; \end{cases}$$

$h(v_{2k+2}) = 3k+6l$; $h(v_{2k+4}) = 3k+6l+1$; $h(v_{2k+6l-2}) = 3k+3l+2$; $h(v_{2k+6l}) = 3k+3l+3$.

Subcase 5.5. For $m = 4k+3$ and $n = 2k+6l+2$, where $k \geq 1$ and $l \geq \lceil (k+1)/3 \rceil$, let $h : V(F) \to \{1,2,\ldots,6k+6l+5\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 3k+3l-3i+6, & \text{if } j=2i-1 \text{ and } 1 \leq i \leq k+1; \\ 3l-3k+3i-4, & \text{if } j=2i-1 \text{ and } k+2 \leq i \leq 2k+2; \\ 6k+6l-3i+7, & \text{if } j=2i \text{ and } 1 \leq i \leq k+1; \\ 6l+3i, & \text{if } j=2i \text{ and } k+2 \leq i \leq 2k+1; \end{cases}$$

$$h(v_j) = \begin{cases} 6k+6l-3i+8, & \text{if } j=2i-1 \text{ and } 1 \leq i \leq k+1; \\ 3k+3l-3i+4, & \text{if } j=2i \text{ and } 1 \leq i \leq k; \\ 3l-3i+3, & \text{if } j=2k+6i-4 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+6, & \text{if } j=2k+6i-3 \text{ and } 1 \leq i \leq l; \\ 3l-3i+4, & \text{if } j=2k+6i-2 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+4, & \text{if } j=2k+6i-1 \text{ and } 1 \leq i \leq l; \\ 3l-3i+2, & \text{if } j=2k+6i \text{ and } 1 \leq i \leq l-1; \\ 3k+6l-3i+5, & \text{if } j=2k+6i+1 \text{ and } 1 \leq i \leq l; \end{cases}$$

$h(v_{2k+6l}) = 1$; $h(v_{2k+6l+2}) = 2$.

Subcase 5.6. For $m = 4k+3$ and $n = 2k+6l+4$, where $k \geq 1$ and $l \geq \lceil k/3 \rceil$, let $h : V(F) \to \{1, 2, \ldots, 6k+6l+7\}$ be the vertex labeling such that

$$h(u_j) = \begin{cases} 6k+6l-3i+10, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6l+3i, & \text{if } j = 2i-1 \text{ and } k+2 \leq i \leq 2k+2; \\ 3k+3l-3i+5, & \text{if } j = 2i \text{ and } 1 \leq i \leq k+1; \\ 3l-3k+3i-2, & \text{if } j = 2i \text{ and } k+2 \leq i \leq 2k+1; \end{cases}$$

$$h(v_j) = \begin{cases} 3k+3l-3i+6, & \text{if } j = 2i-1 \text{ and } 1 \leq i \leq k+1; \\ 6k+6l-3i+8, & \text{if } j = 2i \text{ and } 1 \leq i \leq k; \\ 3k+6l-3i+7, & \text{if } j = 2k+6i-4 \text{ and } 1 \leq i \leq l+1; \\ 3l-3i+4, & \text{if } j = 2k+6i-3 \text{ and } 1 \leq i \leq l+1; \\ 3k+6l-3i+8, & \text{if } j = 2k+6i-2 \text{ and } 1 \leq i \leq l+1; \\ 3l-3i+2, & \text{if } j = 2k+6i-1 \text{ and } 1 \leq i \leq l; \\ 3k+6l-3i+6, & \text{if } j = 2k+6i \text{ and } 1 \leq i \leq l; \\ 3l-3i+3, & \text{if } j = 2k+6i+1 \text{ and } 1 \leq i \leq l. \end{cases}$$

Thus, by Lemma 1, h extends to a super edge-magic labeling of F with valence $5(m+n-1)/2+2$ for $m+n \equiv 1$ or $3 \pmod 6$ and $5(m+n+1)/2-2$ for $m+n \equiv 5 \pmod 6$.

Therefore, having exhausted all the possible cases, we obtain the desired result. ■

The linear forest $2P_3$ is not super edge-magic as we have shown in the previous theorem, but it is edge-magic (label the vertices of one P_3 with $1-9-2$, and the ones of the other P_3 with $3-4-5$ and let valence be 17). Therefore, the following edge-magic analogue to Theorem 4.3 is obtained.

Theorem 4.4. *The linear forest $P_m \cup P_n$ is edge-magic if and only if $(m, n) \neq (2, 2)$.*

5. CONCLUSIONS

The authors wish to reiterate that the super edge-magic 2-regular graphs, which are studied in this paper are, by virtue of Lemmas 3 and 4, also harmonious, sequential and felicitous. Additionally, as mentioned above, the study of the super edge-magic properties of bipartite graphs can provide a means by which they may be shown to be graceful if they meet one additional condition (see [4]).

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