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DECOMPOSITION TREE AND INDECOMPOSABLE COVERINGS*

Andrew Breiner

Department of Mathematics and Computer Science
Nebraska Wesleyan University
5000 St. Paul Avenue, Lincoln, NE 68504, USA

e-mail: abreiner@nebrwesleyan.edu

Jitender Deogun

Department of Computer Science and Engineering
University of Nebraska – Lincoln
Lincoln, NE 68588–0115, USA

e-mail: deogun@cse.unl.edu

and

Pierre Ille†

C.N.R.S. – UMR 6206
Institut de Mathématiques de Luminy
163, Avenue de Luminy – Case 907
13288 Marseille Cedex 9, France

e-mail: ille@iml.univ-mrs.fr

Abstract

Let $G = (V, A)$ be a directed graph. With any subset X of V is associated the directed subgraph $G[X] = (X, A \cap (X \times X))$ of G induced by X. A subset X of V is an interval of G provided that for $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$,

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and similarly for (x, a) and (x, b). For example $\emptyset, V$, and $\{x\}$, where $x \in V$, are intervals of G which are the trivial intervals. A directed graph is indecomposable if all its intervals are trivial. Given an integer $k > 0$, a directed graph $G = (V, A)$ is called an indecomposable k-covering provided that for every subset X of V with $|X| \leq k$, there exists a subset Y of V such that $X \subseteq Y$, $G[Y]$ is indecomposable with $|Y| \geq 3$. In this paper, the indecomposable k-covering directed graphs are characterized for any $k > 0$.

Keywords: interval, indecomposable, k-covering, decomposition tree.

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1. Introduction

A *directed graph* or simply a *digraph* G consists of a nonempty and finite set V of *vertices* together with a collection A of ordered pairs of distinct vertices, called the set of *arcs* of G. Such a digraph is denoted by (V, A). For example, given a nonempty and finite set $V, (V, \emptyset)$ is the *empty* digraph on V whereas $(V, (V \times V) \setminus \{(x, x); x \in V\})$ is the *complete* digraph on V. Given a digraph $G = (V, A)$, with each nonempty subset X of V associate the *subdigraph* $G[X] = (X, A \cap (X \times X))$ of G induced by X. A digraph $G = (V, A)$ is a *poset* provided that for all $x, y, z \in V$, if $(x, y), (y, z) \in A$, then $(x, z) \in A$. Furthermore, a poset is a *linear ordering*, or is *linear*, if for all $x, y \in V$ with $x \neq y$, either $(x, y) \in A$ or $(y, x) \in A$. Finally, a poset $G = (V, A)$, which admits a maximum vertex, is called a *tree* if for each $x \in V$, $G[\{y \in V : (x, y) \in A\} \cup \{x\}]$ is linear.

Given a digraph $G = (V, A)$, a subset X of V is an *interval* [6] (or an *autonomous set* [4, 7, 8] or a *clan* [3] or a *homogeneous set* [2, 5] or a *module* [10]) of G provided that for any $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$, and $(x, a) \in A$ if and only if $(x, b) \in A$. This generalizes the classic notion of the interval of a linear ordering. As recalled by the following well known proposition, the intervals of a digraph and the usual intervals of a linear ordering share the same properties.

Proposition 1. *Let $G = (V, A)$ be a digraph.*

1. *$\emptyset, V$, and $\{x\}$, where $x \in V$, are intervals of G.*
2. *Given subsets X and W of V, if X is an interval of G, then $X \cap W$ is an interval of $G[W]$.*

3. *Given an interval X of G, an interval of $G[X]$ is an interval of G as well.*
4. *If X and Y are intervals of G, then $X \cap Y$ is an interval of G.*
5. *If X and Y are intervals of G such that $X \cap Y \neq \emptyset$, then $X \cup Y$ is an interval of G.*
6. *If X and Y are intervals of G such that $X \setminus Y \neq \emptyset$, then $Y \setminus X$ is an interval of G.*
7. *Given intervals X and Y of G such that $X \cap Y = \emptyset$, for any $x, x' \in X$ and $y, y' \in Y$, $(x, y) \in A$ if and only if $(x', y') \in A$.*

As indicated in the first assertion of the previous result, for every digraph $G = (V, A)$, $\emptyset, V$, and $\{x\}$, where $x \in V$, are intervals of G which are the *trivial intervals.* A digraph is then said to be *indecomposable* [6, 9] (or *prime* [2] or *primitive* [3]) if all its intervals are trivial; otherwise, it is *decomposable.* Among the simplest instances of decomposable digraphs are the complete, empty or linear digraphs having at least 3 vertices.

Given a digraph $G = (V, A)$, $I(G)$ denotes the family of the subsets S of V such that $G[S]$ is indecomposable with $|S| \geq 3$. We are interested in the subsets of V which are covered by an element of $I(G)$.

Observation 1. *A digraph $G = (V, A)$ is indecomposable if and only if for every $X \subseteq V$ such that $|X| \leq 3$, there exists $S \in I(G)$ such that $X \subseteq S$.*

Proof. Obviously, if G is indecomposable with $|V| \geq 3$, then $V \in I(G)$ and hence all the subsets of V are covered by an element of $I(G)$. For the converse, consider an interval I of G such that $|I| \geq 2$. We must show that $I = V$. Let $a \neq b \in I$. For each $x \in V$, there is $S_x \in I(G)$ such that $a, b, x \in S_x$. It follows from the second assertion of Proposition 1 that $I \cap S_x$ is an interval of $G[S_x]$. As $G[S_x]$ is indecomposable and as $a, b \in I \cap S_x$, $I \cap S_x = S_x$ and in particular $x \in I$. Therefore $I = V$. ■

To be more precise, we introduce the following. Given an integer $k > 0$, a digraph $G = (V, A)$ is an *indecomposable k-covering*, or simply is *k-covering*, provided that for every subset X of V with $|X| \leq k$, there exists $Y \in I(G)$ such that $X \subseteq Y$. Given $k \geq 3$, it follows from Observation 1 that a digraph is indecomposable if and only if it is k-covering. In what follows, we characterize the 1-covering digraphs and the 2-covering digraphs in terms of decomposition tree defined as follows (see [1] for details). We need the following strengthening of the notion of interval. Given a digraph $G =$

(V, A), a subset X of V is a *strong interval* [4, 8] of G provided that X is an interval of G and for every interval Y of G, if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. The family of the nonempty strong intervals of a digraph G, ordered by inclusion, constitutes a tree, called the decomposition tree of G and denoted by $\mathbb{D}(G)$.

2. Preliminaries

We use the following property of strong intervals (for instance, see [3, Lemma 4.10]).

Proposition 2. *Let X be a strong interval of a digraph $G = (V, A)$. For every $Y \subseteq X$, Y is a strong interval of $G[X]$ if and only if Y is a strong interval of G.*

The last assertion of Proposition 1 permits to define the quotient of a digraph by an interval partition. Given a digraph $G = (V, A)$, a partition P of V is an *interval partition* of G if all its elements are intervals of G. For such a partition P, the *quotient* of G by P is the digraph $G/P = (P, A/P)$ defined in the following way. Given $X \neq Y \in P$, $(X, Y) \in A/P$ if there exist $x \in X$ and $y \in Y$ such that $(x, y) \in A$.

In the sequel, for a family $\mathcal{F}$ of sets, $\bigcup \mathcal{F}$ denotes the union of the elements of $\mathcal{F}$. As shown by the following, the notions of interval and of quotient are compatible (for instance, see [3, Theorem 4.17]).

Proposition 3. *Given an interval partition P of a digraph $G = (V, A)$, both assertions below are satisfied.*

1. *If X is an interval of G, then $\{Y \in P : Y \cap X \neq \emptyset\}$ is an interval of G/P.*
2. *If Q is an interval of G/P, then $\bigcup Q$ is an interval of G.*

Let P be an interval partition of a digraph $G = (V, A)$. A subset S of V is called *transversal* according to P if for every $X \in P$, $|X \cap S| = 1$. Clearly, for any transversal subset S of V according to P, $G[S]$ and G/P are isomorphic. More generally, let S be a subset of V such that for all $X \in P$, $|X \cap S| \leq 1$. Then, $G[S]$ and $(G/P)[Q]$ are isomorphic, where Q is the family of the elements of P which intersect S. Gallai [4, 8] succeeded in associating in an intrinsic manner a unique quotient with each digraph.

Given a digraph $G = (V, A)$ with $|V| \geq 2$, $P(G)$ denotes the family of the maximal strong intervals of G, with respect to inclusion, which are distinct from V. The Gallai decomposition theorem is stated as follows.

Theorem 1 (Gallai [4, 8]). *Given a digraph $G = (V, A)$ with $|V| \geq 2$, $P(G)$ realizes an interval partition of G and the corresponding quotient $G/P(G)$ is complete, empty, linear or indecomposable.*

To complete the section, we review easily verified properties of the decomposition tree. Given a digraph $G = (V, A)$, $\mathbb{I}(G)$ denotes the family of the elements X of $\mathbb{D}(G)$ satisfying $|P(G[X])| \geq 3$ and $G[X]/P(G[X])$ is indecomposable. For every nonempty subset S of V, $\mathbb{D}_S(G)$ denotes the family of the elements of $\mathbb{D}(G)$ that contain S. It results from the definition of a strong interval that $\mathbb{D}_S(G)$ is linearly ordered by inclusion. Consequently, it admits a minimum element denoted by $\overline{S}$. The result below precises the Gallai decomposition of $G[\overline{S}]$ whenever $S \in I(G)$.

Lemma 1. *Let $G = (V, A)$ be a digraph. For every subset S of V, if $S \in I(G)$, then $\overline{S} \in \mathbb{I}(G)$ and S is included in a transversal subset of $\overline{S}$ according to $P(G[\overline{S}])$.*

Proof. Let S be an element of $I(G)$. By the second assertion of Proposition 1, for every $X \in P(G[\overline{S}])$, $X \cap S$ is an interval of $G[S]$. It follows from the indecomposability of $G[S]$ that $S \subseteq X$ or $|X \cap S| \leq 1$. Since $\overline{S}$ is the minimum element of $\mathbb{D}_S(G)$ under inclusion, $X \notin \mathbb{D}_S(G)$ and hence $|X \cap S| \leq 1$. Consequently, there exists a transversal subset S' of $\overline{S}$ according to $P(G[\overline{S}])$ such that $S \subseteq S'$. As previously mentioned, $G[S']$ and $G[\overline{S}]/P(G[\overline{S}])$ are isomorphic. By Theorem 1, $G[S']$ is complete, empty, linear or indecomposable. Since $G[S]$ is indecomposable with $|S| \geq 3$, $G[S']$ is also and thus $\overline{S} \in \mathbb{I}(G)$. ■

3. Indecomposable 1-coverings and 2-coverings

We begin with an easy characterization of 1-covering digraphs.

Proposition 4. *Given a digraph $G = (V, A)$ with $|V| \geq 2$, G is 1-covering if and only if $\bigcup \mathbb{I}(G) = V$.*

Proof. If G is 1-covering, then for every $x \in V$, there is $S \in I(G)$ such that $x \in S$. Consequently, $x \in \overline{S}$ and, by Lemma 1, $\overline{S} \in \mathbb{I}(G)$. The converse is immediate as well. Indeed, given $x \in V$, there is $X \in \mathbb{I}(G)$ such that $x \in X$. It suffices to consider a transversal subset of X according to $P(G[X])$ which contains x. ■

Now, we investigate the 2-covering digraphs that bear the main results.

Theorem 2. *Given a digraph $G = (V, A)$ with $|V| \geq 2$, G is 2-covering if and only if $\mathbb{I}(G) = \mathbb{D}(G) \setminus \{\{x\}; x \in V\}$.*

Proof. Assume that G is 2-covering and consider $X \in \mathbb{D}(G)$ such that $|X| \geq 2$. Let C and D be distinct elements of $P(G[X])$ and consider $c \in C$ and $d \in D$. Since G is 2-covering, there exists an element S of $I(G)$ which contains c and d. As $X \cap S$ is an interval of $G[S]$ and as $c \neq d \in X \cap S$, $X \cap S = S$. It follows that $\overline{S} = X$ and, by Lemma 1, $X \in \mathbb{I}(G)$.

Conversely, let x and y be distinct vertices of G. By the minimality of $\overline{\{x, y\}}$, x and y do not belong to the same element of $P(G[\overline{\{x, y\}}])$. Thus, there exists a transversal subset S of $\overline{\{x, y\}}$ according to $P(G[\overline{\{x, y\}}])$ which includes $\{x, y\}$. Since $G[S]$ and $G[\overline{\{x, y\}}]/P(G[\overline{\{x, y\}}])$ are isomorphic and since $\overline{\{x, y\}} \in \mathbb{I}(G)$, $S \in I(G)$. ■

Theorem 2 and the next proposition provide a characterization of 2-covering digraphs in terms of intervals.

Proposition 5. *Given a digraph $G = (V, A)$ with $|V| \geq 2$, $\mathbb{I}(G) = \mathbb{D}(G) \setminus \{\{x\}; x \in V\}$ if and only if both assertions below are satisfied*

1. *all the intervals of G are strong intervals of G;*
2. *for each $X \in \mathbb{D}(G) \setminus \{\{x\}; x \in V\}$, $|P(G[X])| \geq 3$.*

Proof. Assume that $\mathbb{I}(G) = \mathbb{D}(G) \setminus \{\{x\}; x \in V\}$. Consider an interval I of G such that $|I| \geq 2$. Denote by Q the family of the elements of $P(G[\overline{I}])$ which intersect I. For every $X \in Q$, X is a strong interval of $G[\overline{I}]$ and hence X is a strong interval of G by Proposition 2. Therefore $X \subseteq I$ or $I \subseteq X$. Since $P(G[\overline{I}]) \subseteq \mathbb{D}(G)$, it follows from the minimality of $\overline{I}$ that $|Q| \geq 2$. Consequently, for all $X \in Q$, $X \subseteq I$ and thus $I = \bigcup Q$. By Proposition 3, Q is an interval of $G[\overline{I}]/P(G[\overline{I}])$. As $\mathbb{I}(G) = \mathbb{D}(G) \setminus \{\{x\}; x \in V\}$, $G[\overline{I}]/P(G[\overline{I}])$ is indecomposable. Since $|Q| \geq 2$, $Q = P(G[\overline{I}])$. It follows that $I = \overline{I}$ and I is a strong interval of G. The second assertion is immediate.

Conversely, given $X \in \mathbb{D}(G) \setminus \{\{x\}; x \in V\}$, we want to show that $X \in \mathbb{I}(G)$. By contradiction, assume that $X \notin \mathbb{I}(G)$. By Theorem 1, $G[X]/P(G[X])$ is complete, empty or linear. Since $|P(G[X])| \geq 3$, $G[X]/P(G[X])$ admits a non-trivial interval Q. By Proposition 3, $\bigcup Q$ is an interval of $G[X]$. It follows from the maximality of the elements of $P(G[X])$ that $\bigcup Q$ is not a strong interval of $G[X]$. Since X is a strong interval of G, it follows from Proposition 2 that $\bigcup Q$ is not a strong interval of G. But $\bigcup Q$ is an interval of G by the third assertion of Proposition 1 because $\bigcup Q$ is an interval of $G[X]$ and X is an interval of G. ■

Lastly, we study the decomposable and 2-covering digraphs. Given a digraph $G = (V, A)$, we utilize the family $I^\star(G)$ of the elements X of $I(G)$ satisfying: for every $Y \in I(G)$, if $|X \cap Y| \geq 2$, then $Y \subseteq X$. In terms of decomposition tree, $I^\star(G)$ is expressed as follows.

Proposition 6. *For every digraph $G = (V, A)$ with $|V| \geq 2$, $I^\star(G) = I(G) \cap \mathbb{D}(G)$.*

Proof. Given $X \in I(G) \cap \mathbb{D}(G)$, consider $Y \in I(G)$ such that $|X \cap Y| \geq 2$. Since X is an interval of G, $X \cap Y$ is an interval of $G[Y]$ by the second assertion of Proposition 1. As $G[Y]$ is indecomposable, $X \cap Y = Y$. Therefore $X \in I^\star(G)$.

Conversely, let $X \in I^\star(G)$. By Lemma 1, $\overline{X} \in \mathbb{I}(G)$ and there exists a transversal subset S of $\overline{X}$ according to $P(G[\overline{X}])$ such that $X \subseteq S$. As shown previously, all the transversal subsets of $\overline{X}$ with respect to $P(G[\overline{X}])$ induce indecomposable subdigraphs of G and hence belong to $I(G)$. In particular, $S \in I(G)$ and, since $|X \cap S| \geq 2$, $S \subseteq X$ and so $X = S$. For each $x \in \overline{X}$, there is $y \in S$ such that x and y are contained in the same element of $P(G[\overline{X}])$. Clearly, $(S \setminus \{y\}) \cup \{x\}$ is a transversal subset of $\overline{X}$ according to $P(G[\overline{X}])$ with $|X \cap ((S \setminus \{y\}) \cup \{x\})| \geq 2$. Consequently, $(S \setminus \{y\}) \cup \{x\} = X$ for all $x \in \overline{X}$ and thus $X = \overline{X}$ belongs to $\mathbb{D}(G)$. ■

For decomposable and 2-covering digraphs, we obtain the following.

Theorem 3. *Given a 2-covering digraph $G = (V, A)$, G is decomposable if and only if $I^\star(G)$ contains a proper subset of V.*

Proof. To begin, assume that G is decomposable. Let X be a minimal non-trivial interval of G under inclusion. We prove that $X \in I^\star(G)$. By Proposition 6, it suffices to show that $X \in I(G) \cap \mathbb{D}(G)$. Since G is

2-covering, it follows from Theorem 2 and Proposition 5 that $X \in \mathbb{I}(G)$. Furthermore, all the elements of $P(G[X])$ are intervals of G by the third assertion of Proposition 1. As X is a minimal non-trivial interval of G, we obtain $P(G[X]) = \{\{x\}; x \in X\}$ so that $G[X]$ and $G[X]/P(G[X])$ are isomorphic. Since $X \in \mathbb{I}(G)$, $G[X]/P(G[X])$ is indecomposable and hence $X \in I(G)$.

Conversely, consider $X \in I^{\star}(G)$ such that $X \subsetneq V$. By Proposition 6, X is a strong interval of G. As $X \in I(G)$, $|X| \geq 3$. Therefore, X is a non-trivial interval of G. ■

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