Discussiones Mathematicae
Graph Theory 31 (2011) 5–23

PAIRED DOMINATION IN PRISMS OF GRAPHS

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Abstract

The paired domination number $\gamma_{\rm pr}(G)$ of a graph G is the smallest cardinality of a dominating set S of G such that $\langle S\rangle$ has a perfect matching. The generalized prisms πG of G are the graphs obtained by joining the vertices of two disjoint copies of G by $|V(G)|$ independent edges. We provide characterizations of the following three classes of graphs: $\gamma_{\rm pr}(\pi G) = 2\gamma_{\rm pr}(G)$ for all πG; $\gamma_{\rm pr}(K_2 \square G) = 2\gamma_{\rm pr}(G)$; $\gamma_{\rm pr}(K_2 \square G) = \gamma_{\rm pr}(G)$.

Keywords: domination, paired domination, prism of a graph, Cartesian product.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

The *paired domination number* of a graph G is the smallest cardinality of a dominating set S of G such that $\langle S\rangle$ has a perfect matching, and is denoted by $\gamma_{\rm pr}(G)$. The paired domination number of the Cartesian product $G \square H$ of two isolate-free graphs G and H was first investigated by Brešar, Henning and Rall [1], who obtained upper bounds on $\gamma_{\rm pr}(G)\gamma_{\rm pr}(H)$ in terms

*Supported by the Natural Sciences and Engineering Research Council of Canada.

of $\gamma_{\text{pr}}(G \,\Box\, H)$. They showed, i.e., that for any nontrivial tree T and any isolate-free graph H, $\gamma_{\text{pr}}(T)\gamma_{\text{pr}}(H) \leq 2\gamma_{\text{pr}}(T \,\Box\, H)$.

We compare the paired domination number of a graph G with the paired domination numbers of its generalized prisms πG; i.e., the graphs obtained by joining the vertices of two disjoint copies of G by $|V(G)|$ independent edges. Obviously, $\gamma_{\text{pr}}(\pi G) \leq 2\gamma_{\text{pr}}(G)$. Graphs G for which $\gamma_{\text{pr}}(\pi G) = 2\gamma_{\text{pr}}(G)$ regardless of how the two copies of G are joined are called *universal* γ_{pr}-doublers.

After providing background information in Section 2, we give necessary and sufficient conditions for a graph to be a universal γ_{pr}-doubler in Section 3. We also give necessary and sufficient conditions for a graph to be a *prism* γ_{pr}-*doubler*, i.e., $\gamma_{\text{pr}}(K_2 \,\Box\, G) = 2\gamma_{\text{pr}}(G)$ (Section 4), and a *prism* γ_{pr}-*fixer*, i.e., $\gamma_{\text{pr}}(K_2 \,\Box\, G) = \gamma_{\text{pr}}(G)$ (Section 5). Open problems related to this topic are mentioned in Section 6.

2. Definitions and Background

For any permutation π of $V(G)$, the *prism of G with respect to π* is the graph πG obtained from two copies G_1 and G_2 of G by joining $u \in V(G_1)$ and $v \in V(G_2)$ if and only if $v = \pi(u)$. If π is the identity $\mathbf{1}_G$, then $\pi G = K_2 \Box G$, the *Cartesian product* of G and K_2. The graph $K_2 \,\Box\, G$ is called the *prism of* (or *over*) G and, in general, πG is a *generalized prism* of G.

We shall abbreviate $V(G)$, $E(G)$ and $V(G_i)$ to V, E and V_i, respectively. Let $u \in V$ and $S \subseteq V$. In πG we denote the counterparts of u (or S) in G_1 and G_2 by u_1 and u_2 (or S_1 and S_2) respectively. Conversely, the vertex u_1 and set S_1 in G_1 (or u_2 and S_2 in G_2) are denoted by u and S respectively when considered in G.

For $v \in V$, the *open neighbourhood* $N(v)$ of v is defined by $N(v) = \{u \in V : uv \in E\}$, and the *closed neighbourhood* $N[v]$ of v is the set $N(v) \cup \{v\}$. For $S \subseteq V$, $N(S) = \bigcup_{s \in S} N(s)$, $N[S] = \bigcup_{s \in S} N[s]$ and $N\{S\} = N[S] - S$. For $v \in S$ we call $w \in V - S$ an *S-external private neighbour* of v if $N(w) \cap S = \{v\}$. Denote the set of all S-external private neighbours of v by $\text{epn}(v, S)$.

A set $S \subseteq V$ *dominates* G or is a *dominating set* of G if every vertex in $V - S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ of G is defined by $\gamma(G) = \min\{|S| : S \text{ dominates } G\}$. A dominating set S is a *paired dominating set* (*PDS*) if $\langle S \rangle$ has a perfect matching. A vertex v is an *$\overline{M}$-vertex* of a matching M if v does not belong to any edge of M. If S

is a PDS and M is a perfect matching of $\langle S\rangle$, we call M an S-*matching*. A γ-*set* of G is a dominating set of G of cardinality $\gamma(G)$; a $\gamma_{\rm pr}$-*set* is defined similarly. We follow [9] for domination terminology.

It is easy to see that $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$ for all permutations π of V. If $\gamma(K_2\Box G) = \gamma(G)$, then G is called a *prism fixer*, and if $\gamma(K_2\Box G) = 2\gamma(G)$, then G is a *prism doubler*. If $\gamma(\pi G) = \gamma(G)$ for all permutations π of V, then G is a *universal fixer*, and if $\gamma(\pi G) = 2\gamma(G)$ for all π, then G is a *universal doubler*.

Prism fixers we first studied by Hartnell and Rall [7, 8] in connection with Vizing's conjecture on the domination number of the Cartesian product of graphs. Prism and universal doublers were studied in [3], while fixers and doublers for other domination parameters, such as total and paired domination, were investigated in [11]. The graphs $\overline{K_n}$, $n \geq 1$, are universal fixers because $\pi\overline{K_n} = nK_2$ for all permutations π of V. Moreover, these graphs are the only universal fixers known to date. The following conjecture was formulated in [10] and also studied in [2, 4, 6].

Conjecture 1. *The graphs* $\overline{K_n}$, $n \geq 1$, *are the only universal fixers.*

It is obvious that $\gamma_{\rm pr}(\pi G) \leq 2\gamma_{\rm pr}(G)$ for any graph G and any permutation π of V. Unlike the case for the domination number, though, the paired domination number of πG is not bounded below by the paired domination number of G. For the graph G in Figure 1, $\gamma_{\rm pr}(G) = 6$, but for any πG obtained by adding enough edges to the graph shown, $\gamma_{\rm pr}(\pi G) = 4$. However, if π is the identity, then the above-mentioned lower bound follows from the work in [1]. We give a direct proof below.

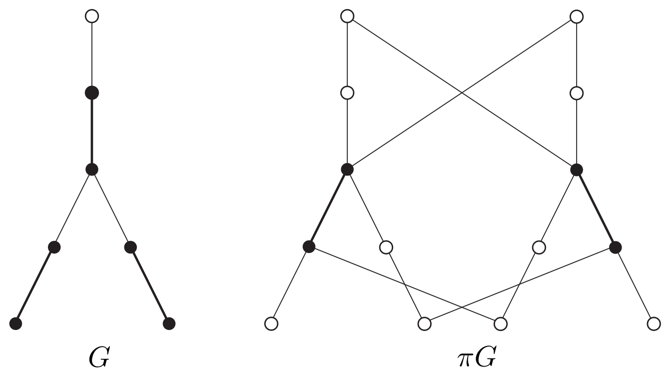

Figure 1. $\gamma_{\rm pr}(\pi G) < \gamma_{\rm pr}(G)$

Proposition 1. *For any isolate-free graph G, $\gamma_{\mathrm{pr}}(G) \leq \gamma_{\mathrm{pr}}(K_2 \,\square\, G) \leq 2\gamma(G)$.*

Proof. For the upper bound, note that if D is a γ-set of G, then $D_1 \cup D_2$ is a PDS of $K_2 \,\square\, G$. For the lower bound, let W be a γ_{pr}-set of $K_2 \,\square\, G$ with $X_1 = W \cap V_1$ and $D_2 = W \cap V_2$ and let $S = X \cup D$. Then S dominates G and $|S| = |X| + |D| - |X \cap D|$.

If $X \cap D = \emptyset$, then $\langle S \rangle$ contains a perfect matching (the matching corresponding to the perfect matching of $\langle W \rangle$) and S is a PDS of G with $|S| = |W|$, so we are done.

Assume $X \cap D \neq \emptyset$. Let M be a maximum matching of $\langle S \rangle$ and $Z = \{z^1, \ldots, z^k\}$ the set of $\overline{M}$-vertices; note that $k \leq |X \cap D|$. Let $S^0 = S$ and for $i = 1, \ldots, k$, construct S^i recursively as follows.

- If z^i is adjacent to $s^i \in V - S^{i-1}$, let $S^i = S^{i-1} \cup \{s^i\}$. Otherwise, z^i is adjacent to $x \in S^{i-1}$ because G is isolate-free; hence $N[z^i] \subseteq S^{i-1}$. Let $S^i = S^{i-1} - \{z^i\}$.

Then S^k dominates G, $\left\langle S^k \right\rangle$ has a perfect matching and thus S^k is a PDS of G. Moreover, $|S^k| \leq |S| + |Z| \leq |X| + |D| = |W|$ and the result follows. ■

Corollary 2. *If a graph G is a prism γ_{pr}-doubler, then $\gamma_{\mathrm{pr}}(G) = \gamma(G)$.*

3. Universal Doublers

Suppose D' is a γ_{pr}-set of a graph G in which u is paired with v, and $\mathrm{epn}(v, D') = \emptyset$. Then $D = D' - \{v\}$ dominates G, and $D_1 \cup D_2$ is a γ_{pr}-set of $K_2 \,\square\, G$ in which u_1 is paired with u_2. Thus G is not a prism γ_{pr}-doubler and thus not a universal γ_{pr}-doubler. A similar argument (but with another permutation) shows that if G has a γ_{pr}-set D in which $|\,\mathrm{epn}(v, D)|$ is small compared to $\gamma_{\mathrm{pr}}(G)$ for some vertex $v \in D$, then G is not a universal γ_{pr}-doubler. These cases suggest that vertices contained in γ_{pr}-sets of universal γ_{pr}-doublers have large degrees relative to $\gamma_{\mathrm{pr}}(G)$, and hence that $\gamma_{\mathrm{pr}}(G)$ is small compared to the order of G, which we denote throughout by n.

In this section we obtain necessary and sufficient conditions for a graph to be a universal γ_{pr}-doubler. These conditions easily lead to an upper bound on the paired domination number of a universal γ_{pr}-doubler G, and lower bounds on the degrees and number of external private neighbours of the vertices in γ_{pr}-sets of G.

We begin with a simple lemma.

Lemma 3. *If* $\gamma(G) = \gamma_{\rm pr}(G)$*, then* $n \geq 2\gamma_{\rm pr}(G)$ *and* G *has a PDS of cardinality* $\gamma_{\rm pr}(G) + 2i$ *for each* $1 \leq i \leq \gamma_{\rm pr}(G)/2$.

Proof. It is well known [9, Theorem 2.1] that $n \geq 2\gamma(G)$, so $n \geq 2\gamma_{\rm pr}(G)$. The latter part of the statement follows because each pair of vertices in a $\gamma_{\rm pr}$-set X which is also a γ-set can be split into two pairs since each vertex of X has an external private neighbour [9, Theorem 1.1]. ■

We next define notation that will be used throughout this section. Let

$$(1) \qquad \begin{cases} X \subseteq V \text{ such that } 0 < |X| < \gamma_{\rm pr}(G); \\ Y = V - N[X]; \\ M \text{ be a matching of } \langle X \rangle\,; \\ Z = X - V(M), \text{ i.e., } Z \text{ is the set of } \overline{M}\text{-vertices in } X; \\ k = |Z|. \end{cases}$$

We now characterize universal $\gamma_{\rm pr}$-doublers in terms of the cardinalities of the sets X, Y and Z as defined in (1).

Theorem 4. *A graph* G *is a universal* $\gamma_{\rm pr}$*-doubler if and only if, for each set* $X \subseteq V$ *with* $0 < |X| < \gamma_{\rm pr}(G)$*, a maximum matching* M *of* $\langle X \rangle$*, and* Y *and* k *as defined in* (1),

$$|Y| \geq 2\gamma_{\rm pr}(G) - |X| - k - 1.$$

Proof. Suppose that for some $X \subseteq V$ with $0 < |X| < \gamma_{\rm pr}(G)$,

$$|Y| < 2\gamma_{\rm pr}(G) - |X| - k - 1.$$

We consider two cases, depending on the parity of k.

Case 1. k is even.
Then by definition of Z, $|X|$ is even. Choose a PDS D of G as follows.

(i) If $|Y| + k \leq \gamma_{\rm pr}(G)$, then let D be any $\gamma_{\rm pr}$-set of G.

(ii) Otherwise, let D be any PDS of G with $|D| = |Y| + k$ if $|Y|$ is even, or $|D| = |Y| + k + 1$ if $|Y|$ is odd. (A PDS of this size exists by Lemma 3.)

Let π be any permutation of V such that $\pi(Y \cup Z) \subseteq D$ and $\langle \pi(Z) \rangle$ has a perfect matching M' that is contained in a D-matching. Then $W = X_1 \cup D_2$ dominates πG and $\langle W \rangle$ has a W-matching in which each edge u_2v_2 in M_2' is replaced by two edges z_1u_2 and $z_1'v_2$, where $z, z' \in Z$. (See Figure 2.)

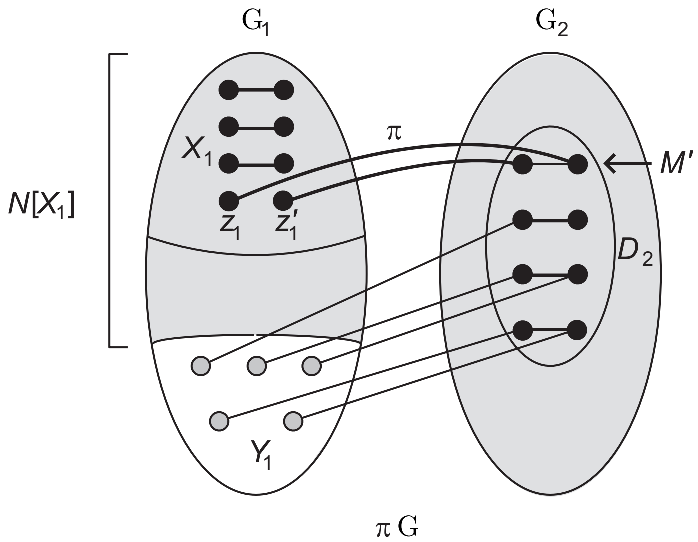

Figure 2. $X_1 \cup D_2$ dominates πG.

Therefore W is a PDS of πG. If D is a γ_{pr}-set of G (i.e., if D was defined in (i)), then

$$|W| = |X| + |D| < 2\gamma_{\mathrm{pr}}(G),$$

i.e., G is not a universal γ_{pr}-doubler. If D was defined in (ii), then

$$\begin{aligned} |W| &= |X| + |D| \\ &\leq |X| + |Y| + k + 1 \\ &< |X| + (2\gamma_{\mathrm{pr}}(G) - |X| - k - 1) + k + 1 \\ &= 2\gamma_{\mathrm{pr}}(G), \end{aligned}$$

and G is not a universal doubler in this case either.

Case 2. k is odd.
Then $|X|$ is odd. If $|X| = \gamma_{\mathrm{pr}}(G) - 1$, then $|Y| \leq \gamma_{\mathrm{pr}}(G) - k - 1 = |X - Z|$.

Let π be any permutation of V such that

$$\pi(Y) \subseteq X - Z, \quad \pi(Z) = Z \quad \text{and} \quad Y \subseteq \pi(X - Z).$$

Then $W = X_1 \cup X_2$ dominates $\pi(G)$ and it is easy to see that $\langle W \rangle$ has a perfect matching. Therefore W is a PDS of πG and

$$|W| = 2|X| = 2\gamma_{\mathrm{pr}}(G) - 2,$$

so G is not a universal doubler.

Thus we assume that $0 < |X| < \gamma_{\mathrm{pr}}(G) - 2$. Similar to Case 1, we choose the PDS D of G as follows.

(iii) If $|Y| + k \le \gamma_{\mathrm{pr}}(G)$, let D be any γ_{pr}-set of G.

(iv) Otherwise, let D be any PDS of G with $|D| = |Y| + k - 1$ if $|Y|$ is even, or $|D| = |Y| + k$ if $|Y|$ is odd.

Let $w \in Z$ and let π be any permutation of V such that $\pi(Y \cup Z - \{w\}) \subseteq D$, $\pi(Z - \{w\})$ has a perfect matching M' which is contained in a D-matching, and $\pi(w) = w' \in V - D$. Let $W = X_1 \cup D_2 \cup \{w_2'\}$. Since X_1 dominates $G_1 - Y_1$ and D_2 dominates G_2 and Y_1, it follows that W dominates πG. Also, $\langle W \rangle$ has a perfect matching in which w_1 is paired with w_2', and each edge $u_2 v_2$ in M_2' is replaced by two edges $z_1 u_2$ and $z_1' v_2$, where $z, z' \in Z - \{w\}$. Therefore W is a PDS of G. If D was chosen in (iii) and thus is a γ_{pr}-set of G, then

$$|W| = |X| + |D| + 1 < \gamma_{\mathrm{pr}}(G) - 2 + \gamma_{\mathrm{pr}}(G) + 1 = 2\gamma_{\mathrm{pr}}(G) - 1$$

and G is not a universal γ_{pr}-doubler. On the other hand, if D was chosen in (iv), then $|D| \le |Y| + k$, so

$$\begin{aligned}|W| &= |X| + |D| + 1 \\ &< |X| + 2\gamma_{\mathrm{pr}}(G) - |X| - k - 1 + k + 1 \\ &= 2\gamma_{\mathrm{pr}}(G)\end{aligned}$$

and once again G is not a universal γ_{pr}-doubler.

Conversely, let π be a permutation of V such that $\gamma_{\mathrm{pr}}(\pi G) < 2\gamma_{\mathrm{pr}}(G) - 1$ and consider any γ_{pr}-set W of πG. Define

$$X_1 = W \cap V_1 \quad \text{and} \quad D_2 = W \cap V_2.$$

Assume without loss of generality that $|X_1| < \gamma_{\mathrm{pr}}(G)$. Let M' be a W-matching and let D_2' be the set of vertices in D_2 which are not paired with another vertex in D_2 under M'. Say $|D_2'| = k'$. Also, let k be the number of vertices not paired in a maximum matching of $\langle X_1 \rangle$. Note that $k \leq k'$.

If $X_1 \neq \emptyset$, then $|D_2| < 2\gamma_{\mathrm{pr}}(G) - |X| - 1$ and each vertex of $D_2 - D_2'$ dominates at most one vertex in Y_1, while no vertex in D_2' dominates a vertex in Y_1. Therefore $|Y_1| \leq |D_2 - D_2'|$, which implies that

$$|Y| < 2\gamma_{\mathrm{pr}}(G) - |X| - k' - 1 \leq 2\gamma_{\mathrm{pr}}(G) - |X| - k - 1.$$

If $X_1 = \emptyset$, then D_2 dominates V_1 and so $D_2 = V_2$. Therefore $n = |D_2| < 2\gamma_{\mathrm{pr}}(G)$, so that by Lemma 3, $\gamma(G) < \gamma_{\mathrm{pr}}(G)$. Let X' be a γ-set of G, $Y' = V - N[X']$ and k' be the number of vertices not paired in a maximum matching of $\langle X' \rangle$. Since $k' \leq |X'| < \gamma_{\mathrm{pr}}(G)$,

$$|Y'| = 0 < 2\gamma_{\mathrm{pr}}(G) - |X'| - k' - 1. \tag{2}$$

■

As an example of universal γ_{pr}-doublers, consider the following family $\mathcal{F}$ of graphs. Form the graph $F_{2n} \in \mathcal{F}$ by joining each vertex of C_{2n} to $2n-1$ new vertices. Note that $\gamma_{\mathrm{pr}}(F_{2n}) = \gamma(F_{2n}) = 2n$. Figure 3 shows the graph F_4.

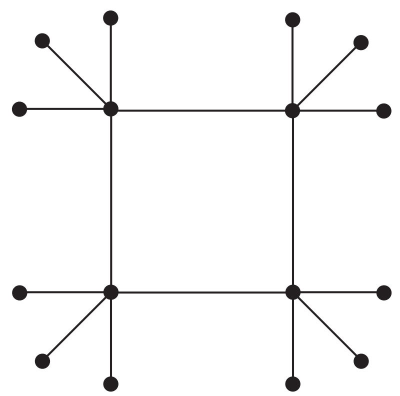

Figure 3. $F_4 \in \mathcal{F}$: An example of a universal γ_{pr}-doubler.

By Theorem 4, to prove that F_{2n} is a universal γ_{pr}-doubler, we must show that for each pair of sets $X, Y \subseteq V(F_{2n})$ as defined in (1), $|Y| \geq 2\gamma_{\mathrm{pr}}(F_{2n}) - |X| - k - 1$. Suppose $|X| = 2n - d$, where $1 \leq d \leq 2n - 1$. It is easy to see that $|Y| \geq d(2n-1)$. If $d = 1$, then $k \geq 1$, hence

$$2\gamma_{\rm pr}(F_{2n}) - |X| - k - 1 \leq 4n - (2n-1) - 1 - 1 = 2n - 1 \leq |Y|.$$

If $2 \leq d \leq 2n-1$, then $k \geq 0$, hence

$$\begin{aligned} 2\gamma_{\rm pr}(F_{2n}) - |X| - k - 1 &\leq 4n - (2n-d) - 1 \\ &= 2n + d - 1 \\ &\leq 2n + (2n-1) - 1 \\ &= 2(2n-1) \\ &\leq d(2n-1) \\ &\leq |Y|. \end{aligned}$$

Note that to construct a universal $\gamma_{\rm pr}$-doubler G from C_{2n} by adding pendant edges at vertices of C_{2n}, at least $2n-1$ pendant edges must be added at each vertex of C_{2n}. If some vertices of C_{2n} are joined to more than $2n-1$ new vertices, the resulting graph is also a universal $\gamma_{\rm pr}$-doubler.

Corollary 5. *If $\gamma(G) = \gamma_{\rm pr}(G) = 2$, then G is a universal $\gamma_{\rm pr}$-doubler.*

Proof. Suppose $\gamma(G) = \gamma_{\rm pr}(G) = 2$. Let $x \in V$ and $Y = V - N[x]$. Since $\gamma(G) = 2$, $|Y| \geq 1$. The result follows from Theorem 4. ■

We use Theorem 4 to obtain the promised results on the degrees and number of external private neighbours of the vertices in $\gamma_{\rm pr}$-sets of a universal $\gamma_{\rm pr}$-doubler.

Corollary 6. *Let G be a universal $\gamma_{\rm pr}$-doubler and D any $\gamma_{\rm pr}$-set of G. Then $|\operatorname{epn}(v, D)| \geq \gamma_{\rm pr}(G) - 1$ for each $v \in D$.*

Proof. Let $X = D - \{v\}$. Then $X \neq \emptyset$ because $\gamma_{\rm pr}(G) \geq 2$, and $k = 1$ because there is only one vertex in X that is not paired. By Theorem 4,

$$|V - N[X]| \geq 2\gamma_{\rm pr}(G) - |X| - k - 1 = \gamma_{\rm pr}(G) - 1.$$

Since D is a dominating set, v dominates $V - N[X]$. Moreover, $v \notin V - N[X]$ because v is dominated by its partner in D. Hence $\operatorname{epn}(v, D) = V - N[X]$ and the result follows. ■

The converse of Corollary 6 is shown to be false by the counterexample in Figure 4. The black vertices form the set D, which is the only $\gamma_{\rm pr}$-set of G, and for all $v \in D$, $|\operatorname{epn}(v, D)| = 3 = \gamma_{\rm pr}(G) - 1$. Let X consist of the circled vertices. Then

$$|Y| = |V - N[X]| = 2 < 2\gamma_{\rm pr}(G) - |X| - k - 1 = 3,$$

so by Theorem 4, G is not a universal $\gamma_{\rm pr}$-doubler.

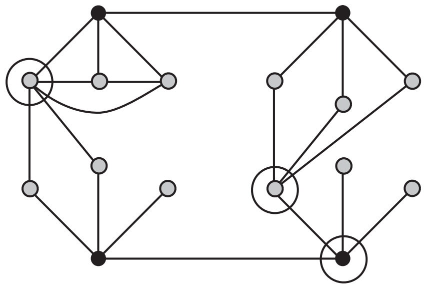

Figure 4. A counterexample to the converse of Corollary 6.

Corollary 7. *If G is a universal $\gamma_{\rm pr}$-doubler and $v \in V$ is contained in a $\gamma_{\rm pr}$-set of G, then $\deg v \geq \gamma_{\rm pr}(G)$.*

Proof. Suppose D is a $\gamma_{\rm pr}$-set of G and $v \in D$. By Corollary 6, $|\operatorname{epn}(v, D)| \geq \gamma_{\rm pr}(G) - 1$. Since v is paired with some vertex in D, the result follows. ■

The complete graphs of order at least three show that the converse of Corollary 7 is not true.

Corollary 8. *If G is a universal $\gamma_{\rm pr}$-doubler of order n, then $\gamma_{\rm pr}(G) \leq \sqrt{n}$.*

Proof. By Corollary 7, $\deg v \geq \gamma_{\rm pr}(G)$ for any vertex v of any $\gamma_{\rm pr}$-set D of G. Hence $n \geq [\gamma_{\rm pr}(G)]^2$. ■

We conclude this section by obtaining a sufficient condition for regular graphs to be universal $\gamma_{\rm pr}$-doublers. This allows us to construct a family of universal $\gamma_{\rm pr}$-doublers.

The PDS D is an *efficient paired dominating set* (*EPDS*) if $N(u) \cap N(v) = \emptyset$ for any two vertices $u, v \in D$.

Lemma 9. *If G is regular and has an EPDS D, then $\gamma_{\rm pr}(G) = |D|$.*

Proof. Let X be a $\gamma_{\rm pr}$-set of G. Then $|X| \leq |D|$ and by regularity, $n \leq r|X|$. Since D is an EPDS, $n = r|D|$. Hence $|D| \leq |X|$ and so D is a $\gamma_{\rm pr}$-set of G. ∎

Corollary 10. *If G is r-regular with $r \geq \gamma_{\rm pr}(G)$ and G has an EPDS, then G is a universal $\gamma_{\rm pr}$-doubler.*

Proof. Let $X \subseteq V$ with $0 < |X| < \gamma_{\rm pr}(G)$ and define Y and k as in (1). Then $|N[X]| \leq r|X| + k$. Since G has an EPDS, $n = r\gamma_{\rm pr}(G)$. Then

$$|Y| \geq r\gamma_{\rm pr}(G) - r|X| - k \geq \gamma_{\rm pr}(G)(\gamma_{\rm pr}(G) - |X|) - k.$$

If $|X| = \gamma_{\rm pr}(G) - 1$, then

$$|Y| \geq \gamma_{\rm pr}(G) - k = 2\gamma_{\rm pr}(G) - |X| - k - 1,$$

and if $|X| \leq \gamma_{\rm pr}(G) - 2$, then

$$|Y| \geq 2\gamma_{\rm pr}(G) - k.$$

In either case the hypothesis of Theorem 4 is satisfied and it follows that G is a universal $\gamma_{\rm pr}$-doubler. ∎

Corollary 10 allows us to construct a family $\mathcal{H}$ of regular universal $\gamma_{\rm pr}$-doublers. Label the vertices of C_{2m} consecutively by $u_1, v_1, u_2, v_2, \ldots, u_m, v_m$. Construct each $H_{2m,r} \in \mathcal{H}$ by replacing alternate edges u_iv_i, $i = 1, \ldots, m$, of C_{2m} by a copy of $B_i \cong K_{r-1,r-1}$, $r \geq 2m$, joining u_i to each vertex in one partite set, and v_i to each vertex in the other partite set of B_i. See Figure 5 for $H_{4,4}$.

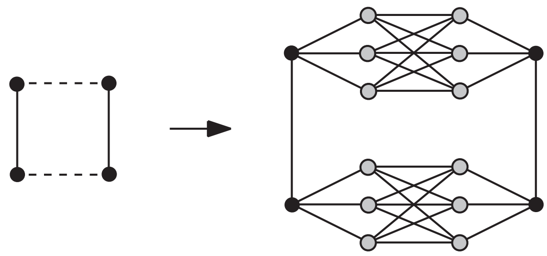

Figure 5. The 4-regular universal $\gamma_{\rm pr}$-doubler $H_{4,4}$.

Clearly, $H_{2m,r}$ is r-regular. It is also easy to see that $\bigcup_{i=1}^{m}\{u_i, v_i\}$ forms an efficient $\gamma_{\rm pr}$-set of $H_{2m,r}$ (in which each v_i is partnered by $u_{i+1 \pmod{m}}$). By Corollary 10, $H_{2m,r}$ is a universal $\gamma_{\rm pr}$-doubler.

4. Prism Doublers

It is reasonable to expect that there are graphs that are prism $\gamma_{\rm pr}$-doublers but not universal $\gamma_{\rm pr}$-doublers. In this section we first supply necessary and sufficient conditions in Theorem 11, and then a simpler sufficient condition in Proposition 12, for a graph to be a prism doubler. The latter result combined with Corollary 6 allows us to construct prism $\gamma_{\rm pr}$-doublers that are not universal $\gamma_{\rm pr}$-doublers.

Theorem 11. *A graph G is a prism $\gamma_{\rm pr}$-doubler if and only if for each set $X \subseteq V$ with $0 < |X| < \gamma_{\rm pr}(G)$, any matching M of $\langle X\rangle$, and Y and k as defined in* (1)*, either*

(i) $|Y| \geq 2\gamma_{\rm pr}(G) - |X| - k - 1$, *or*

(ii) $|Y| = 2\gamma_{\rm pr}(G) - |X| - k - d - 1$, *where $d \geq 1$, and if $A \subseteq N[X] - Z$ dominates $N\{X\} - N[Y] - N[Z]$ and $\langle A \cup Y\rangle$ has a perfect matching, then $|A| \geq d$.*

Proof. Assume $\gamma_{\rm pr}(K_2 \,\square\, G) = 2\gamma_{\rm pr}(G)$ and consider any pair of sets X, Y as defined in (1) and a matching M of $\langle X\rangle$. If $|Y| \geq 2\gamma_{\rm pr}(G) - |X| - k - 1$ then we are done, so assume $|Y| = 2\gamma_{\rm pr}(G) - |X| - k - d - 1$ for some $d \geq 1$.

Suppose to the contrary that there exists a set $A \subseteq N[X] - Z$ such that A dominates $N\{X\} - N[Y] - N[Z]$ and $\langle A \cup Y\rangle$ has a perfect matching M^*, but $|A| \leq d - 1$. Define the set $W \subseteq V(K_2 \,\square\, G)$ by $W = X_1 \cup Y_2 \cup A_2 \cup Z_2$. By the definition of X and Y, $X_1 \cup Y_2$ dominates G_1. Since A_2 dominates $N\{X_2\} - N[Y_2] - N[Z_2]$, W also dominates G_2. Thus W dominates $K_2 \,\square\, G$. Moreover, $M \cup M^* \cup \{z_1 z_2 : z \in Z\}$ is a W-matching, so W is a PDS of $K_2 \,\square\, G$. But

$$\begin{aligned}|W| &= |X| + |Y| + |Z| + |A| \\ &\leq |X| + (2\gamma_{\rm pr}(G) - |X| - k - d - 1) + k + (d - 1) \\ &= 2\gamma_{\rm pr}(G) - 2,\end{aligned}$$

a contradiction. Thus (ii) holds.

Conversely, assume $\gamma_{\mathrm{pr}}(K_2 \,\Box\, G) < 2\gamma_{\mathrm{pr}}(G) - 1$ and let $W = X_1 \cup D_2$ be a γ_{pr}-set of $K_2 \,\Box\, G$. We may assume without loss of generality that $|X| < \gamma_{\mathrm{pr}}(G)$.

We consider two cases, depending on whether $X = \emptyset$ or $X \neq \emptyset$.

Case 1. $X = \emptyset$.
Then $D_2 = V_2$ to dominate G_1. Therefore

$$|W| = |D| = n \leq 2\gamma_{\mathrm{pr}}(G) - 2.$$

By Lemma 3, $\gamma(G) < \gamma_{\mathrm{pr}}(G)$. Let X' be a γ-set of G, M' be a maximum matching of $\langle X' \rangle$, Z' the set of $\overline{M'}$-vertices in X' and $k' = |Z'|$. Then $k' > 0$ because X' is not a PDS of G, and $Y' = V - N[X'] = \emptyset$ because X' dominates G. But

$$2\gamma_{\mathrm{pr}}(G) - |X'| - k' - 1 \geq 2\gamma_{\mathrm{pr}}(G) - 2|X'| - 1 > 0 = |Y'|$$

and so (i) does not hold. Hence there exists a positive integer d such that

$$\begin{aligned} 0 = |Y'| &= 2\gamma_{\mathrm{pr}}(G) - |X'| - k' - d - 1, \\ \text{i.e.,}\quad d &= 2\gamma_{\mathrm{pr}}(G) - |X'| - k' - 1. \end{aligned}$$

Let $A' = X' - Z'$. Then $A' \subseteq N[X'] - Z'$, A' dominates $N\{X'\} - N[Y'] - N[Z']$ and, since $Y' = \emptyset$, M' is a perfect matching of $\langle A' \cup Y' \rangle$. But

$$|A'| = |X'| - k' = 2|X'| - |X'| - k' < 2\gamma_{\mathrm{pr}}(G) - |X'| - k' - 1 = d,$$

thus (ii) also does not hold.

Case 2. $X \neq \emptyset$.
Let M^* be a W-matching, let M_1 be the matching of $\langle X_1 \rangle$ induced by M^*, and let Z_1 be the set of vertices in X_1 which are paired with vertices in D_2 (i.e., the vertices in Z_2) under M^*. Then in G, Z is the set of $\overline{M}$-vertices in X, and $Z \subseteq D$. Define Y and k as in (1). Since D_2 dominates Y_1, $Y_2 \subseteq D_2$ and so $Y \subseteq D$. Moreover, $Y \cap Z = \emptyset$. Hence

$$|Y| \leq |D| - |Z| < 2\gamma_{\mathrm{pr}}(G) - |X| - 1 - k.$$

Therefore (i) does not hold.

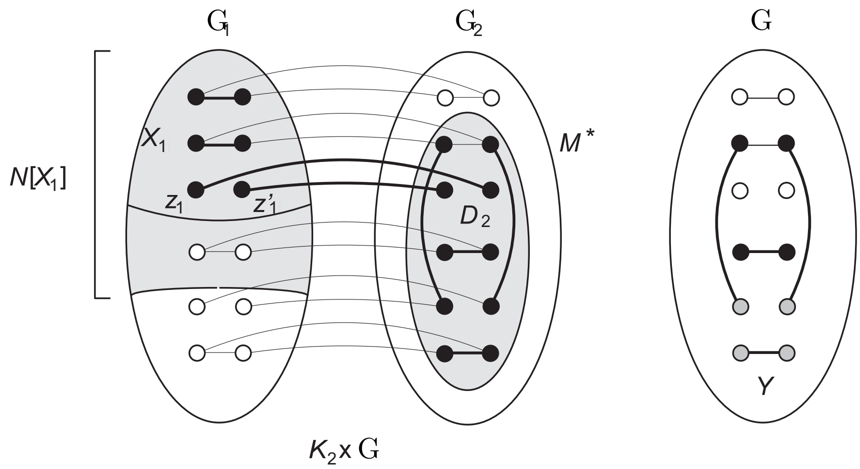

Figure 6. $\langle A \cup Y \rangle$ has a perfect matching in G.

Let $A = D - Z - Y$. Then $A \subseteq N[X] - Z$. Since D_2 dominates all vertices of G_2 except possibly the vertices in $X_2 - D_2$, D dominates $N\{X\}$, and so A dominates $N\{X\} - N[Y] - N[Z]$. Moreover, $A \cup Y = D - Z$ and so $\langle A \cup Y \rangle$ has a perfect matching (corresponding to the edges of M^* with both endvertices in D_2). (See Figure 6, where the black vertices indicate X_1 in G_1, D_2 in G_2, and A in G, the grey vertices indicate Y in G, and the dark edges indicate the matching M^* in $K_2 \,\square\, G$ and the perfect matching in $A \cup Y$.) Since

$$Y = D - Z - A, \quad A \cup Z \subseteq D \quad \text{and} \quad A \cap Z = \emptyset,$$

it follows that

$$|Y| = |D| - |Z| - |A| < 2\gamma_{\mathrm{pr}}(G) - 1 - |X| - k - |A|.$$

Thus

$$|Y| = 2\gamma_{\mathrm{pr}}(G) - |X| - k - d - 1$$

for some $d > |A|$, and so (ii) also does not hold. ■

The following proposition enables us to describe classes of prism γ_{pr}-doublers that are not universal γ_{pr}-doublers.

Proposition 12. *If every vertex that is contained in a γ_{pr}-set of $G \neq K_2$ is adjacent to at least one leaf, then G is a prism γ_{pr}-doubler.*

Proof. It is obvious that any support vertex of a graph G is contained in each PDS of G. Thus, if G satisfies the hypothesis, then $\gamma_{\text{pr}}(G) = k$, where k is the number of support vertices of G. Say $u \in V$ is adjacent to the leaf v. Then in $K_2 \,\square\, G$, u_1, v_1, v_2, u_2, u_1 is an induced 4-cycle, and $\deg v_1 = \deg v_2 = 2$. Thus any PDS of $K_2 \,\square\, G$ contains at least two of these vertices, so that $\gamma_{\text{pr}}(K_2 \,\square\, G) \geq 2k$, and the result follows. ∎

Now let H be a graph of order $k \geq 4$ that has a perfect matching and let G be any graph obtained by joining each vertex of H to at least one leaf, and some vertex v to at most $k-2$ leaves. By Proposition 12, G is a prism γ_{pr}-doubler with γ_{pr}-set $V(H)$ and $\gamma_{\text{pr}}(G) = k$. However, by Corollary 6, G is not a universal γ_{pr}-doubler, because $|\operatorname{epn}(v, V(H))| \leq k-2 < \gamma_{\text{pr}}(G) - 1 = k-1$.

5. Prism Fixers

Since $\gamma_{\text{pr}}(K_2 \,\square\, G) \leq 2\gamma(G)$ for any graph G, it is immediately clear that if $\gamma_{\text{pr}}(G) = 2\gamma(G)$, then G is a prism γ_{pr}-fixer. Examples of such graphs include nontrivial complete graphs, P_5, C_5 and C_6. We now extend this result to determine a necessary and sufficient condition for a graph to be a prism γ_{pr}-fixer.

Let $S \subseteq V$ such that $\langle S \rangle$ has a perfect matching M. A *paired partition* of S is a partition $S_1, \ldots, S_k$ such that each edge of M is contained in $\langle S_i \rangle$ for some i. A *weak* paired partition is a paired partition in which some of the sets may be empty. A *split* of S is a partition $S = S_1 \cup S_2$ such that each edge of M has one endvertex in S_1 and the other one in S_2.

In our next theorem we consider a weak paired partition $S = D \cup Y \cup Z$ of a γ_{pr}-set S of G, and define $U = (V-S) \cap N[D] \cap N[Z]$ and $X = V-S-U$. Note that each vertex in U is adjacent to a vertex in D and to a vertex in Z, each vertex in X is adjacent to vertices in at most one of D and Z, and any vertex of $G-S$ may or may not be adjacent to a vertex in Y. See Figure 7, where S consists of the black vertices, U of the grey vertices and X of the white vertices, and where the vertices in D are indicated by circles, those in Z by squares, and those in Y by triangles.

Theorem 13. *A graph G is a prism γ_{pr}-fixer if and only if G has a γ_{pr}-set S with a weak paired partition $S = D \cup Y \cup Z$ in which Y has a split $Y = Y' \cup Y''$ such that Y' dominates $X = V - S - (N[D] \cap N[Z])$.*

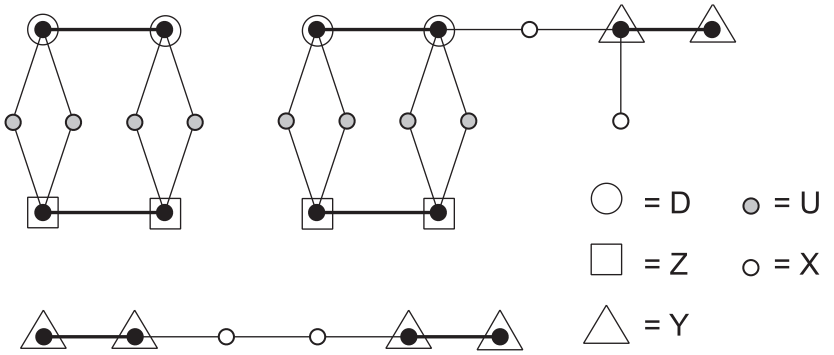

Figure 7. Examples of weak paired partitions.

Proof. Suppose G is a prism γ_{pr}-fixer and let W be a γ_{pr}-set of $K_2 \square G$. Say $D_1' = W \cap V_1$ and $Z_2' = W \cap V_2$. Let M^* be a W-matching in which as few vertices as possible are matched with their own image. Let

$$\begin{aligned}
&S' = D' \cup Z',\\
&Y' = D' \cap Z',\\
&M' \text{ be the matching of } \langle S' \rangle \text{ induced by } M^*,\\
&R \text{ be the set of } \overline{M'}\text{-vertices.}
\end{aligned}$$

Then S' dominates G, $R \subseteq Y'$, and if $u \in R$, then $u_1u_2 \in E(M^*)$. Say $R = \{u^1, \ldots, u^k\}$, let $S^0 = S'$ and for $i = 1, \ldots, k$, construct S^i recursively as follows.

(i) If u^i is adjacent to $s^i \in V - S^{i-1}$, let $S^i = S^{i-1} \cup \{s^i\}$.

(ii) Otherwise, u^i is adjacent to some vertex in S^{i-1} because G is isolate-free, hence $N[u^i] \subseteq S^{i-1}$; let $S^i = S^{i-1} - \{u^i\}$.

Then S^k dominates G, $\langle S^k \rangle$ has a perfect matching and thus S^k is a PDS of G. Moreover,

$$|S^k| \leq |D'| + |Z'| - |Y'| + |R| \leq |W|. \tag{3}$$

But G is a prism γ_{pr}-fixer, so equality holds in (3). In particular, $R = Y'$ and each S^i is constructed as described in (i). Moreover, Y' is independent, for if $u, v \in Y'$ and $uv \in E$, then $u_1u_2,\ v_1v_2 \in E(M^*)$ (since u and v are $\overline{M'}$-vertices) and $(M^* - \{u_1u_2, v_1v_2\}) \cup \{u_1v_1, u_2v_2\}$ is a W-matching in which fewer vertices are mapped to their own images than in M^*, contradicting the choice of M^*.

Let

$$S = S^k, \quad D = D' - Y', \quad Z = Z - Y', \quad Y = S - D - Z \text{ and } Y'' = Y - Y'.$$

Then $D \cup Z \cup Y$ is a weak paired partition of S and $Y' \cup Y''$ is a split of Y and we only need to prove that Y' dominates X. Suppose $x \in X$. We assume that $x \notin N[D]$; the case $x \notin N[Z]$ is similar. Since $x \notin S$, $x_1 \notin D_1'$ and $x_2 \notin Z_2'$. Thus x_1 is dominated in G_1 by a vertex in $D_1' - D_1$, i.e., by a vertex in Y_1'. Therefore x is dominated by a vertex in Y' as required.

Conversely, assume G has a $\gamma_{\rm pr}$-set S that satisfies the conditions of the theorem. Then $D_1 \cup Y_1'$ dominates $(G_1 - Z_1) \cup D_2$, and $Z_2 \cup Y_2'$ dominates $(G_2 - D_2) \cup Z_1$. Hence $W = D_1 \cup Z_2 \cup Y_1' \cup Y_2'$ is a PDS of $K_2 \,\square\, G$ and $|W| = |S| = \gamma_{\rm pr}(G)$. By Proposition 1, W is a $\gamma_{\rm pr}$-set of $K_2 \,\square\, G$. ■

The three graphs in Figure 7 are examples of prism $\gamma_{\rm pr}$-fixers. Other examples of prism fixers include K_n for $n \geq 2$, P_n for $n \in \{3, 5, 6, 9\}$ and C_n for $n \in \{5, 6, 9\}$. (This list contains all paths and cycles that are prism $\gamma_{\rm pr}$-fixers.)

6. Problems

We conclude with open problems related to the above material. The graph G in Figure 1 illustrates that the paired domination number of a graph may exceed the paired domination number of some of its generalized prisms. Note that this graph is $\gamma_{\rm pr}$-edge-critical, i.e., $\gamma_{\rm pr}(G + e) < \gamma_{\rm pr}(G)$ for each edge $e \in E(\overline{G})$. (See [5], for example.)

Problem 1.

(i) Characterize the class of graphs G with $\gamma_{\rm pr}(\pi G) < \gamma_{\rm pr}(G)$ for some permutation π of V.

(ii) If $\gamma_{\rm pr}(\pi G) < \gamma_{\rm pr}(G)$ for some permutation π of V, what is $\max_{\pi \in S_n}\{\gamma_{\rm pr}(\pi G)\}$?

(iii) What is $\min_{\pi \in S_n}\{\gamma_{\rm pr}(\pi G)/\gamma_{\rm pr}(G)\}$?

(iv) If $\gamma_{\rm pr}(\pi G) < \gamma_{\rm pr}(G)$ for some permutation π of V, does it follow that G is $\gamma_{\rm pr}$-edge-critical? (The converse is not true—consider C_5.)

For the usual domination number γ, it is still an open problem to find a nontrivial connected universal fixer, or to show that no such graph exists. The corresponding problem for the paired domination number (for graphs G with $\gamma_{\rm pr}(G) \leq \gamma_{\rm pr}(\pi G)$ for all permutations π of V) has not been studied at all. It is easy to see that nontrivial complete graphs are universal $\gamma_{\rm pr}$-fixers, but none of the other graphs listed at the end of Section 5 is a universal $\gamma_{\rm pr}$-fixer.

Problem 2. Prove or disprove Conjecture 1: The graphs $\overline{K_n}$, $n \geq 1$, are the only universal γ-fixers.

Problem 3.

(i) Characterize the class of universal $\gamma_{\rm pr}$-fixers.

(ii) Failing (i), find examples of noncomplete universal $\gamma_{\rm pr}$-fixers.

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Received 29 January 2009
Revised 27 July 2009
Accepted 27 July 2009