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NOTE

THE NP-COMPLETENESS OF AUTOMORPHIC COLORINGS

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Abstract

Given a graph G, an automorphic edge(vertex)-coloring of G is a proper edge(vertex)-coloring such that each automorphism of the graph preserves the coloring. The automorphic chromatic index (number) is the least integer k for which G admits an automorphic edge(vertex)-coloring with k colors. We show that it is NP-complete to determine the automorphic chromatic index and the automorphic chromatic number of an arbitrary graph.

Keywords: NP-complete problems, chromatic parameters, graph coloring, computational complexity.

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1. Introduction

We assume the reader to be familiar with the terminology and results of NP-completeness as presented in Garey and Johnson [2]. The *automorphic edge-colorings* of a graph have been defined in [1] as proper edge-colorings preserved by each automorphism of the graph. Thereby the *automorphic chromatic index* is the minimum number of colors requires for the existence of an automorphic edge-coloring. Similarly, we can define an *automorphic vertex-coloring* and the *automorphic chromatic number* of a graph.

It is natural to ask about the computational complexity of determining these automorphic chromatic parameters. More precisely, consider the following problems:

INSTANCE: Graph $G = (V, E)$, automorphism group of G, positive integer $k \leq |E|$,($|V|$).

QUESTION: Does G admit an automorphic edge(vertex)-coloring with k colors?

Preliminarly we remark that both problems are NP-problems: it is clear that one can check whether a coloring is proper in polynomial time. The residual task is to verify if the coloring is preserved by the automorphism group of the graph. That can be done in polynomial time just by a brute-force check over all generators of the automorphism group, and a result of Jerrum [4] ensures that the number of these generators is at most equal to the number of vertices. Our aim is to prove that the problems are NP-complete and then it is NP-complete to determine the automorphic chromatic index and the automorphic chromatic number of an arbitrary graph.

It is well known that the corresponding problems of determining the classical chromatic parameters of a graph are NP-complete (see [3] and [5]). Furthermore, it is a trivial consideration that automorphic parameters coincide with the classical ones for each *rigid* graph, that is a graph admitting no non-trivial automorphism. One is willing to believe that the edge-coloring and vertex-coloring problems, which are NP-complete, remain such when restricted to the subclass of rigid graphs: our strategy will consist precisely in furnishing a rigorous proof of this circumstance.

2. Automorphic Edge-Coloring

We will make use of the standard operation on cubic graphs known as Y-reduction and of its inverse, Y-extension, defined as in Figure 1.

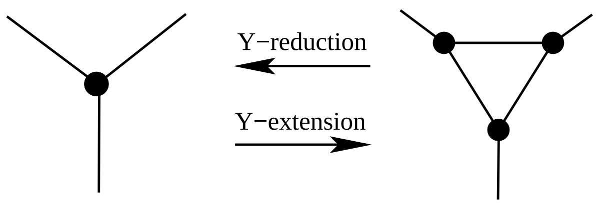

Figure 1. Y-operations.

It is straightforward that the chromatic index is invariant under Y-reduction and Y-extension (see for instance [7]).

By Theorem 6 in [6], the problem to decide whether a 3-connected cubic graph with girth at least 6 has a 3-edge-coloring is NP-complete. For our aim it is sufficient to limit our attention on the class of 3-connected cubic graphs of girth at least 4. We now show a polynomial reduction from this problem to the same problem restricted to the subclass of rigid graphs.

Lemma 1. *Graph* 3*-edge-colorability is NP-complete even when restricted to rigid* 3*-connected cubic graphs of girth at least* 4*.*

Proof. Let G be an arbitrary 3-connected cubic graph of girth at least 4 and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex-set of G. Construct a new graph $\overline{G}$ obtained by G in the following way: we substitute each vertex v_i with a graph H_i formed by a 3-cycle, namely T_i, and $i-1$ 4-cycles, namely Q_i^k for $k = 1, \ldots, i-1$, see Figure 2.

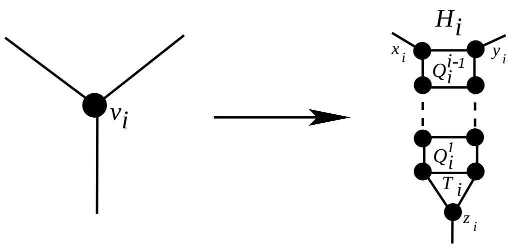

Figure 2. Step of the polynomial reduction.

Note that each of this substitution can be realized by repeated application of i Y-extensions, then the reduction to $\overline{G}$ is polynomial and the chromatic index of $\overline{G}$ is equal to the chromatic index of G.

To conclude the proof we have to prove that $\overline{G}$ is rigid. The graph G is 3-connected of girth greater than 3, then T_i are the unique 3-cycles in $\overline{G}$ and Q_i^k are the unique 4-cycles in $\overline{G}$. Let σ be an automorphism of $\overline{G}$. Let $\sigma(T_i) = T_j$, this implies $\sigma(Q_i^1) = Q_j^1$ and more in general $\sigma(Q_i^k) = Q_j^k$ for each k, thus $i = j$ that is $T_i = T_j$. This proves that each subgraph H_i is fixed by σ. The three vertices x_i, y_i, z_i in H_i are pointwise fixed by σ due to the fact that each H_i is fixed by σ. In particular, since x_i and y_i are fixed by σ then also the vertices of H_i adjecent to them are fixed. By iteration on all vertices of H_i we conclude that each vertex of H_i is fixed by σ. This proves that $\overline{G}$ is rigid. ■

The following theorem easily follows by Lemma 1 and by the coincidence of automorphic chromatic index and chromatic index for rigid graphs.

Theorem 1. *It is NP-complete to determine the automorphic chromatic index of an arbitrary graph.*

3. Automorphic Vertex-Coloring

In what follows the terminology "appending a path of length t to the vertex v" means adding t new vertices $\{w_1, \dots, w_t\}$ and t new edges $[v, w_1], [w_1, w_2], [w_2, w_3], \dots, [w_{t-1}, w_t]$ to a graph. The operation of appending paths will have the same role of the Y-reduction in the previous section: we use it to obtain a rigid graph having the same chromatic number of G. The following polynomial reduction proves that graph k-vertex-colorability, with $k > 2$, is NP-complete in the class of rigid graphs.

Lemma 2. *Graph k-vertex-colorability, with $k > 2$, is NP-complete even when restricted to rigid graphs.*

Proof. Graph k-vertex colorability, with $k > 2$, is NP-complete by a result of Karp [5]. Let G be an arbitrary graph and let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex-set of G. Without loss of generality suppose G to have at least a vertex of degree greater than 2 and to be 2-edge-connected, otherwise either the problem is trivially polynomial or we can delete pendant vertices without modifications of the chromatic number of G. Construct a new graph $\overline{G}$ obtained by G appending a path of length i to the vertex v_i. We denote by w_i^j, for $j = 1, \dots, i$, the vertices in the path appended to v_i. Note that vertices w_i^i are the unique vertices of $\overline{G}$ of degree 1, by the 2-edge-connectivity of G. Moreover, if the chromatic number of G is at least 2 then it is equal to the chromatic number of $\overline{G}$: it is sufficient to color each path appended to v_i alternating the color of v_i and another color. Let σ be an automorphism of $\overline{G}$. Each vertex w_i^i is fixed by σ since its the unique vertex of degree 1 at distance i to a vertex of degree greater than 2. Since w_i^i are fixed then all the vertices w_i^j are fixed. Each vertex w_i^1 has at most (exactly for $i > 1$) two neighbors: the vertex v_i and the vertex w_i^2 (for $i > 1$). Since we have proved that w_i^2 is fixed by σ then v_i is also fixed by σ. Hence the vertex-set of $\overline{G}$ is pointwise fixed by σ. This proves that $\overline{G}$ is rigid. ∎

As already remarked automorphic chromatic number is equal to chromatic number into the class of rigid graphs. Hence we can state the following theorem:

Theorem 2. *It is NP-complete to determine the automorphic chromatic number of an arbitrary graph.*

4. Final Remarks

In this note we have proved that the problem of determining automorphic parameters is NP-complete for an arbitrary graph. As the matter of fact the proofs are achieved within the "trivial" subclass of rigid graphs. One can ask what happens in more symmetric classes of graphs, for which automorphic parameters could be indeed different from the classical ones. For instance: is the problem still NP-hard for vertex-transitive or edge-transitive graphs? In alternative is it conceivable that a careful use of the automorphism group may yield a proper coloring in polynomial time?

A well-known conjecture of Lovász (1970) states that every finite connected vertex-transitive graph contains a Hamiltonian cycle except five known examples. If this conjecture is true, the problem to establish if a vertex-transitive 3-regular graph is 3-edge-colorable is trivial. Is it also possible to deduce that determining the automorphic chromatic index in the class of vertex-transitive 3-regular graphs is trivial?

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