n-ary TRANSIT FUNCTIONS IN GRAPHS *

Manoj Changat, Joseph Mathews

Department of Futures Studies
University of Kerala, Trivandrum, India

e-mail: mchangat@gmail.com
e-mail: jose_chingam@yahoo.co.in

Iztok Peterin

Institute of Mathematics and Physics, FEECS
University of Maribor, Smetanova 17, 2000 Maribor, Slovenia

e-mail: iztok.peterin@uni-mb.si

and

Prasanth G. Narasimha-Shenoi

Department of Mathematics, Government College, Chittur
Palakkad – 678 104, India

e-mail: gnprasanth@gmail.com

Abstract

n-ary transit functions are introduced as a generalization of binary (2-ary) transit functions. We show that they can be associated with convexities in natural way and discuss the Steiner convexity as a natural n-ary generalization of geodesicaly convexity. Furthermore, we generalize the betweenness axioms to n-ary transit functions and discuss the connectivity conditions for underlying hypergraph. Also n-ary all paths transit function is considered.

Keywords: n-arity, transit function, betweenness, Steiner convexity.

2000 Mathematics Subject Classification: 52A01, O5C12.

1. Introduction and Preliminaries

Transit functions in graphs where introduced in [15] to study three basic notions in metric graph theory, namely the interval, convexity and betweenness. It is a set function defined for every pair of points on a set V satisfying three simple axioms. Well studied transit functions on graphs are the geodesic, the induced and the all paths transit functions. These three transit functions and their convexities are extensively studied; some relevant references are: for the geodesic transit function [11, 14, 16, 17, 19], for the induced-path transit function [10, 13, 3, 4], and for the all paths transit function [2, 9, 18]. Path transit functions and their associated convexities are surveyed in [7]. A *transit function* on a set V is a function $R : V \times V \to 2^V$ satisfying the following axioms for any u, v in V.

- $(t1)$ $u \in R(u,v)$;
- $(t2)$ $R(u,v) = R(v,u)$;
- $(t3)$ $R(u,u) = \{u\}$.

The three axioms of the transit function defines three basic betweenness properties, namely, $(t1)$ implies that every element u in V is between u and any other element v, $(t2)$ implies that if x is between u and v, then x is also between v and u and $(t3)$ implies that no element different from u is between u and u for any element u in V. Non-trivial betweenness axioms where introduced by Mulder in [15] and considered also in [8]. Let R be any transit function on a non-empty set V. The betweenness axioms are:

- $(b1)$ $x \in R(u,v) \Rightarrow v \notin R(u,x)$;
- $(b2)$ $x \in R(u,v) \Rightarrow R(u,x) \subseteq R(u,v)$;
- $(b3)$ $x \in R(u,v), y \in R(u,x) \Rightarrow x \in R(y,v)$.

Another stronger betweenness axiom is the monotone axiom (m) defined as:

- (m) $x, y \in R(u,v) \Rightarrow R(x,y) \subseteq R(u,v)$.

A transit function satisfying the axioms $(b1)$ and $(b2)$ is known as a *betweenness* by Mulder [15], see also [7, 6, 5], while that satisfying $(b2)$ and $(b3)$ is known as *geometric transit function* by van de Vel [19] and Nebesky [16]. It is easy to observe that for a given nonempty set V, a function $R : V \times V \to 2^V$, satisfying the axioms $(t1)$, $(t2)$ and $(b1)$ is a transit function in the sense that it satisfies the idempotent axiom $(t3)$.

A family $\mathcal{C}$ of subsets of a nonempty set V is *convexity* or *alignment* of V, if $\mathcal{C}$ fulfills the following conditions:

(1) $\emptyset, V \in \mathcal{C}$;

(2) if $C_i \in \mathcal{C}$, $i = 1, 2, \ldots$, then $\cap_i C_i \in \mathcal{C}$;

(3) for a sequence $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n$ in $\mathcal{C}$, $n \geq 1$, also $\cup_{i=1}^n C_i \in \mathcal{C}$.

The elements of $\mathcal{C}$ are known as *convex sets*. For a subset W of V the smallest convex set containing W is called the $\mathcal{C}$-*convex hull* of W or *convex hull* of W denoted as $\langle W \rangle_{\mathcal{C}}$. Let R be a transit function on a nonempty set V. A set $W \subseteq V$ is called as R-*convex* if $R(u, v) \subseteq W$, for every $u, v \in W$. Every $R(u, v)$ is clearly R-convex, if R satisfies the monotone axiom (m). It is easy to verify that the family of all R-convex sets $\mathcal{C}_R$ on V is a convexity on V in which every singleton set is convex. Convexity in which every singelton is convex is called an S_1-*convexity* [19].

The associated convexity of a transit function R is known as interval convexities (van de Vel [19]) or 2-ary convexities in the sense that the convex hull of a subset A of V (the smallest convex subset containing A) can be generated ultimately by 2 point sets. That is, the convex hull of A can be obtained by taking the union of $R(u, v)$, for every $u, v \in A$ and iterating this operation until we obtain no more new vertices and at that stage we have the R-convex hull of A. An interesting question is that whether an n-ary convexity $(n > 2)$ is the associated convexity of a transit function of arity $n > 2$? Trying to answer this question, we will end up in defining a transit function of arity more than 2. Once we define an n-ary transit function R on V, we can also think of how the betweenness axioms can be extended for R. One can extend meaningfully the axioms $(b1)$, $(b2)$ and (m), but we could not find an analogue of $(b3)$ for $n \geq 3$. The study of transit functions of higher arity is also motivated by natural examples in graphs. The n-ary analogue of the geodesic interval function $I(u, v)$, namely the Steiner interval $S(u_1, u_2, \ldots, u_n)$ and its betweenness properties has been initiated recently in [1]. There specifically, the class of graphs for which the Steiner interval $S(u_1, u_2, \ldots, u_n)$ satisfies the relation $S(u_1, \ldots, u_n) = \cup_{i \neq j} I(u_i, u_j)$ is characterized. This property is defined as the *the union property* of the n-Steiner interval in [1]. When $n = 2$ the union property trivially holds in all graphs. It turns out that, as soon as $n > 3$, this class coincides with the graphs in which the n-Steiner interval enjoys the betweenness axiom $(b2)$ and the monotone axiom (m). For $n = 3$, it is proved in [1] that the class of graphs for which the n-Steiner interval satisfy the union property is properly contained in the class of graphs for which the n-Steiner interval satisfy the monotone axiom (m) which is properly contained in the class of graphs satisfying the betweenness axiom $(b2)$. It would be interesting to

study the n-ary analogues of the other well studied binary transit functions in graphs, namely the induced path transit function and the all paths transit function, in a similar way as attempted for n-Steiner intervals in [1].

In this paper, we make an attempt to study the n-ary analogue of the all paths transit functions for $n \geq 2$. It turns out that the union property, the betweenness axiom $(b2)$ and the monotone axiom (m) holds for any connected graph for the n-ary all paths transit function. The paper is organized as follows. We consider only finite connected graphs.

In Section 2, we introduce the n-ary transit function and its associated convexity. We consider the most prominent example of the n-ary transit functions in graphs, namely the n-Steiner intervals and its associated convexity. In Section 3, we define the betweenness axioms analogous to that of the binary transit functions and consider the implications of these betweenness axioms with specific examples. We also introduce the concept of an underlying hypergraph of an n-ary transit function and discuss a preliminary result on its connectivity using the betweenness axioms. In Section 4, we discuss the n-ary analogue of the binary all paths transit function, its union property, the betweenness axiom $(b2)$ and montone axiom (m) and prove that all these properties hold for any connected graph.

2. n-ary Transit Functions and Associated Convexities

One may call the transit function R defined using the axioms $(t1)$, $(t2)$ and $(t3)$ as "binary transit function", since it associates with every pair of points u, v the function $R(u, v)$. In this section, we generalize the transit function from binary to n-ary, $n > 2$. By relaxing the idempotent axiom, we may obtain non-trivial convex sets even though the singletons are not a convex set. Let V be a non-empty set. Then a function R: $\underbrace{V \times V \times \ldots \times V}_{n \text{ times}} \to 2^V$

is a *transit function of arity n* (or *n-ary transit function*) on V if R satisfies the following axioms.

$(t1)$ $u_1 \in R(u_1, u_2, \ldots, u_n)$;

$(t2)$ $R(u_1, u_2, \ldots, u_n) = R(\pi(u_1, u_2, \ldots, u_n))$ for all $u_i \in V$, where $\pi(u_1, u_2, \ldots, u_n)$ is any permutation of $(u_1, u_2, \ldots, u_n)$;

$(t3)$ $R(u, u, \ldots, u) = \{u\}$ for all $u \in V$.

Note that from $(t1)$ and $(t2)$ it immediately follows that $u_i \in R(u_1, u_2, \ldots, u_n)$, for all $u_i \in V$, $i = 1, 2, \ldots, n$. Similar to the binary case, if V is the

vertex set of a graph G and R an n-ary transit function on V, then R is called an n-ary transit function on G. We observe some basic results on the convexity associated with an n-ary transit function.

If R is a transit function on V, a subset W of V is R-convex if $R(u_1, u_2, \ldots, u_n) \subseteq W$ for any $u_1, u_2, \ldots, u_n \in W$.

Let $\mathcal{C}$ be a convexity on V. We say that $\mathcal{C}$ is of $arity \leq n$ if $\mathcal{C} = \{C \subseteq V \mid F \subseteq C, |F| \leq n \Rightarrow \langle F \rangle_{\mathcal{C}} \subseteq C\}$. Again refer van de Vel [19], for a discussion on n-ary convexities.

Proposition 1. $\mathcal{C}$ is an S_1-convexity on V of arity $\leq n$ if and only if $\mathcal{C}$ is an R-convexity for some n-ary transit function R on V.

Proof. By definition of an n-ary transit function R, the family of R-convex sets $\mathcal{C}_R$ of R is an S_1-convexity of arity at most n.

Conversely, we prove that any S_1-convexity $\mathcal{C}$ of arity at most n is an R-convexity for an n-ary transit function R. For that, let $\mathcal{C}$ be an S_1-convexity on V with arity at most n on V. Define $R(u_1, u_2, \ldots, u_n) = \langle \{u_1, u_2, \ldots, u_n\} \rangle_{\mathcal{C}}$. Then R is clearly an n-ary transit function on V. We prove that $\mathcal{C}_R = \mathcal{C}$. Let first $C \in \mathcal{C}_R$ and $F \subseteq C$ with $|F| \leq n$. By the definition of R we have $\langle F \rangle_{\mathcal{C}_R} \subseteq C$. Therefore $C \in \mathcal{C}$ and $\mathcal{C}_R \subseteq \mathcal{C}$.

Now suppose $C \in \mathcal{C}$. Let $u_1, u_2, \ldots, u_n \in C$. Then $u_1, u_2, \ldots, u_n \in \langle \{u_1, u_2, \ldots, u_n\} \rangle_{\mathcal{C}}$. Now $R(u_1, u_2, \ldots, u_n) = \langle \{u_1, u_2, \ldots, u_n\} \rangle_{\mathcal{C}} \subseteq \mathcal{C}$. Therefore $R(u_1, u_2, \ldots, u_n) \subseteq \mathcal{C}$ which implies that $C \in \mathcal{C}_R$. Therefore $\mathcal{C} \subseteq \mathcal{C}_R$ and hence $\mathcal{C} = \mathcal{C}_R$. ∎

The prime example of a binary transit function on a graph is the geodesic interval function I. The n-ary analogue of I, the Steiner interval function is considered in [1]. We briefly discuss the definition of n-Steiner interval and consider n-Steiner convex sets of a graph G.

Example 2. Let G be a graph. A *Steiner tree* of a multiset $W \subseteq V(G)$, is a minimum order tree in G that contains all vertices of W. The *n-Steiner interval* $S(u_1, u_2, \ldots, u_n)$ consists of all vertices in G that lie on some Steiner tree with respect to $(u_1, u_2, \ldots, u_n)$. Furthermore, a set $A \subseteq V(G)$ is *n-Steiner convex* if it is closed for n-Steiner intervals, i.e. $S(u_1, u_2, \ldots, u_n) \subseteq A$ for every n-multi subset $\{u_1, u_2, \ldots, u_n\}$ of A.

We can easily verify that 2-Steiner intervals are precisely the geodesic intervals I and thus the n-Steiner interval S naturally generalize I.

It can be seen that every $n{+}1$-Steiner convex subset is n-Steiner convex, but converse need not hold.

Theorem 3. *If a set W is $(n+1)$-Steiner convex, then it is also n-Steiner convex.*

Proof. First note that the n-Steiner interval $S(u_1, u_2, \ldots, u_n)$ is precisely the $(n+1)$-Steiner interval $S(u_1, u_1, u_2, \ldots, u_n)$. Thus if W is $(n+1)$-Steiner convex, we have $S(u_1, u_2, \ldots, u_{n+1}) \subseteq W$ for all $u_1, u_2, \ldots, u_{n+1} \in W$ and $S(u_1, u_2, \ldots, u_n) = S(u_1, u_1, u_2, \ldots, u_n) \subseteq W$ for every $u_1, u_2, \ldots, u_n \in W$. Hence W is n-Steiner convex. ∎

We prove that the converse is not true by constructing an example from the subdivision of the complete graph K_{n+1}. In the case for $n = 3$ presented on the following figure, the outer cycle is 2-Steiner convex but not 3-Steiner convex, since for $W = \{u, v, w\}$, $S(W)$ contains x, y, t, s.
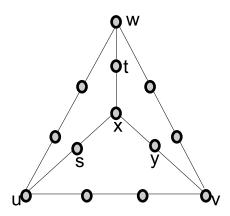

Figure 1. 2-Steiner convex but not 3-Steiner convex.

The general case is treated in the following theorem.

Theorem 4. *There exists a graph G with vertex set V that contains an n-Steiner convex subset which is not $(n+1)$-Steiner convex for every $n \geq 2$.*

Proof. Consider the complete graph K_{n+1} with vertex set $U = \{u_1, u_2, \ldots, u_n, u_{n+1}\}$ and replace each edge of K_{n+1} by a path of length $n + 1$. Let the resulting graph be H. Then add a vertex u_{n+2} and join u_{n+2} with u_i for every $i = 1, \ldots, n + 1$ by paths of length n. Let the new graph be G.

We claim the following: H is a subgraph of G which is n-Steiner convex, but not $(n+1)$-Steiner convex. (Note that the graph on the above figure is the example for $n = 2$.)

Let A be an n-subset of U. Note that for $i \neq j$ there is unique u_i, u_j-shortest path of length $n+1$ and unique shortest u_i, u_j-path that contains u_{n+2} is of length $2n$. Let $u_i \in A$. A subdivided star with the center u_i and leaves in $A - \{u_i\}$ contains $(n-1)(n+1) = n^2 - 1$ edges. Clearly this is a Steiner tree for A, since the distance between any two vertices of A is $n+1$. Thus a Steiner tree for A has $n^2 - 1$ edges. Suppose that a Steiner tree for T contains a vertex $x \in V(G) - V(H)$. Let u_j be a vertex of A that is closest to x. If a tree T with vertices of A contains only u_j, x-path P outside of H, T is not a Steiner tree since $T - P$ still contains all vertices of A. Hence T contains a path $u_i \rightarrow u_{n+2} \rightarrow u_j$, for some $u_i, u_j \in A$.

Let k be the number of u_i, u_{n+2}-shortest paths in T. By the above, $2 \leq k \leq n$. If $k = n$ we get T as a subdivided star with center u_{n+2} and n leaves with vertices from A. However T has n^2 edges which is not possible for a Steiner tree on A. If T has $2 \leq k < n$ such paths, it contains kn edges in $G - H$ and some edges in H that must cover $n - k$ vertices of A, not covered by these paths. For these vertices we need additional $(n-k)(n+1)$ edges, altogether $kn + n^2 - kn + n - k = n^2 + n - k > n^2 - 1$ edges, which is again not possible for a Steiner tree for A.

Let now B be a multi set with n elements that are all in $V(H)$. We have already proved this, when $B \subseteq U$ and all the vertices in B are different. The same arguments hold when some vertices of $B \subseteq U$ are repeated, only the number of edges is smaller in a Steiner tree for B. Let $y \in B$ be from $V(H) - U$. By the structure of G, y lies on a shortest u_i, u_j-path, $i \neq j$. Again by the same reasoning as before no vertex of $V(G) - V(H)$ can be in a Steiner tree for B, since otherwise we can construct (as before) the tree with less edges that contains B.

Thus H (as a subgraph of G) is closed for n-Steiner trees and hence n-Steiner convex. However it is not hard to see that $(V(G) - V(H)) \cup U$ form unique Steiner tree for U and the Steiner-convex hull of $S(u_1, u_2, \ldots, u_{n+1})$, $\langle S(u_1, u_2, \ldots, u_{n+1}) \rangle_{SC} = V(G)$, which is not a subset of $V(H)$. Hence $V(H)$ is n-Steiner convex but not $(n+1)$-Steiner convex. $\blacksquare$

3. Betweenness

We generalize the betweenness axioms $(b1)$, $(b2)$ and (m) in the case of an n-ary transit function. The following betweenness axioms can be considered for an n-ary transit function R. For any $u_1, u_2, \ldots, u_n, x, x_1, x_2, \ldots, x_n \in V$, define

$(b1)$ $x \in R(u_1, u_2, \ldots, u_n), x \neq u_k \Rightarrow u_k \notin R(u_1, u_2, \ldots, x, \ldots, u_n)$.

$(b2)$ $x \in R(u_1, u_2, \ldots, u_n) \Rightarrow R(x, u_2, \ldots, u_n) \subseteq R(u_1, u_2, \ldots, u_n)$.

(m) $\forall\ x_1, x_2, \ldots, x_n \in R(u_1, u_2, \ldots, u_n) \Rightarrow R(x_1, x_2, \ldots, x_n) \subseteq R(u_1, u_2, \ldots, u_n)$.

Observation 5. $(m) \Rightarrow (b2)$ *for any n-ary transit function.*

Analogous to the binary case which is explained in [8], we have examples of transit functions to show that the above betweenness axioms taken one at a time are independent, except when (m) implies $(b2)$. The illustration is given in the Table 1.

Table 1. n-ary transit functions with possible betweenness relations.
$\times$ denotes that the corresponding axiom is satisfied.

$(b1)$	$(b2)$	(m)	$Example$
$\times$			6
	$\times$		7
$\times$	$\times$		8
	$\times$	$\times$	9
$\times$	$\times$	$\times$	10
			11

In the following examples R will be an n-ary transit function on V with $|V| > n + 2$. Thus $R(u, u, \ldots, u) = \{u\}$ holds and $R(u_1, u_2, \ldots, u_n)$ equals to $R(\pi(u_1, u_2, \ldots, u_n))$ for all possible permutations π. Also $v_1, v_2, \ldots, v_n$ are some fixed vertices of V.

Example 6. Let $A = R(v_1, v_2, \ldots, v_n) = \{v_1, v_2, \ldots, v_{n+1}\}$, $B = R(v_2, \ldots, v_{n+1}) = \{v_2, v_3, \ldots, v_{n+2}\}$ and $R(u_1, u_2, \ldots, u_n) = \{u_1, u_2, \ldots, u_n\}$ for any other n-tuple. We can see that R satisfies $(b1)$ but not $(b2)$, since $v_{n+1} \in A$ but $B \nsubseteq A$. By the same reason also (m) is not fulfilled.

Example 7. Define $R(v_1, v_2, \ldots, v_n) = V$, $R(v_2, \ldots, v_{n+1}) = \{v_2, \ldots, v_{n+1}\}$, $C = R(v_1, v_2, \ldots, v_2) = \{v_1, v_2, v_3\}$, $A = R(v_1, v_3, \ldots, v_3) = \{v_1, v_2, v_3\}$, $B = R(v_2, v_3, \ldots, v_3) = \{v_1, v_2, v_3\}$ and in all the other cases $R(u_1, u_2, \ldots, u_n) = \{u_1, u_2, \ldots, u_n\}$. We can see that R satisfies $(b2)$ but not $(b1)$, see A and B, and not (m), since $v_3 \notin R(u_1, u_2, \ldots, u_{n-2}, v_1, v_2) = D$ if every $u_i \neq v_3$, but $C \nsubseteq D$.

Example 8. Let $R(v_1, v_2, \ldots, v_2) = \{v_1, v_2, v_3\}$, $R(v_1, v_3, \ldots, v_3) = \{v_1, v_3\}$, $R(v_1, v_4, \ldots, v_4) = \{v_1, v_2, v_3, v_4\}$, $R(v_2, v_3, \ldots, v_5) = \{v_2, v_3, v_5\}$ and in all the other cases $R(u_1, u_2, \ldots, u_n) = \{u_1, u_2, \ldots, u_n\}$. We can see that R satisfies $(b2)$ and $(b1)$ but not (m) by a similar reason as in previous example.

Example 9. We can see that for $R(u_1, u_2, \ldots, u_n) = V$, R satisfies (m) and $(b2)$ but not $(b1)$, since $|V| > n + 2 > n$.

Example 10. For $R(v_1, v_2, \ldots, v_n) = V$ and $R(u_1, u_2, \ldots, u_n) = \{u_1, u_2, \ldots, u_n\}$ for the rest, we can see that R satisfies (m), $(b2)$ and $(b1)$.

Example 11. Let $A = R(v_1, v_2, \ldots, v_n) = \{v_1, v_2, \ldots, v_{n+1}\}$, $R(v_2, \ldots, v_{n+1}) = V$ and $R(u_1, u_2, \ldots, u_n) = \{u_1, u_2, \ldots, u_n\}$ for any other n-tuple. We can see that R does not satisfy any of the axioms $(b1)$, $(b2)$ and (m). For $v_{n+1} \neq v_1$ and $v_{n+1} \in A$, but $v_1 \in R(v_{n+1}, v_2 \ldots, v_n)$, hence $(b1)$ is not fulfilled. Also $v_{n+1} \in A$, but $V \nsubseteq A$ and hence $(b2)$ and (m) are not fulfilled.

Given a binary transit function R on a non-empty set V, one can define an underlying graph G_R with vertex set V and uv is an edge of G_R if and only if $R(u, v) = \{u, v\}$. The underlying graphs of ordinary transit functions are discussed in [6, 14, 2]. It is proved in [6] that if R is a betweenness on V, then the underlying graph G_R of V is connected and both axioms $(b1)$ and $(b2)$ are necessary for the connectedness of G_R. Analogous to the binary case, for an n-ary transit function R, we obtain the underlying hypergraph which we denote as H_R. We first define a hypergraph. A *hypergraph H* is a pair $(V, \mathcal{E})$ where V is a nonempty finite set and $\mathcal{E}$ is a collection of subsets of V. The members of $\mathcal{E}$ are called *edges* of H. Let R be any transit function on V. Then the underlying hypergraph H_R on V is defined as follows. V is the vertex set of H_R and E is an edge of H_R if and only if $E = R(u_1, u_2, \ldots, u_n) = \{u_1, u_2, \ldots, u_n\}$. Two vertices u and v of a hypergraph H are *connected* if there exists a sequence $(E_1, E_2, \ldots, E_n)$ $n \geq 1$, of edges of H such that $u \in E_1$, $v \in E_n$ and $E_i \cap E_{i+1} \neq \emptyset$ for

$1 \leq i \leq n - 1$. Hypergraph H is *connected* if every pair of vertices u and v are connected.

Let R be any transit function on V. For $n = 2$, we can easily see that the hypergraph H_R becomes the underlying graph G_R of R.

For an n-ary transit function R, $n > 2$, we have the following proposition, which is an analogue of the binary transit proved in [6].

Proposition 12. *Let R be an n-ary transit function on V with $|V| > n$. If R satisfies ($b1$) and ($b2$), then H_R is connected.*

Proof. Let $V = \{v_1, v_2, \ldots, v_m\}$, $m > n$. Consider $u, v \in V(G)$. Let us denote $u = u_1$ and $v = u_p$ for some positive integer p. Consider $R(v, u, \ldots, u)$. Suppose $|R(v, u, \ldots, u)| > 2$, then there exists $u_2 \in R(v, u, \ldots, u)$ such that $u_2 \neq u, v$. Since R satisfies ($b2$) axiom, $R(v, u_2, \ldots, u_2) \subseteq R(v, u, \ldots, u)$ and $R(u, u_2, \ldots, u_2) \subseteq R(v, u, \ldots, u)$. Also since R satisfies ($b1$) axiom, both $R(v, u_2, \ldots, u_2)$ and $R(u, u_2, \ldots, u_2)$ are proper subsets of $R(v, u, \ldots, u)$. If any one of these proper subsets contains more than 2 elements, by applying the construction described above, we can divide it into two proper subsets. Since $R(v, u, \ldots, u)$ is finite, this process cannot continue indefinitely. Hence after a finite number of steps we get distinct vertices $u_2, u_3, \ldots, u_{p-1} \in R(v, u, \ldots, u)$ such that $|R(u_{i+1}, u_i, \ldots, u_i)| = 2$ for $i = 0, 1, \ldots, p-1$. Since $|R(u_{i+1}, u_i, \ldots, u_i)| = 2$, $E_i = \{u_i, u_{i+1}\}$ is an edge, for $i = 0, 1, \ldots, n$ and also $E_i \cap E_{i+1} = \{u_{i+1}\}$. Hence $E_0, E_1, \ldots, E_p$ is a u, v-path in H_R. Hence H_R is connected. ∎

As in the binary case, we can show that both axioms ($b1$) and ($b2$) are necessary for the connectedness of H_R. As an application of the celebrated Hall's matching theorem (marriage theorem) for bipartite graphs [12], it is possible to establish the existence of an n-ary transit function R satisfying ($b1$), but not ($b2$) with disconnected underlying hypergraph H_R. For completeness, we state the Hall's theorems and prove the existence of such a transit function.

Note that in a bipartite graph G, a *matching* in G is a set of edges S such that any two edges in S are pairwise disjoint and a matching is *maximal* if for any edge in $e \in E \setminus S$, $S \cup \{e\}$ is not a matching. A *perfect matching* is a matching such that every vertex is incident to some edge in S.

Theorem 13 (Hall's Marriage Theorem [12])**.** *Let G be a bipartite graph with bipartition U and V. Then there is a maximal matching from U to V*

if and only if Hall's condition is satisfied: $|N(A)| \geq |A|$ for all subsets A of U. Here $N(A)$ denotes the set of neighbors of the vertices in A.

Theorem 14 (Hall's marriage theorem on regular bipartite graphs [12]). *If G is a k-regular bipartite graph for $k > 1$, then G has a perfect matching.*

We construct $(n-1)$-bipartite graphs as follows. Let U and V be the set of all k-tuples and $(k+1)$-tuples, respectively, of distinct integers $1, 2, \ldots, 2n+1$ (for $k = 2, 3, \ldots, n$, respectively) so that $(a_1, a_2, \ldots, a_k)$ is such that $i \leq j \Rightarrow a_i \leq a_j$. Define edges of G as follows. If $u \in U$ and $v \in V$, then (u, v) is an edge of G if and only if u is a proper subset of v. Then G is a bipartite graph. Thus we have $(n-1)$-bipartite graphs. Each of these bipartite graphs satisfies the conditions of the Halls's marriage theorem, because each vertex in U has more than one neighbor. Therefore for any subset A of U, we have $|N(A)| \geq |A|$. Thus by Hall's theorem, there exists a maximal matching from U to V. In fact for $k = n$, we have that both U and V have the same number of elements and the degree of a vertex in U is the same as the degree of a vertex in V which is equal to $k+1$. Hence when $k = n$, G has a perfect matching by Hall's theorem on regular bipartite graphs. Therefore when $k = n$, two distinct n-tuples are mapped to two distinct $(n+1)$-tuples.

Now, we use the edges of the maximal matching in all the bipartite graphs for $k = 1, 2, \ldots, n-1$, as well as the edges of the regular bipartite graph when $k = n$ to define the required n-ary transit function R. Therefore the vertex set (U, V) of the bipartite graph G can be considered as U consists of all n-tuples and V consists of subsets of cardinality $3 \leq n+1$ (by considering the tuples of V as subsets).

Let uv be an edge of the maximal matching given by Hall's theorem, for $k = 2, 3, \ldots, n$, with coresponding tuples $u = (u_1, u_2, \ldots, u_n)$ and $v = (y_1, y_2, \ldots, y_{n+1})$. Define R as $R(u, u, \ldots, u) = \{u\}$ and $R(u_1, u_2, \ldots, u_n) = \{y_1, y_2, \ldots, y_{n+1}\}$.

Clearly R is an n-ary transit function satisfying the $(b1)$ axiom and the underlying hypergraph H_R of R has no edges and so H_R is disconnected. It is to be noted that R does not satisfy the $(b2)$ axiom, as any two distinct vertices of U when $k = n$ are mapped by R into two distinct subsets of equal cardinality and hence one cannot be a subset of the other. Therefore R fails to satisfy the $(b2)$ axiom. Thus, we have proved the following theorem.

Theorem 15. *There exists an n-ary transit function R satisfying the $(b1)$-axiom, but not the $(b2)$-axiom, with disconnected underlying hypergraph H_R.*

We illustrate the above theorem with an example for $n = 3$. The actual construction of an arbitrary n-ary transit function may not be easy.

Example 16. (R satisfies ($b1$) but not ($b2$)); and G_R is not connected.
Let $V = \{1, 2, 3, 4, 5, 6, 7\}$ and let R be a 3-ary transit function, defined on V as follows. There are 35 distinct 3 tuples and 4 tuples with distinct elements. We define R on distinct 3 tuples as follows.

$R(1, 2, 3) = \{1, 2, 3, 4\}$, $R(1, 2, 4) = \{1, 2, 4, 5\}$, $R(1, 2, 5) = \{1, 2, 3, 5\}$,
$R(1, 2, 6) = \{1, 2, 3, 6\}$, $R(1, 2, 7) = \{1, 2, 3, 7\}$, $R(1, 3, 4) = \{1, 3, 4, 5\}$,
$R(1, 3, 5) = \{1, 3, 5, 6\}$, $R(1, 3, 6) = \{1, 3, 6, 7\}$, $R(1, 3, 7) = \{1, 3, 5, 7\}$,
$R(1, 4, 5) = \{1, 4, 5, 6\}$, $R(1, 4, 6) = \{1, 4, 6, 7\}$, $R(1, 4, 7) = \{1, 2, 4, 7\}$,
$R(1, 5, 6) = \{1, 2, 5, 6\}$, $R(1, 5, 7) = \{1, 2, 5, 7\}$, $R(1, 6, 7) = \{1, 2, 6, 7\}$,
$R(2, 3, 4) = \{2, 3, 4, 6\}$, $R(2, 3, 5) = \{2, 3, 4, 5\}$, $R(2, 3, 6) = \{2, 3, 5, 6\}$,
$R(2, 3, 7) = \{2, 3, 4, 7\}$, $R(2, 4, 5) = \{2, 4, 5, 7\}$, $R(2, 4, 6) = \{1, 2, 4, 6\}$,
$R(2, 4, 7) = \{2, 4, 6, 7\}$, $R(2, 5, 6) = \{2, 5, 6, 7\}$, $R(2, 5, 7) = \{2, 3, 5, 7\}$,
$R(2, 6, 7) = \{2, 3, 6, 7\}$, $R(3, 4, 5) = \{3, 4, 5, 6\}$, $R(3, 4, 6) = \{1, 3, 4, 6\}$,
$R(3, 4, 7) = \{1, 3, 4, 7\}$, $R(3, 5, 6) = \{3, 5, 6, 7\}$, $R(3, 5, 7) = \{3, 4, 5, 7\}$,
$R(3, 6, 7) = \{3, 4, 6, 7\}$, $R(4, 5, 6) = \{2, 4, 5, 6\}$, $R(4, 5, 7) = \{1, 4, 5, 7\}$,
$R(4, 6, 7) = \{4, 5, 6, 7\}$, $R(5, 6, 7) = \{1, 5, 6, 7\}$.

There are 21 distinct 3 tuples with one element repeating. R on such 3 tuples as follows.

$R(1, 1, 2) = R(1, 2, 2) = \{1, 2, 3\}$, $R(1, 1, 3) = R(1, 3, 3) = \{1, 3, 4\}$,
$R(1, 1, 4) = R(1, 4, 4) = \{1, 4, 5\}$, $R(1, 1, 5) = R(1, 5, 5) = \{1, 5, 6\}$,
$R(1, 1, 6) = R(1, 6, 6) = \{1, 6, 7\}$, $R(1, 1, 7) = R(1, 7, 7) = \{1, 2, 7\}$,
$R(2, 2, 3) = R(2, 3, 3) = \{2, 3, 4\}$, $R(2, 2, 4) = R(2, 4, 4) = \{2, 4, 5\}$,
$R(2, 2, 5) = R(2, 5, 5) = \{2, 5, 6\}$, $R(2, 2, 6) = R(2, 6, 6) = \{2, 6, 7\}$,
$R(2, 2, 7) = R(2, 7, 7) = \{2, 3, 7\}$, $R(3, 3, 4) = R(3, 4, 4) = \{3, 4, 5\}$,
$R(3, 3, 5) = R(3, 5, 5) = \{3, 5, 6\}$, $R(3, 3, 6) = R(3, 6, 6) = \{1, 3, 6\}$,
$R(3, 3, 7) = R(3, 7, 7) = \{3, 4, 7\}$, $R(4, 4, 5) = R(4, 5, 5) = \{4, 5, 6\}$,
$R(4, 4, 6) = R(4, 6, 6) = \{4, 6, 7\}$, $R(4, 4, 7) = R(4, 7, 7) = \{4, 5, 7\}$,
$R(5, 5, 6) = R(5, 6, 6) = \{5, 6, 7\}$, $R(5, 5, 7) = R(5, 7, 7) = \{1, 5, 7\}$,
$R(6, 6, 7) = R(6, 7, 7) = \{3, 6, 7\}$; and $R(u, u, u) = \{u\}$ for all $u \in V$.

We have a simple example of a transit function R satisfying ($b2$), but not ($b1$) with disconnected H_R.

Example 17. Define $R(u_1, u_2, \ldots, u_n) = V(G)$ for all $u_1, u_2, \ldots, u_n \in V(G)$ and $R(u, u, \ldots, u) = \{u\}$ with $|V(G)| > n$. It can be verified that R satisfies ($b2$) but not ($b1$) and that H_R is totally disconnected.

4. n-ary All Paths Transit Function of a Graph

Let $G = (V, E)$ be a connected graph. The binary *all paths transit function* A of G is defined as $A(u, v) = \{w \in V | w$ lies on some u, v-path in $G\}$. A *block* of a graph is a maximal connected subgraph without a cut vertex. A graph G is called a *block graph* if and only if every block of G is a clique. A *tree of blocks* in G is a connected subgraph such that, whenever it contains two distinct vertices u and v of some block of G, it contains the whole block. Let $(u_1, u_2, \ldots, u_n)$ be an n-tuple of $V \times V \times \cdots \times V$. The n-ary analogue of the all paths transit function denoted as $A_n(u_1, u_2, \ldots, u_n)$ is defined as the set of vertices belonging to the smallest tree of blocks containing the multi set $\{u_1, u_2, \ldots, u_n\}$. We can easily verify that when $n = 2$, A_2 is precisely the all paths transit function A of G. If every element of $\{u_1, u_2, \ldots, u_n\}$ belongs to the same block of G, then $A_n(u_1, u_2, \ldots, u_n)$ induces the block of G containing the multi set $\{u_1, u_2, \ldots, u_n\}$. Note that if G is itself a block, then the family of A_n-convex sets in G is just the trivial convexity consisting of the empty set $\emptyset$, all the singletons and V just as the case of A-convex sets. Note also that for every set V the n-ary transit function A_n satisfies the monotone axiom (m). Analogous to the union property of n-Steiner interval S, we say that the n-ary all paths transit function $A_n(u_1, u_2, \ldots, u_n)$ satisfy the union property if $A_n(u_1, \ldots, u_n) = \cup_{i \neq j} A(u_i, u_j)$, where A is the binary all paths transit function of G.

To understand the structure of $A_n(u_1, u_2, \ldots, u_n)$ more clearly, we use the intersection graph of blocks of the graph G, denoted as $B(G)$, which is defined as the graph whose vertices correspond to the blocks of G and two blocks will form an edge in the intersection graph if the corresponding blocks have a common cut vertex. It is well-known and also is trivial to verify that the graph $B(G)$ is a block graph in the sense that every block of $B(G)$ forms a clique. It can be easily verified that a tree of blocks in G correspond to some Steiner tree in $B(G)$ and conversely. For n-Steiner intervals, the following theorem is proved in [1]:

Theorem 18 [1]**.** *Let G be a connected graph and $n > 3$. The following statements are equivalent:*

 (i) *G is a block graph,*

 (ii) *the n-Steiner interval on G satisfies (m),*

 (iii) *the n-Steiner interval on G satisfies $(b2)$,*

 (iv) *the n-Steiner interval on G satisfies the union property.*

We have the following theorem.

Theorem 19. *Let G be a connected graph and $n \geq 2$ and A_n be the n-ary all paths transit function of G. The following statements are equivalent:*

(i) *the A_n on G satisfies (m),*

(ii) *the A_n on G satisfies $(b2)$,*

(iii) *the A_n on G satisfies the union property.*

Proof. The theorem can be proved using Theorem 18. For this, consider the graph $B(G)$ of G. Since $B(G)$ is a block graph and $A_n(u_1, u_2, \ldots, u_n)$ correspond to an n-Steiner interval in $B(G)$ and by Theorem 18, the condition (i), (ii) and (iii) are equivalent for $n > 3$. It can be easily verified that for $n = 3$ also the conditions (i), (ii) and (iii) hold. Hence the theorem follows. ∎

We conclude the paper with an interesting question, namely, what will be the n-ary analogue of the induced path transit function of a connected graph and what about its betweenness and convexity properties?

References

[1] B. Brešar, M. Changat, J. Mathews, I. Peterin, P.G. Narasimha-Shenoi and A. Tepeh Horvat, *Steiner intervals, geodesic intervals, and betweenness*, Discrete Math. **309** (2009) 6114–6125.

[2] M. Changat, S. Klavžar and H.M. Mulder, *The All-Paths Transit Function of a Graph*, Czechoslovak Math. J. **51** (126) (2001) 439–448.

[3] M. Changat and J. Mathew, *Induced path transit function, monotone and Peano axioms*, Discrete Math. **286** (2004) 185–194.

[4] M. Changat and J. Mathew, *Characterizations of J-monotone graphs*, in: Convexity in Discrete Structures (M. Changat, S. Klavžar, H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 47–55.

[5] M. Changat, J. Mathew and H.M. Mulder, *Induced path function, betweenness and monotonicity*, Discrete Appl. Math. **158** (2010) 426–433.

[6] M. Changat, J. Mathew and H.M. Mulder, *Induced path transit function, betweenness and monotonicity*, Elect. Notes Discrete Math. **15** (2003).

[7] M. Changat, H.M. Mulder and G. Sierksma, *Convexities Related to Path Properties on Graphs*, Discrete Math. **290** (2005) 117–131.

[8] M. Changat, P.G. Narasimha-Shenoi and I.M. Pelayo, *The longest path transit function and its betweenness*, to appear in Util. Math.

[9] P. Duchet, *Convexity in combinatorial structures*, Rend. Circ. Mat. Palermo (2) Suppl. **14** (1987) 261–293.

[10] P. Duchet, *Convex sets in graphs* II. *Minimal path convexity*, J. Combin. Theory (B) **44** (1988) 307–316.

[11] P. Duchet, *Discrete convexity: retractions, morphisms and partition problem*, in: Proceedings of the conference on graph connections, India, (1998), Allied Publishers, New Delhi, 10–18.

[12] P. Hall, *On representation of subsets*, J. Lon. Mat. Sc. **10** (1935) 26–30.

[13] M.A. Morgana and H.M. Mulder, *The induced path convexity, betweenness and svelte graphs*, Discrete Math. **254** (2002) 349–370.

[14] H.M. Mulder, The Interval Function of a Graph. Mathematical Centre Tracts 132, Mathematisch Centrum (Amsterdam, 1980).

[15] H.M. Mulder, *Transit functions on graphs* (*and posets*), in: Convexity in Discrete Structures (M. Changat, S. Klavžar, H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 117–130.

[16] L. Nebeský, *A characterization of the interval function of a connected graph*, Czechoslovak Math. J. **44(119)** (1994) 173–178.

[17] L. Nebeský, *A Characterization of the interval function of a* (*finite or infinite*) *connected graph*, Czechoslovak Math. J. **51(126)** (2001) 635–642.

[18] E. Sampathkumar, *Convex sets in graphs*, Indian J. Pure Appl. Math. **15** (1984) 1065–1071.

[19] M.L.J. van de Vel, Theory of Convex Structures (North Holland, Amsterdam, 1993).