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ON RAMSEY $(K_{1,2}, C_4)$-MINIMAL GRAPHS

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Abstract

For graphs F, G and H, we write $F \to (G, H)$ to mean that any red-blue coloring of the edges of F contains a red copy of G or a blue copy of H. The graph F is Ramsey (G, H)-minimal if $F \to (G, H)$ but $F^* \nrightarrow (G, H)$ for any proper subgraph $F^* \subset F$. We present an infinite family of Ramsey $(K_{1,2}, C_4)$-minimal graphs of any diameter ≥ 4.

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1. Introduction

All graphs considered in this paper are finite, undirected, without loops and multiple edges. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The *distance* $d_G(u, v)$ between two vertices u and v in a graph G is the length of the shortest path connecting them. The *eccentricity* of a vertex u is the greatest distance between u and any other vertex in G. The *diameter* of a connected graph G is the maximum distance between two vertices in G. If G contains vertices v_1, v_2, v_3, v_4 and edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3$, we say that the edge v_1v_3 lies inside 4-cycle $v_1v_2v_3v_4$.

Let F, G and H be graphs. We say that F *contains* G if F contains a subgraph isomorphic to G. We write $F \to (G, H)$ if whenever each edge of F is colored either red or blue, then F contains a red copy of G or a blue copy of H. A graph F is *Ramsey* (G, H)*-minimal* if $F \to (G, H)$ but $F^* \nrightarrow (G, H)$ for any proper subgraph $F^* \subset F$. The class of all Ramsey (G, H)-minimal graphs is denoted by $\mathcal{R}(G, H)$.

Numerous papers study the problem of determining the set $\mathcal{R}(G, H)$. Burr, Erdös and Lovász [5] showed that $\mathcal{R}(K_{1,2}, K_{1,2}) = \{K_{1,3}, C_{2n+1}\}$ where $n \geq 1$. Later, Burr *et al.* [4] proved that if m, n are odd, then $\mathcal{R}(K_{1,m}, K_{1,n}) = \{K_{1,m+n-1}\}$. All graphs belonging to $\mathcal{R}(2K_2, K_{1,n})$ for $n \geq 3$ were presented by Mengersen and Oeckermann [7]. Borowiecki, Hałuszczak and Sidorowicz [2] determined the class $\mathcal{R}(K_{1,2}, K_{1,n})$ for $n \geq 3$.

Łuczak [6] proved that if G is a forest other than a matching and H is a graph containing at least one cycle, then $\mathcal{R}(G, H)$ is infinite. It follows that the set $\mathcal{R}(K_{1,2}, C_n)$ is infinite for any $n \geq 3$. Borowiecki, Schiermeyer and Sidorowicz [3] found all graphs in $\mathcal{R}(K_{1,2}, C_3)$. Recently, Baskoro, Yulianti and Assiyatun [1] gave a family of graphs belonging to $\mathcal{R}(K_{1,2}, C_4)$, where an infinite family of Ramsey $(K_{1,2}, C_4)$-minimal graphs was stated only for diameter 2. We present an infinite class of Ramsey $(K_{1,2}, C_4)$-minimal graphs for any diameter ≥ 4.

2. Graphs of Diameter 4

We define some classes of graphs. Let $t \geq 6$ be an even integer. Let $G(t)$ be a graph with the vertex set $V(G(t)) = \{v, v_1, v_2, \ldots, v_t = v_0\}$ and with the edge set $E(G(t)) = \{vv_{2i} : i = 1, 2, \ldots, \frac{t}{2}\} \cup \{v_jv_{j+1} : j = 0, 1, \ldots, t-1\}$.

Let $A_1(t)$ be a graph with $V(A_1(t)) = V(G(t)) \cup \{v', v'_0\}$ and $E(A_1(t)) = E(G(t)) \cup \{vv_1, vv'_0, v_0v'_0, v'v_0, v'v_1, v_2v_4\}$.
Let $A_2(t)$ be a graph with $V(A_2(t)) = V(G(t)) \cup \{v'_1, v'_p\}$ for odd $p \in \{3, 5, \ldots, t-1\}$ and $E(A_2(t)) = E(G(t)) \cup \{v_0v'_1, v'_1v_2, v_{p-1}v'_p, v'_pv_{p+1}\}$.

Let $A_3(t)$ be a graph with $V(A_3(t)) = V(G(t)) \cup \{v'_1, v'_2\}$ and $E(A_3(t)) = E(G(t)) \cup \{vv_3, vv'_2, v_2v'_2, v_0v'_1, v'_1v_2\}$.

We show that $A_1(t), A_2(t)$ and $A_3(t)$ are Ramsey $(K_{1,2}, C_4)$-minimal graphs.

Assertion 1. $A_1(t) \in \mathcal{R}(K_{1,2}, C_4)$.

Proof. First we prove that $A_1(t) \to (K_{1,2}, C_4)$. Consider any red-blue coloring of the edges of $A_1(t)$. Suppose that there is no red copy of $K_{1,2}$ in the coloring. Since the edges vv_0, vv_1, vv_2, v_0v_1 and v_2v_4 lie inside 4-cycles, we can not color them by red, because otherwise, we would have blue copies of C_4 in our coloring. We must color by red the edge v_1v_2 to avoid blue 4-cycle $vv_0v_1v_2$. Next, we must color by red the edge $v'v_0$ to avoid blue 4-cycle $vv_0v'v_1$ and the edge vv_0' to avoid blue 4-cycle $vv_0'v_0v_1$. Then all the edges $vv_i, i = 0, 2, \ldots, t-2$ must be blue. It follows that to avoid blue 4-cycles $vv_jv_{j+1}v_{j+2}, j = 2, 4, \ldots, t-4$, the edges $v_{j+1}v_{j+2}$ must be red and v_jv_{j+1} are blue. But since $v_{t-3}v_{t-2}$ and $v'v_0$ are red, we are not able to avoid blue 4-cycle $vv_{t-2}v_{t-1}v_0$ which means that $A_1(t) \to (K_{1,2}, C_4)$.

Now let us show that $A_1^*(t) \nrightarrow (K_{1,2}, C_4)$ for the graph $A_1^*(t) \simeq A_1(t)\setminus \{e\}$, where e is any fixed edge of $A_1(t)$. Let $e = v_lv_{l+1}, l = 2, 3, \ldots, t-1$. We can color by red the edges $vv_0', v'v_0$ and v_iv_{i+1}, where $i = 1, 3, \ldots, l-1; l+2, l+4, \ldots, t-2$ if l is even, and $i = 1, 3, \ldots, l-2; l+1, l+3, \ldots, t-2$ if l is odd. We color by blue all the edges of $A_1^*(t)$ that are not colored by red.

If $e = vv_l, l = 2, 4, \ldots, t$, color by red the edges $vv_0', v'v_0, v_1v_2$ and $v_iv_{i+1}, i = 3, 5, \ldots, l-3; l+2, l+4, \ldots, t-2$. If $e = vv_0', v_0v_0'$ or v_0v_1, the edges colored by red are vv_0 and v_iv_{i+1}, where $i = 1, 3, \ldots, t-3$. If $e = vv_1, v_1v_2$ or v_2v_4, we can color by red $v'v_1, v_0v_0', vv_2$ and $v_iv_{i+1}, i = 4, 6, \ldots, t-2$. Finally, if $e = v'v_0$ or $v'v_1$, we color by red vv_4, v_0v_1 and v_iv_{i+1}, $i = 6, 8, \ldots, t-2$. The other edges will be colored by blue. These colorings of $A_1^*(t)$ contain neither a red copy of $K_{1,2}$ nor a blue copy if C_4. The proof is complete. ■

Assertion 2. $A_2(t) \in \mathcal{R}(K_{1,2}, C_4)$.

Proof. Let us show that $A_2(t) \to (K_{1,2}, C_4)$. We consider any red-blue coloring of the edges of $A_2(t)$ such that there is no red copy of $K_{1,2}$ in the coloring. In order to avoid blue 4-cycles containing at least one of the vertices v_1, v_1', v_p or v_p', we must color by red one of the edges v_iv, v_iv_1, v_iv_1' for $i = 0, 2$ and one of the edges v_jv, v_jv_p, v_jv_p' for $j = p-1, p+1$. Note that if $p = 3$ or $p = t-1$, we must color by red the edge vv_2 or vv_0. There can be at most one red edge $vv_i, i \in \{2, 4, \ldots, t\}$ in our coloring. It can be seen that if all the edges $vv_i, i = 2, 4, \ldots, p-1$ are blue, we can not avoid blue 4-cycle $vv_jv_{j+1}v_{j+2}$ for some $j \in \{2, 4, \ldots, p-3\}$, and if all the edges

$vv_i, i = p+1, p+3, \ldots, t$ are blue, it is not possible to avoid blue 4-cycle $vv_jv_{j+1}v_{j+2}$ for a $j \in \{p+1, p+3, \ldots, t-2\}$. Therefore, $A_2(t) \to (K_{1,2}, C_4)$.

To prove the minimality of $A_2(t)$, consider the graph $A_2^*(t) \simeq A_2(t) \backslash \{e\}$ for any fixed edge $e \in E(A_2(t))$. Let $e = v_lv_{l+1}, l = 0, 1, \ldots, p$. We can color by red the edges $vv_0, v_p'v_{p+1}$, $v_iv_{i+1}, i = p+2, p+4, \ldots, t-3$ and v_jv_{j+1}, where $j = 1, 3, \ldots, l-1; l+2, l+4, \ldots, p-1$ if l is even, and $j = 1, 3, \ldots, l-2; l+1, l+3, \ldots, p-1$ if l is odd. If $e = vv_l, l = 2, 4, \ldots, p+1$, the edges colored by red are $vv_0, v_1'v_2, v_p'v_{p+1}$ and $v_iv_{i+1}, i = 3, 5, \ldots, l-3; l+2, l+4, \ldots, p-1; p+2, p+4, \ldots, t-3$. The rest of the edges of $A_2^*(t)$ will be colored by blue. There is no red copy of $K_{1,2}$ and no blue copy of C_4 in these colorings. The cases $e = v_0v_1', v_1'v_2, v_{p-1}v_p', v_p'v_{p+1}, v_jv_{j+1}, j = p+1, p+2, \ldots, t-1$ or $e = vv_i, i = p+3, p+5, \ldots, t$ are similar. ■

Assertion 3. $A_3(t) \in \mathcal{R}(K_{1,2}, C_4)$.

Proof. We show that $A_3(t) \to (K_{1,2}, C_4)$. Let us consider any red-blue coloring of $A_3(t)$. Assume there is no red $K_{1,2}$ in the coloring. We can not color by red the edges vv_2 and vv_3, because they lie inside 4-cycles $vv_2'v_2v_3$ and $vv_2v_3v_4$. We also can not color by red the edges v_2v_2' and v_2v_3, because then, we would not be able to avoid blue 4-cycle $vv_0v_1v_2$, $vv_0v_1'v_2$ or $v_0v_1v_2v_1'$. It follows that to avoid blue 4-cycle $vv_2'v_2v_3$, we must color by red the edge vv_2'. Then the edges $vv_i, i = 2, 4, \ldots, t$ must be blue. Consequently, if we want to avoid blue cycles $vv_0v_1v_2$ and $vv_0v_1'v_2$, we must color by red either the edges $v_0v_1, v_1'v_2$ or the edges v_0v_1', v_1v_2. The edges $v_jv_{j+1}, j = 2, 3, \ldots, t-3$ must be colored alternatingly by blue and red. It follows that we can not avoid blue 4-cycle $vv_{t-2}v_{t-1}v_t$. Hence, $A_3(t) \to (K_{1,2}, C_4)$.

In order to prove the minimality of $A_3(t)$ we consider $A_3^*(t) \simeq A_3(t) \backslash \{e\}$, where e is any fixed edge of $A_3(t)$. Let $e = v_lv_{l+1}, l = 0, 1, \ldots, t-1$. We can color by red the edges vv_2', v_0v_1' and v_iv_{i+1}, where $i = 1, 3, \ldots, l-1; l+2, l+4, \ldots, t-2$ if l is even (where $i = 1, 3, \ldots, l-2; l+1, l+3, \ldots, t-2$ if l is odd). If $e = vv_l, l = 2, 4, \ldots, t$, the edges colored by red are vv_2', v_0v_1', v_1v_2 and $v_iv_{i+1}, i = 1, 3, \ldots, l-3; l+2, l+4, \ldots, t-2$. If $e = vv_3, vv_2'$ or v_2v_2', color by red vv_2, v_0v_1' and v_iv_{i+1}, where $i = 4, 6, \ldots, t-2$, and if $e = v_0v_1'$ or $v_1'v_2$, color by red vv_0, v_2v_2' and $v_iv_{i+1}, i = 3, 5, \ldots, t-3$. The other edges will be colored by blue. The colorings of $A_3^*(t)$ contain neither a red $K_{1,2}$ nor a blue C_4. This finishes the proof. ■

It is easy to verify that the graphs $A_i(t), i = 1, 2, 3$ have diameter 4 for $t \geq 8$, and 3 if $t = 6$.

3. Auxiliary Results

Let us introduce Definitions 1 and 2.

Definition 1. Let F be a graph with $U \subset V(F)$. For any given graphs G and H, provided that the vertices in U are not incident to red edges, we write $F \to (G(U), H)$ to mean that any red-blue coloring of the edges of F contains a red copy of G or a blue copy of H.

Definition 2. Let $U_0 \subset V(F)$ where $|U_0| = p$. For $i \in \{0, 1, \ldots, p-1\}$ a graph F is Ramsey $(G(U_0)_i, H)$-minimal if

(i) $F \to (G(U_i), H)$, where U_i is any subset of U_0 such that $|U_i| = p - i$,

(ii) $F^* \nrightarrow (G(U_i), H)$ for any proper subgraph $F^* \subset F$,

(iii) $F \nrightarrow (G(U_{i+1}), H)$, where U_{i+1} is any subset of U_i such that $|U_{i+1}| = p - i - 1$.

Vertices in U_0 will be called *roots* of F and the class of all Ramsey $(G(U_0)_i, H)$-minimal graphs will be denoted by $\mathcal{R}(G(U_0)_i, H)$.

If F is Ramsey $(G(U_0)_0, H)$-minimal, we write $F \in \mathcal{R}(G(U_0), H)$. Particularly, for $U_0 = \emptyset$, F is a Ramsey (G, H)-minimal graph.

We need to define the following families of graphs:

$L_1(t)$ is a graph with $V(L_1(t)) = V(G(t)) \cup \{v'\}$ and $E(L_1(t)) = E(G(t)) \cup \{vv_1, v'v_0, v'v_1, v_2v_4\}$. Let us remind that $G(t)$ is defined for an even integer $t \geq 6$.

$L_2(t)$ is a graph with $V(L_2(t)) = V(G(t)) \cup \{v_1'\}$ and $E(L_2(t)) = E(G(t)) \cup \{v_0v_1', v_1'v_2\}$.

$L_3(t)$ is a graph with $V(L_3(t)) = V(G(t)) \cup \{v_0'\}$ and $E(L_3(t)) = E(G(t)) \cup \{vv_1, vv_0', v_0v_0'\}$.

$M_2(t) = G(t)$ and $M_3(t)$ is a graph with $V(M_3(t)) = V(G(t))$ and $E(M_3(t)) = E(G(t)) \backslash \{v_0v_1, v_1v_2\} \cup \{vv_1, v_1v_4, vv_5\}$.

Let $s \geq 5$ be odd. $M_1(s)$ is a graph with the vertex set $V(M_1(s)) = \{v, v_1, v_2, \ldots, v_s = v_0\}$ and with the edge set $E(M_1(s)) = \{vv_i, i = 1, 2, \ldots, s\} \cup \{v_jv_{j+1}, j = 1, 2, \ldots, s-1\}$.

We prove some lemmas characterizing the graphs defined above.

Lemma 1. (i) *Let $t \geq 8$ and $p \in \{6, 8, \ldots, t-2\}$. Then $L_1(t) \in \mathcal{R}(K_{1,2}(v_p), C_4)$.*
(ii) *Let $t \geq 10$ and $r, s \in \{6, 8, \ldots, t-2\}$, $r \neq s$. Then $L_1(t) \in \mathcal{R}(K_{1,2}(v_r, v_s)_1, C_4)$.*

Proof. (i) First we show that $L_1(t) \to (K_{1,2}(v_p), C_4)$ for even integers t, p, where $t \geq 8$ and $p \in \{6, 8, \ldots, t-2\}$. Provided that there are no red edges incident to the vertex v_p, let us consider any red-blue coloring of the edges of $L_1(t)$ such that we have no red copy of $K_{1,2}$ in the coloring. Since the edges vv_1, vv_2, v_0v_1 and v_2v_4 lie inside 4-cycles, we can not color them by red, because then, we would have blue copies of C_4 in our coloring. We must color by red one of the edges vv_0, v_1v_2 and one of the edges vv_4, v_1v_2 to avoid blue 4-cycles $vv_0v_1v_2$ and $vv_1v_2v_4$, which means that v_1v_2 must be red in any case. Consequently, we color by red one of the edges vv_4, v_3v_4 and one of the edges $vv_0, v'v_0$ to avoid blue 4-cycles $vv_2v_3v_4$ and $vv_0v'v_1$.

Since there can be at most one red edge $vv_i, i \in \{4, 6 \ldots, t\}, i \neq p$, without lose of generality we can assume that all the edges $vv_j, j = 4, 6, \ldots, p-2$ are blue. In order to avoid blue 4-cycles $vv_jv_{j+1}v_{j+2}, j = 2, 4, \ldots, p-4$, we must color the edges $v_{j+1}v_{j+2}$ by red. Clearly, the edges v_jv_{j+1} are blue. Then, since $v_{p-3}v_{p-2}$ is red and no red edge can be incident to v_p, we have blue 4-cycle $vv_{p-2}v_{p-1}v_p$ in our coloring. Hence, $L_1(t) \to (K_{1,2}(v_p), C_4)$.

Now we prove that $L_1^*(t) \nrightarrow (K_{1,2}(v_p), C_4)$, where $L_1^*(t) \simeq L_1(t) \backslash \{e\}$ for any fixed edge $e \in E(L_1(t))$. Let $e = v_lv_{l+1}, l = 2, 3, \ldots, p-1$. The edges colored by red are $vv_0, v_iv_{i+1}, i = p+1, p+3, \ldots, t-3$ and v_jv_{j+1}, where $j = 1, 3, \ldots, l-1; l+2, l+4, \ldots, p-2$ if l is even, and $j = 1, 3, \ldots, l-2$; $l+1, l+3, \ldots, p-2$ if l is odd.

If $e = vv_l, l = 2, 4, \ldots, p$, we can color by red the edges vv_0, v_1v_2, v_iv_{i+1}, $i = 3, 5, \ldots, l-3; l+2, l+4, \ldots, p-2; p+1, p+3, \ldots, t-3$. If $e = vv_0, vv_1, v'v_0$ or $v'v_1$, color by red the edges vv_{p-2} and v_iv_{i+1}, where $i = 1, 3, \ldots, p-5$; $p+1, p+3, \ldots, t-1$. If $e = v_0v_1$ or v_1v_2, the edges colored by red are vv_1 and $v_iv_{i+1}, i = 2, 4, \ldots, p-2; p+1, p+3, \ldots, t-1$. If $e = v_2v_4$, we color by red $v'v_1, vv_2$ and v_iv_{i+1}, $i = 4, 6, \ldots, p-2; p+1, p+3, \ldots, t-1$. The rest of the edges will be colored by blue. If $e = v_lv_{l+1}, l = p, p+1, \ldots, t-1$ or $e = vv_k, k = p+2, p+4, \ldots, t-2$, we can analogously show that there exists a red-blue coloring of $L_1^*(t)$ containing neither a red $K_{1,2}$ nor a blue C_4 such that there is no red edge incident to the vertex v_p.

Clearly, $L_1(t) \nrightarrow (K_{1,2}, C_4)$, because $L_1(t) \subset A_1(t)$. Hence, $L_1(t) \in \mathcal{R}(K_{1,2}(v_p), C_4)$.

(ii) From the proof of part (i) we get $L_1(t) \to (K_{1,2}(v_p), C_4)$ for $p \in \{6, 8, \ldots, t-2\}$, $L_1^*(t) \nrightarrow (K_{1,2}(v_p), C_4)$ for $L_1^*(t) \simeq L_1(t) \backslash \{e\}$, where e is any fixed edge of $L_1(t)$, and $L_1(t) \nrightarrow (K_{1,2}, C_4)$. This shows that for $t \geq 10$ one has $L_1(t) \in \mathcal{R}(K_{1,2}(v_r, v_s)_1, C_4)$, where $r, s \in \{6, 8, \ldots, t-2\}, r \neq s$. The proof is complete. ∎

Lemma 2. (i) *Let* $t \geq 6$ *and* $p \in \{4, 6, \ldots, t-2\}$. *Then* $L_2(t) \in \mathcal{R}(K_{1,2}(v_p), C_4)$.
(ii) *Let* $t \geq 8$ *and* $r, s \in \{4, 6, \ldots, t-2\}$, $r \neq s$. *Then* $L_2(t) \in \mathcal{R}(K_{1,2}(v_r, v_s)_1, C_4)$.

Proof. (i) We prove that $L_2(t) \to (K_{1,2}(v_p), C_4)$. Consider any red-blue coloring of the edges of $L_2(t)$ such that there is no red edge incident to the vertex v_p. Assume that we have no red $K_{1,2}$ in the coloring. We must color by red one of the edges v_iv, v_iv_1, v_iv_1' for $i = 0, 2$ to avoid blue 4-cycles containing at least one of the vertices v_1, v_1'. Note that there can be at most one red edge $vv_i, i \in \{2, 4, \ldots, t\}, i \neq p$ in our coloring. It is easy to show that if all the edges $vv_i, i = 2, 4, \ldots, p-2$ are blue, we are not able to avoid blue 4-cycle $vv_jv_{j+1}v_{j+2}$ for some $j \in \{2, 4, \ldots, p-2\}$, and if $vv_i, i = p+2, p+4, \ldots, t$ are blue, we can not avoid blue 4-cycle $vv_jv_{j+1}v_{j+2}$ for a $j \in \{p, p+2, \ldots, t-2\}$. $L_2(t) \to (K_{1,2}(v_p), C_4)$.

Consider $L_2^*(t) \simeq L_2(t) \backslash \{e\}$ for any fixed edge $e \in E(L_2(t))$. We show that $L_2^*(t) \nrightarrow (K_{1,2}(v_p), C_4)$. Let $e = v_lv_{l+1}, l = 0, 1, \ldots, p-1$. We can color by red the edges $vv_0, v_iv_{i+1}, i = p+1, p+3, \ldots, t-3$ and the edges v_jv_{j+1}, where $j = 1, 3, \ldots, l-1; l+2, l+4, \ldots, p-2$ if l is even, and $j = 1, 3, \ldots, l-2; l+1, l+3, \ldots, p-2$ if l is odd. If $e = vv_l, l = 2, 4, \ldots, p$, the edges colored by red are $vv_0, v_1v_2, v_iv_{i+1}, i = 3, 5, \ldots, l-3; l+2, l+4, \ldots, p-2; p+1, p+3, \ldots, t-3$. The other edges are colored by blue. The cases $e = v_0v_1', v_1'v_2, v_lv_{l+1}, l = p, p+1 \ldots, t-1$ and $e = vv_k, k = p+2, p+4, \ldots, t$ are similar.

Finally, since $L_2(t) \subset A_2(t)$, it is evident that $L_2(t) \nrightarrow (K_{1,2}, C_4)$.
(ii) The proof follows from the previous part. ■

Lemma 3. (i) *Let* $t \geq 6$ *and* $p = 0$ *or* $t-2$. *Then* $L_3(t) \in \mathcal{R}(K_{1,2}(v_p), C_4)$.
(ii) *Let* $t \geq 6$. *Then* $L_3(t) \in \mathcal{R}(K_{1,2}(v_0, v_{t-2})_1, C_4)$.

The proof is analogous to the proofs of Lemma 1 and Lemma 2.

Lemma 4. *Let* $s \geq 5$. *Then* $M_1(s) \in \mathcal{R}(K_{1,2}(v_1, v_s), C_4)$.

Proof. Let us show that $M_1(s) \to (K_{1,2}(v_1, v_s), C_4)$. Provided that the vertices v_1, v_s are not incident to red edges, we consider any red-blue coloring of $M_1(s)$ such that there is no red copy of $K_{1,2}$ in the coloring. If we color by red some edge $vv_i, i \in \{2, 3, \ldots, s-1\}$, we have blue 4-cycle $vv_{i-1}v_iv_{i+1}$. Therefore, all the edges $vv_i, i = 1, 2, \ldots, s$ must be blue. In order to avoid blue 4-cycles $vv_{j-1}v_jv_{j+1}$ and $vv_jv_{j+1}v_{j+2}, j = 2, 4, \ldots, s-3$, the edges v_jv_{j+1} must be red. Then we are not able to avoid blue 4-cycle $vv_{s-2}v_{s-1}v_s$.

We prove that $M_1^*(s) \nrightarrow (K_{1,2}(v_1, v_s), C_4)$, where $M_1^*(s) \simeq M_1(s)\backslash\{e\}$ for any fixed edge $e \in E(M_1(s))$. Let $e = v_lv_{l+1}, l = 1, 2, \ldots, s-1$. We can color by red the edges v_iv_{i+1}, where $i = 2, 4, \ldots, l-2; l+1, l+3, \ldots, s-2$ if l is even, and $i = 2, 4, \ldots, l-1; l+2, l+4, \ldots, s-2$ if l is odd. Let $e = vv_l, l = 3, 4, \ldots, s$. The edges colored by red are vv_{l-1} and v_iv_{i+1}, where $i = 2, 4, \ldots, l-4, l+1, l+3, \ldots, s-2$ if l is even, and $i = 2, 4, \ldots, l-3$, $l+2, l+4, \ldots, s-2$ if l is odd. We color by blue all the edges of $M_1^*(s)$ that are not colored by red. The cases $e = vv_1$ or vv_2 can be handled similarly.

Finally, $M_1(s) \nrightarrow (K_{1,2}(v_p), C_4)$ for $p = 1$ (for $p = s$), since there exists a red-blue coloring of $M_1(s)$ containing neither a red $K_{1,2}$ nor a blue C_4 such that there is no red edge incident to v_p. It is enough to color by red the edges v_iv_{i+1}, where $i = 2, 4, \ldots, s-1$ (where $i = 1, 3, \ldots, s-2$) and color by blue the rest of the edges. This finishes the proof. ■

Lemma 5. *Let $t \geq 6$. Then $M_3(t) \in \mathcal{R}(K_{1,2}(v_0, v_2), C_4)$.*

Proof. Let us consider any red-blue coloring of $M_3(t)$ such that the vertices v_0, v_2 are not incident to any red edges. We show that $M_3(t) \rightarrow (K_{1,2}(v_0, v_2), C_4)$. Suppose that we have no red $K_{1,2}$ in the coloring. We can not color by red the edges vv_4 and vv_5, because they lie inside 4-cycles $vv_1v_4v_5$ and $vv_4v_5v_6$. It follows that we must color by red the edge v_3v_4 to avoid blue cycle $vv_2v_3v_4$, and the edge vv_1 to avoid blue cycle $vv_1v_4v_5$. But then, it is not possible to avoid blue 4-cycle $vv_jv_{j+1}v_{j+2}$ for some $j \in \{4, 6, \ldots, t-2\}$, which shows that $M_3(t) \rightarrow (K_{1,2}(v_0, v_2), C_4)$.

Now consider the graph $M_3^*(t) \simeq M_3(t)\backslash\{e\}$, where e is any fixed edge of $M_3(t)$. Let us prove that $M_3^*(t) \nrightarrow (K_{1,2}(v_0, v_2), C_4)$, Let $e = v_lv_{l+1}, l = 2, 3, \ldots, t-1$. We can color by red the edges vv_1 and v_iv_{i+1}, where $i = 3, 5, \ldots, l-1; l+2, l+4, \ldots, t-2$ if l is even, and $i = 3, 5, \ldots, l-2$; $l+1, l+3, \ldots, t-2$ if l is odd. If $e = vv_l, l = 2, 4, \ldots, t$, the edges colored by red are vv_1 and v_iv_{i+1}, where $i = 3, 5, \ldots, l-3; l+2, l+4, \ldots, t-2$. If $e = vv_1, vv_5$ or v_1v_4, we color by red the edges vv_4 and $v_iv_{i+1}, i = 6, 8, \ldots, t-2$. The rest of the edges will be colored by blue. The colorings of $M_3^*(t)$ contain neither a red copy of $K_{1,2}$ nor a blue copy of C_4.

In order to show that $M_3(t) \nrightarrow (K_{1,2}(v_p), C_4)$ for $p = 0$ (for $p = 2$) it suffices to color by red the edges $v_iv_{i+1}, i = 2, 4, \ldots, t-2$ (the edges vv_1 and $v_iv_{i+1}, i = 3, 5, \ldots, t-1$) and color by blue all the other edges. ■

Lemma 6. (i) *Let* $t \geq 6$ *and* $p \in \{2, 4, \ldots, t\}$. *Then* $M_2(t) \in \mathcal{R}(K_{1,2}(v, v_p), C_4)$.
(ii) *Let* $t \geq 8$ *and* $p \in \{4, 6, \ldots, t-4\}$. *Then* $M_2(t) \in \mathcal{R}(K_{1,2}(v_0, v_p), C_4)$.

The proof is similar to the previous proofs.

4. Main Results

Let $n \geq 4$. Let $M_{a_j}, j = 1, 2, \ldots, k$ be any graphs with roots $r_{a_j,1}, r_{a_j,2}$ such that $M_{a_j} \in \mathcal{R}(K_{1,2}(r_{a_j,1}, r_{a_j,2}), C_n)$. Let $L_{b_i}, i = 1, 2$ be any graphs with a root r_{b_i} such that $L_{b_i} \in \mathcal{R}(K_{1,2}(r_{b_i}), C_n)$ and let L be any graph with roots r_1, r_2, where $L \in \mathcal{R}(K_{1,2}(r_1, r_2)_1, C_n)$.

Let $P(a_1, a_2, \ldots, a_k)$ be a graph which consists of k graphs $M_{a_1}, M_{a_2}, \ldots, M_{a_k}$, where the vertex $r_{a_j,2}$ is stuck to the vertex $r_{a_{j+1},1}, j = 1, 2, \ldots, k-1$. A graph $C(a_1, a_2, \ldots, a_k)$ is defined in the same way with the only difference that $r_{a_1,1}$ is stuck to $r_{a_k,2}$ as well.

Finally, we define the following families of graphs:
$B_1(C(a'_1, a'_2, \ldots, a'_{k_1}), P(a_1, a_2, \ldots, a_{k_2})), k_1 \geq n+1, k_2 \geq 1$, is a graph that consists of the graphs $C(a'_1, a'_2, \ldots, a'_{k_1})$ and $P(a_1, a_2, \ldots, a_{k_2})$, where the first root of M_{a_1} is stuck to any root x of $C(a'_1, a'_2, \ldots, a'_{k_1})$ and the second root of $M_{a_{k_2}}$ is stuck to any root y of $C(a'_1, a'_2, \ldots, a'_{k_1})$, where $d_{C(a'_1, a'_2, \ldots, a'_{k_1})}(x, y) + d_{P(a_1, a_2, \ldots, a_{k_2})}(x, y) \geq n+1$.

$B_2(L, P(a_1, a_2, \ldots, a_k)), k \geq n$, is a graph which consists of the graphs L and $P(a_1, a_2, \ldots, a_k)$, where the first root of M_{a_1} is stuck to the first root of L and the second root of M_{a_k} is stuck to the second root of L.

$B_3(L_{b_1}, P(a_1, a_2, \ldots, a_k), L_{b_2}), k \geq 0$, is obtained by sticking the first root of M_{a_1} to the root of L_{b_1} and the second root of M_{a_k} is stuck to the root of L_{b_2}.

$B_4(C(a'_1, a'_2, \ldots, a'_{k_1}),\ P(a_1, a_2, \ldots, a_{k_2}),\ C(a''_1, a''_2, \ldots, a''_{k_3}));\ k_1,\ k_3 \geq n+1, k_2 \geq 0$, is constructed by sticking the first root of M_{a_1} to any root of $C(a'_1, a'_2, \ldots, a'_{k_1})$ and the second root of $M_{a_{k_2}}$ is stuck to any root of $C(a''_1, a''_2, \ldots, a''_{k_3})$.

$B_5(L_{b_1}, P(a_1, a_2, \ldots, a_{k_1}), C(a'_1, a'_2, \ldots, a'_{k_2})), k_1 \geq 0, k_2 \geq n+1$, is obtained by sticking the first root of M_{a_1} to the root of L_{b_1} and the second root of $M_{a_{k_1}}$ is stuck to any root of $C(a'_1, a'_2, \ldots, a'_{k_3})$.

The graphs defined above will be also denoted briefly by $B_1, B_2, \ldots, B_5$. The graphs $M_{a'_i}, i = 1, 2, \ldots, k_1$ and $M_{a_j}, j = 1, 2, \ldots, k_2$ will be called *seeds* of B_1. Seeds of B_2, B_3, B_4 and B_5 can be defined analogously. We show that $B_1, B_2, \ldots, B_5$ are Ramsey $(K_{1,2}, C_n)$-minimal graphs.

Theorem 1. $B_1 \in \mathcal{R}(K_{1,2}, C_n)$.

Proof. First let us show by contradiction that $B_1 \to (K_{1,2}, C_n)$. Assume that $B_1 \nrightarrow (K_{1,2}, C_n)$. Since $M_{a'_i} \in \mathcal{R}(K_{1,2}(r_{a'_i,1}, r_{a'_i,2}), C_n), i = 1, 2, \ldots, k_1$ and $M_{a_j} \in \mathcal{R}(K_{1,2}(r_{a_j,1}, r_{a_j,2}), C_n), j = 1, 2, \ldots, k_2$, by part (i) of Definition 2, we must color by red at least one edge incident to some root in $M_{a'_i}$ (in M_{a_j}) to have a red-blue coloring of the edges of $M_{a'_i}$ (of M_{a_j}) that contains neither a red copy of $K_{1,2}$ nor a blue copy of C_n. But then, we have at least $k_1 + k_2$ red edges incident to roots in B_1. Because the number of different roots in B_1 is $k_1 + k_2 - 1$, there must be a red copy of $K_{1,2}$ in any coloring of B_1. A contradiction.

In order to prove the minimality of B_1 it suffices to show that $B_1^* \nrightarrow (K_{1,2}, C_n)$, where $B_1^* \simeq B_1 \backslash \{e\}$ for any fixed edge $e \in E(B_1)$. Suppose $e \in E(M_{a'_i})$ where $i \in \{1, 2, \ldots, k_1\}$. (The case $e \in E(M_{a_j}), j \in \{1, 2, \ldots, k_2\}$ can be handled similarly). Then $M^*_{a'_i} \simeq M_{a'_i} \backslash \{e\}$. We know that $M^*_{a'_i} \nrightarrow (K_{1,2}(r_{a'_i,1}, r_{a'_i,2}), C_n)$, which means that there exists a red-blue coloring of the edges of $M^*_{a'_i}$ containing neither a red copy of $K_{1,2}$ nor a blue copy of C_n such that the roots $r_{a'_i,1}, r_{a'_i,2}$ are not incident to red edges in $M^*_{a'_i}$.

From Definition 2 it follows that in any other seed of B_1^* we must color by red some edges incident to any fixed root, while the second root does not have to be incident to red edges of the seed to have a red-blue coloring of the seed containing no red $K_{1,2}$ and no blue C_n. Note that since the coloring contains no red $K_{1,2}$, there must be just one red edge in the seed which is incident to the fixed root.

Thus, we can color the edges of B_1^* such that every root is incident to exactly one red edge. We do not have any red copy of $K_{1,2}$ in the coloring of B_1^*. Since the number of seeds in $C(a'_1, a'_2, \ldots, a'_{k_1})$ is $k_1 \geq n + 1$ and $d_{C(a'_1, a'_2, \ldots, a'_{k_1})}(x, y) + d_{P(a_1, a_2, \ldots, a_{k_2})}(x, y) \geq n + 1$, we do not have any blue copy of C_n in the coloring of B_1^* as well. This finishes the proof. ■

Theorem 2. $B_2 \in \mathcal{R}(K_{1,2}, C_n)$.

Proof. We show that $B_2 \to (K_{1,2}, C_n)$. Suppose the contrary, let $B_2 \nrightarrow (K_{1,2}, C_n)$. Since $M_{a_i} \in \mathcal{R}(K_{1,2}(r_{a_i,1}, r_{a_i,2}), C_n)$, $i = 1, 2, \ldots, k$ and $L \in \mathcal{R}(K_{1,2}(r_1, r_2)_1, C_n)$, from part (i) of Definition 2 it follows that we must have at least one red edge incident to some root in M_{a_i} to obtain a red-blue coloring of the edges of M_{a_i} containing neither a red copy of $K_{1,2}$ nor a blue copy of C_n.

In any red-blue coloring of L that contains no red $K_{1,2}$ and no blue C_n, there must be at least one red edge e_1 incident to the first root in L and at least one red edge e_2 incident to the second root in L, where the edges e_1, e_2 are not necessarily different. Because the number of different roots in B_2 is $k+1$, there must be a root incident to at least two red edges. We have a red copy of $K_{1,2}$ in the coloring of B_2, a contradiction.

Let us prove that $B_2^* \nrightarrow (K_{1,2}, C_n)$ for $B_2^* \simeq B_2 \backslash \{e\}$, where e is any fixed edge of B_2. We distinguish two cases:

a) Let $e \in E(M_{a_i})$ where $i \in \{1, 2, \ldots, k\}$. Then $M_{a_i}^* \simeq M_{a_i} \backslash \{e\}$ and $M_{a_i}^* \nrightarrow (K_{1,2}(r_{a_i,1}, r_{a_i,2}), C_n)$, which says that there exists a red-blue coloring of $M_{a_i}^*$ containing neither a red $K_{1,2}$ nor a blue C_n, where there are no red edges incident to the roots $r_{a_i,1}, r_{a_i,2}$ in $M_{a_i}^*$.

Now consider all the other seeds $M_{a_j}, j = 1, 2, \ldots, k$, $j \neq i$ and L. By Definition 2, in any seed M_{a_j} we must color by red some edges incident to any fixed root to have a red-blue coloring of M_{a_j} that contains neither a red $K_{1,2}$ nor a blue C_n. The second root does not have to be incident to any red edge of M_{a_j}. Since the coloring does not contain any red $K_{1,2}$, the fixed root is incident to exactly one red edge in M_{a_j}. In the seed L, if we have exactly one red edge incident to the first root and one red edge incident to the second root, there exists a red-blue coloring of L that does not contain any red $K_{1,2}$ and any blue C_n.

It follows that it is possible to color the edges of B_2^* such that every root is incident to exactly one red edge, hence there is no red $K_{1,2}$ in the coloring of B_2^*. Because the number of seeds in B_2^* is $k+1 \geq n+1$, there is also no blue C_n in the coloring.

b) Let $e \in E(L)$. Then $L^* \simeq L \backslash \{e\}$ and $L^* \nrightarrow (K_{1,2}(r_j), C_n), j = 1, 2$, which means that there is a red-blue coloring of L^* that contains neither a red $K_{1,2}$ nor a blue C_n, where there is no red edge incident to r_j in L^*. Note that the other root can be incident to at most one red edge in L^*, otherwise we have a red $K_{1,2}$ in the coloring of L^*.

Consider the seeds M_{a_j}, $j = 1, 2, \ldots, k$. Analogously as in case a) it suffices to color by red exactly one edge of M_{a_j} which is incident to any root, while the second root does not have to be incident to any red edge in M_{a_j} to have a red-blue coloring of M_{a_j} that contains no red $K_{1,2}$ and no blue C_n, Then we are able to color B_2^* such that we have neither a red $K_{1,2}$ nor a blue C_n in the coloring. The proof is complete. ■

Theorem 3. $B_5 \in \mathcal{R}(K_{1,2}, C_n)$.

Proof. Let us prove by contradiction that $B_5 \to (K_{1,2}, C_n)$. Because $M_{a_i} \in \mathcal{R}(K_{1,2}(r_{a_i,1}, r_{a_i,2}), C_n)$, $i = 1, 2, \ldots, k_1$ (because $M_{a'_j} \in \mathcal{R}(K_{1,2}(r_{a'_j,1}, r_{a'_j,2}), C_n)$, $j = 1, 2, \ldots, k_2$ and $L_{b_1} \in \mathcal{R}(K_{1,2}(r_{b_1}), C_n)$), in any red-blue coloring of M_{a_i} (of $M_{a'_j}$, L_{b_1}) that contains no red $K_{1,2}$ and no blue C_n, there must be at least one red edge incident to some root in M_{a_i} (in $M_{a'_j}$, L_{b_1}). Then there are at least $k_1 + k_2 + 1$ red edges incident to roots in B_5. Since the number of roots in B_5 is $k_1 + k_2$, we have a red $K_{1,2}$ in any coloring of B_5. A contradiction.

We show that $B_5^* \nrightarrow (K_{1,2}, C_n)$ for the graph $B_5^* \simeq B_5 \backslash \{e\}$, where e is any fixed edge of B_5. Assume that $e \in E(M_{a_i})$ where $i \in \{1, 2, \ldots, k_1\}$. (The cases $e \in E(M_{a'_j}), j \in \{1, 2, \ldots, k_2\}$ and $e \in E(L_{b_1})$ are similar.) Then $M_{a_i}^* \simeq M_{a_i} \backslash \{e\}$ and $M_{a_i}^* \nrightarrow (K_{1,2}(r_{a_i,1}, r_{a_i,2}), C_n)$, which means that there exists a red-blue coloring of $M_{a_i}^*$ containing neither a red $K_{1,2}$ nor a blue C_n such that $r_{a_i,1}, r_{a_i,2}$ are not incident to red edges in $M_{a_i}^*$.

In any other seed of B_5^*, if one of the roots is not incident to red edges of the seed and the second root is incident to exactly one red edge, there exists a red-blue coloring of the seed that contains neither a red $K_{1,2}$ nor a blue C_n (in $L_{b_1}^*$ we have just one root which is incident to one red edge of $L_{b_1}^*$).

Hence, it is possible to color the edges of B_5^* such that every root is incident to exactly one red edge and there is no red $K_{1,2}$ in the coloring of B_5^*. Because the number of seeds in $C(a'_1, a'_2, \ldots, a'_{k_2})$ is $k_2 \geq n + 1$, there is no blue C_n in the coloring as well. ■

Similarly as Theorem 3, we can prove the next theorem.

Theorem 4. $B_3, B_4 \in \mathcal{R}(K_{1,2}, C_n)$.

Theorems 1–4 in combination with Lemmas 1–6 give infinite families of Ramsey $(K_{1,2}, C_4)$-minimal graphs.

For example, the graph $B_3(L_m(t'_1),\ P(a_1, a_2, \ldots, a_k),\ L_n(t'_2))$, where $P(a_1, a_2, \ldots, a_k)$ consists of the graphs $M_{a_j}(t_j), j = 1, 2, \ldots, k$ and $a_j, m, n \in \{1, 2, 3\}$ is a Ramsey $(K_{1,2}, C_4)$-minimal graph. Values of the parameters t'_1, t'_2, t_j follow from Lemmas 1–6.

Let $B_3(L_m(t'_1),\ P(a_1, a_2, \ldots, a_k),\ L_n(t'_2))$ contains exactly r seeds $M_2(t_j), j \in \{1, 2, \ldots, k\}$ such that the vertex which has degree $t_j/2$ in $M_2(t_j)$ is one of the roots of $M_2(t_j)$ and let $B_3(L_m(t'_1), P(a_1, a_2, \ldots, a_k), L_n(t'_2))$ also contains z seeds $L_3(6)$ with the root denoted by v_4 in $L_3(6)$. Note that $0 \leq r \leq k$ and $0 \leq z \leq 2$. It is easy to show that the diameter of $B_3(L_m(t'_1), P(a_1, a_2, \ldots, a_k), L_n(t'_2))$ is $2k + 6 - r - z$, since

- the eccentricity of the root of $L_i(t')$ is 3 for $i = 1, 2, 3$ and any t' except for the eccentricity of v_4 in $L_3(6)$ that is equal to 2,
- the distance between two roots in $M_i(t)$ is 2 for $i = 1, 3$, while in $M_2(t)$ the roots can be adjacent.

It follows that we found an infinite class of Ramsey $(K_{1,2}, C_4)$-minimal graphs for every diameter ≥ 4. The problem of existence of an infinite family of Ramsey $(K_{1,2}, C_4)$-minimal graphs of diameter 3 remains open.

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