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MATCHINGS AND TOTAL DOMINATION SUBDIVISION NUMBER IN GRAPHS WITH FEW INDUCED 4-CYCLES

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Abstract

A set S of vertices of a graph $G = (V, E)$ without isolated vertex is a *total dominating set* if every vertex of $V(G)$ is adjacent to some vertex in S. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. The *total domination subdivision number* $\mathrm{sd}_{\gamma_t}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the total domination number. Favaron, Karami, Khoeilar and Sheikholeslami (Journal of Combinatorial Optimization, to appear) conjectured that: For any connected graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1$. In this paper we use matchings to prove this conjecture for graphs with at most three induced 4-cycles through each vertex.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n with minimum degree $\delta(G) \geq 1$. The neighborhood of a vertex u is denoted by $N_G(u)$ and its degree $|N_G(u)|$ by $d_G(u)$ (briefly $N(u)$ and $d(u)$ when no ambiguity on the graph is possible). If $S \subseteq V(G)$, $N(S) = \cup_{x \in S} N(x)$. We denote by $N_2(v)$ the set of vertices at distance 2 from the vertex v and put $d_2(v) = |N_2(v)|$ and $\delta_2(G) = \min\{d_2(v); v \in V(G)\}$. A *matching* is a set of edges with no shared endvertices. A perfect matching M of G is a matching with $V(M) = V(G)$. If n is odd, a *near perfect matching* leaves exactly one vertex uncovered, i.e., $|V(M)| = n - 1$. A graph is *factor-critical* if the deletion of any vertex leaves a graph with a perfect matching. Note that factor-critical graphs have odd order. The maximum number of edges of a matching in G is denoted by $\alpha'(G)$ (α' for short). The length of a smallest cycle in a graph G, that contains cycles, is the *girth* of G (denoted $g(G)$). We use [15] for terminology and notation which are not defined here.

A set S of vertices of a graph G with minimum degree $\delta(G) > 0$ is a *total dominating set* if $N(S) = V(G)$. The minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$, is called the *total domination number* of G. A $\gamma_t(G)$-*set* is a total dominating set of G of cardinality $\gamma_t(G)$. The *total domination subdivision number* $\mathrm{sd}_{\gamma_t}(G)$ is the minimum number of edges of G that must be subdivided once in order to increase the total domination number. This kind of concept was first introduced for the domination number by Velammal in his Ph.D. thesis [14]. The total domination subdivision number was considered by Haynes *et al.* in [8] and since then have been studied by several authors (see for example [2, 4, 5, 3, 6, 7, 10, 11]). Since the total domination number of the graph K_2 does not change when its only edge is subdivided, in the study of total domination subdivision number we must assume that the graph has maximum degree at least two.

It is known that the parameter sd_{γ_t} can take arbitrarily large values [6] and an interesting problem is to find good bounds on $\mathrm{sd}_{\gamma_t}(G)$ in terms of other parameters of G. For instance it has been proved that for any graph G of order n, $\mathrm{sd}_{\gamma_t}(G) \leq n - \gamma_t(G) + 1$ [4], $\mathrm{sd}_{\gamma_t}(G) \leq 2n/3$ [5] and $\mathrm{sd}_{\gamma_t}(G) \leq n - \Delta + 2$ [2]. Favaron *et al.* in [3] posed the following conjecture

Conjecture 1. For any connected graph G of order $n \geq 3$,

$$\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1$$

and proved it for some classes of graphs.

Our purpose in this paper is to prove Conjecture 1 for connected graphs with few induced cycles C_4 through each vertex of G. We will use the following results on $\alpha'(G)$, $\gamma_t(G)$ and $\mathrm{sd}_{\gamma_t}(G)$.

Theorem A [6]**.** *For any connected graph G with adjacent vertices u and v, each of degree at least two,*

$$\mathrm{sd}_{\gamma_t}(G) \leq d(u) + d(v) - |N(u) \cap N(v)| - 1 = |N(u) \cup N(v)| - 1.$$

Theorem B [3]**.** *For any connected graph G of order $n \geq 3$ and $\gamma_t(G) \leq \alpha'(G)$,*

$$\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1.$$

Theorem C [3]**.** *For any connected graph G of order $n \geq 3$ with $\delta = 1$,*

$$\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G).$$

Theorem D [2]**.** *Let G be a connected graph of minimum degree at least 2. Then $sd_{\gamma_t}(G) \leq \delta_2(G) + 3$.*

In the proof of Theorem 1 we use the concept of barrier. If S is a separator of a graph G, $o(G)$ denotes the number of odd components of $G - S$, i.e., components of odd order. A *barrier* of G is a separator S such that $o(G - S) = |S| + t$ where $t = n - 2\alpha'$ is the number of vertices of G which are not covered by a maximum matching. By Tutte-Berge's Theorem every connected graph admits barriers. Moreover (see for example exercise 3.3.18 in [12]) if S is a maximal barrier, then all the components $G_1, G_2, \ldots, G_{|S|+t}$ of $G - S$ are factor-critical (hence odd) and every maximum matching of G is formed by a matching pairing S with $|S|$ different components of $G - S$ and a near perfect matching in each component. Therefore, with the notation $|S| + t = \ell$ and $|V(G_i)| = n_i$,

$$\alpha'(G) = |S| + \sum_{i=1}^{\ell} \frac{n_i - 1}{2}. \tag{1}$$

2. Main Result

Theorem 1. *Every connected graph G of order $n \geq 3$ such that each vertex belongs to at most three induced C_4 satisfies*

$$\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1.$$

Proof. By Theorems B and C, we may assume $\delta(G) \geq 2$ and

$$\alpha'(G) \leq \gamma_t(G) - 1. \tag{2}$$

Let S be a maximal barrier of G and $G_1, G_2, \ldots, G_\ell$ the components of $G-S$. Let S_1 be the set of the isolated vertices of $G[S]$.

If $S_1 = \emptyset$ and $G - S$ has only trivial components, then S is a total dominating set of G and $\gamma_t(G) \leq |S| = \alpha'(G)$ by (1), a contradiction with (2). If $S_1 \neq \emptyset$ and if all the neighbors of a vertex x of S_1 belong to trivial components of $G-S$, then $N_2(x) \subseteq S - \{x\}$ and by Theorem D, (1) and (2),

$$\mathrm{sd}_{\gamma_t}(G) \leq \delta_2(G) + 3 \leq |N_2(x)| + 3 \leq |S| + 2 \leq \alpha'(G) + 2 \leq \gamma_t(G) + 1.$$

Therefore we can assume that at least one component of $G-S$ is not trivial and that every isolated vertex of S has at least one neighbor in a non-trivial component of $G-S$.

First suppose that at least two components of $G-S$, say G_1 and G_2 with $n_1 \leq n_2$, are not trivial. Let uv be an edge of G_1. Since $\delta(G) \geq 2$, Theorem A, (1) and (2) imply

$$\begin{aligned}\mathrm{sd}_{\gamma_t}(G) &\leq |N(u) \cup N(v)| - 1 \leq n_1 + |S| - 1 \\ &\leq \frac{n_1 + n_2}{2} + |S| - 1 \leq \alpha'(G) \leq \gamma_t(G) - 1.\end{aligned}$$

Now suppose that G_1 is the unique nontrivial component of $G-S$. Then $\alpha'(G) = |S| + \frac{n_1 - 1}{2}$ and every vertex of S_1 has a neighbor in G_1. Let $G_i = \{y_i\}$ for $2 \leq i \leq \ell$, and $Y = \{y_i \mid 2 \leq i \leq \ell\}$. If $|N(u) \cup N(v)| - 1 \leq \gamma_t(G) + 1$ for some edge uv of G_1, then the result follows from Theorem A. Therefore we may assume that for each edge uv of G_1,

$$|N(u) \cup N(v)| \geq \gamma_t(G) + 3. \tag{3}$$

This implies in particular by (2) that

$$|S| + \frac{n_1 - 1}{2} = \alpha'(G) \leq \gamma_t(G) - 1 \leq |N(u) \cup N(v)| - 4 \leq |S| + n_1 - 4$$

and thus $n_1 \geq 7$.

Claim. $\gamma_t(G_1) \leq \dfrac{n_1 - 1}{2}$.

Proof of the Claim. Let $M = \{u_1v_1, \ldots, u_{\frac{n_1-1}{2}} v_{\frac{n_1-1}{2}}\}$ be a near perfect matching of G_1 and let $\{x\} = V(G_1) - V(M)$. Without loss of generality we assume x is adjacent to $u_{\frac{n_1-1}{2}}$. Let X be a subset of $V(G_1)$ satisfying the following properties:

(a) $u_{\frac{n_1-1}{2}} \in X$ and $x \notin X$,

(b) $|X \cap \{u_i, v_i\}| \leq 1$ for $1 \leq i \leq \frac{n_1-1}{2}$,

(c) $G[X]$ has no isolated vertex if $|X| > 1$.

Choose X to be maximum among all such sets. By Property (b), $|X| \leq \frac{n_1-1}{2}$. Suppose $|X| < \frac{n_1-1}{2}$. Without loss of generality we may assume $X \cap \{u_i, v_i\} = \emptyset$ for $1 \leq i \leq r < \frac{n_1-1}{2}$. Let $R = \{u_iv_i \mid 1 \leq i \leq r\}$, $R' = \{u_i, v_i \mid 1 \leq i \leq r\}$ and $G' = G[R'] - R$. If $uv \in E(G')$, then $X' = X \cup \{u, v\}$ satisfies Properties (a) to (c), a contradiction with the choice of X. Similarly if G contains an edge uv with $u \in X$ and $v \in R'$, then $X' = X \cup \{v\}$ contradicts the choice of X. Hence G' is empty and no edge exists between X and R'. Therefore $N(u_1) \cup N(v_1) \subseteq S \cup \{u_1, v_1\} \cup (V(G_1) \setminus (X \cup R'))$. Since $|V(G_1)| \leq 2|X| + |R'| + 1$, we have by (2),

$$|N(u_1) \cup N(v_1)| \leq |S| + |X| + 3 < \alpha'(G) + 3 \leq \gamma_t(G) + 2,$$

in contradiction with (3). Therefore $|X| = \frac{n_1-1}{2} \geq 3$ and from its construction, it is clear that X is a total dominating set of G_1. □

Let X be a total dominating set of G_1 of order $\frac{n_1-1}{2}$ as in the claim. If $S_1 = \emptyset$ or if every vertex of S_1 has a neighbor in X, then $S \cup X$ is a total dominating set of G and thus $\gamma_t(G) \leq |S| + |X| = \alpha'(G)$, a contradiction with (2). Hence the set S_2 of the isolated vertices of $G[S]$ with no neighbor in X is not empty.

If $N(y_i) \nsubseteq S_2$ for each i with $2 \leq i \leq \ell$, we associate to each vertex x of S_2 one of its neighbors $f(x)$ in $V(G_1) - X$ (recall that each vertex of S_1 has at least one neighbor in G_1) and we let $S_2' = \{f(x) \mid x \in S_2\}$. Clearly $|S_2'| \leq |S_2|$, S_2' dominates S_2, and $X \cup S_2'$ is a total dominating set of $V(G_1) \cup S_1$. Therefore $(S - S_2) \cup X \cup S_2'$ is a total dominating set of G and $\gamma_t(G) \leq |S| + |X| = \alpha'(G)$, a contradiction with (2).

Hence some vertex y_i of Y, say y_2, has all its neighbors in S_2. Since $\delta(G) \geq 2$, $|S_2| \geq 2$. Let uv be an edge of $G[X]$ (such an edge exists since $n_1 \geq 7$). Then $N(u) \cup N(v) \subseteq (S - S_2) \cup V(G_1)$. By (2) and (3),

$$|S| + \frac{n_1 - 1}{2} = \alpha'(G) \leq \gamma_t(G) - 1 \leq |N(u) \cup N(v)| - 4 \leq |S| - |S_2| + n_1 - 4.$$

Therefore

$$n_1 \geq 2|S_2| + 7 \geq 11.$$

Let z_1 and z_2 be two neighbors of y_2. The neighborhoods $N_{G_1}(z_1)$ and $N_{G_1}(z_2)$ are contained in $V(G_1) - X$. Let $|N_{G_1}(z_1) \cap N_{G_1}(z_2)| = p$ and suppose without loss of generality $d_{G_1}(z_1) \leq d_{G_1}(z_2)$. Then

$$2|N_{G_1}(z_1)| \leq |N_{G_1}(z_1)| + |N_{G_1}(z_2)| \leq \frac{n_1 + 1}{2} + p$$

and since $p \geq 0$ and $n_1 \geq 11$,

$$(4) \qquad |N_{G_1}(z_1)| \leq \frac{n_1 + 1}{4} + \frac{p}{2} \leq \frac{n_1 - 1}{2} - 2 + p.$$

Note that each of the p vertices of $N_{G_1}(z_1) \cap N_{G_1}(z_2)$ induces with y_2, z_1, z_2 a cycle C_4 containing one vertex in Y and one vertex in G_1.

Let $A = N_Y(z_1) - \{y_2\}$ and $B = N(y_2) - \{z_1\}$ $(\subseteq S_2)$. For each $a \in A$, let a' be one of its neighbors in $S - \{z_1\}$ (a' exists since $\delta(G) \geq 2$) and let $A' = \{a' \mid a \in A\}$. Then $|A'| \leq |A|$ and $|A| - |A'|$ is at most the number of pairs a_i, a_j of vertices of A such that $a_i' = a_j'$. Note that if $a_i' = a_j'$, then a_i', a_i, a_j, z_1 induce a C_4 not containing y_2. Since the set $B \cup A'$ is contained in $S - \{z_1\}$, $|B \cup A'| \leq |S| - 1$. Each vertex a' of $B \cap A'$ corresponds to at least one induced C_4 of the form $z_1 y_2 a' a z_1$ (possibly more if a' is associated to several vertices of A). Hence if we denote by q the number of induced cycles C_4 containing z_1 and two vertices of Y, we get $|A| - |A'| + |B \cap A'| \leq q$.

Therefore

$$\begin{aligned}|N_Y(z_1)| + |N(y_2)| &= |A| + 1 + |B| + 1 \\ &= |A - A'| + |A'| + |B| + 2 \\ &= |A - A'| + |A' \cup B| + |A' \cap B| + 2 \\ &\leq |A' \cup B| + q + 2 \\ &\leq |S| + q + 1.\end{aligned}$$

Since $N(z_1) \cap N(y_2) = \emptyset$ and by Theorem A, (1), (2) and (4), we get

$$\begin{aligned}\mathrm{sd}_{\gamma_t}(G) &\leq |N(z_1)| + |N(y_2)| - 1 \\ &\leq |N_{G_1}(z_1)| + |N_Y(z_1)| + |N(y_2)| - 1 \\ &\leq (\tfrac{n_1-1}{2} - 2 + p) + (|S| + q + 1) - 1 \\ &\leq \alpha'(G) + p + q - 2 \\ &\leq \gamma_t(G) - 3 + p + q.\end{aligned}$$

By hypothesis, z_1 is contained in at most three induced cycles C_4. Hence $p + q \leq 3$ and $\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1$, which completes the proof of Theorem 1. ■

Corollary 2. *For any connected graph G of order $n \geq 3$ with girth greater than* 4, $\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1$.

Corollary 3. *For any connected chordal graph G of order $n \geq 3$,* $\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1$.

REFERENCES

[1] D. Archdeacon, J. Ellis-Monaghan, D. Fisher, D. Fronček, P.C.B. Lam, S. Seager, B. Wei and R. Yuster, *Some remarks on domination*, J. Graph Theory **46** (2004) 207–210.

[2] O. Favaron, H. Karami, R. Khoeilar and S.M. Sheikholeslami, *A new upper bound for total domination subdivision numbers*, Graphs and Combinatorics **25** (2009) 41-47.

[3] O. Favaron, H. Karami, R. Khoeilar and S.M. Sheikholeslami, *On the total domination subdivision number in some classes of graphs*, Journal of Combinatorial Optimization, (to appear).

[4] O. Favaron, H. Karami and S.M. Sheikholeslami, *Total domination and total domination subdivision numbers of graphs*, Australas. J. Combin. **38** (2007) 229–235.

[5] O. Favaron, H. Karami and S.M. Sheikholeslami, *Bounding the total domination subdivision number of a graph in terms of its order*, Journal of Combinatorial Optimization, (to appear).

[6] T.W. Haynes, M.A. Henning and L.S. Hopkins, *Total domination subdivision numbers of graphs*, Discuss. Math. Graph Theory **24** (2004) 457–467.

[7] T.W. Haynes, M.A. Henning and L.S. Hopkins, *Total domination subdivision numbers of trees*, Discrete Math. **286** (2004) 195–202.

[8] T.W. Haynes, S.T. Hedetniemi and L.C. van der Merwe, *Total domination subdivision numbers*, J. Combin. Math. Combin. Comput. **44** (2003) 115–128.

[9] M.A. Henning, L. Kang, E. Shan and A. Yeo, *On matching and total domination in graphs*, Discrete Math. **308** (2008) 2313–2318.

[10] H. Karami, A. Khodkar and S.M. Sheikholeslami, *An upper bound for total domination subdivision numbers of graphs*, Ars Combin. (to appear).

[11] H. Karami, A. Khodkar, R. Khoeilar and S.M. Sheikholeslami, *Trees whose total domination subdivision number is one*, Bulletin of the Institute of Combinatorics and its Applications, **53** (2008) 57–67.

[12] L. Lovász and M.D. Plummer, Matching Theory, Annals of Discrete Math 29 (North Holland, 1886).

[13] W.T. Tutte, *The factorization of linear graphs*, J. Lond. Math. Soc. **22** (1947) 107–111.

[14] S. Velammal, Studies in Graph Theory: Covering, Independence, Domination and Related Topics, Ph.D. Thesis (Manonmaniam Sundaranar University, Tirunelveli, 1997).

[15] D.B. West, Introduction to Graph Theory (Prentice-Hall, Inc, 2000).

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