

## ON THE EXISTENCE OF A CYCLE OF LENGTH AT LEAST 7 IN A $(1, \leq 2)$ -TWIN-FREE GRAPH

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### Abstract

We consider a simple, undirected graph  $G$ . The ball of a subset  $Y$  of vertices in  $G$  is the set of vertices in  $G$  at distance at most one from a vertex in  $Y$ . Assuming that the balls of all subsets of at most two vertices in  $G$  are distinct, we prove that  $G$  admits a cycle with length at least 7.

**Keywords:** undirected graph, twin subsets, identifiable graph, distinguishable graph, identifying code, maximum length cycle.

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### 1. INTRODUCTION

We consider a finite, undirected, simple graph  $G = (X, E)$ , where  $X$  is the vertex set and  $E$  the edge set.

If  $r$  is a positive integer and  $x$  a vertex in  $G$ , the *ball of  $x$  with radius  $r$* , denoted by  $B_r(x)$ , is the set of vertices in  $G$  which are within distance  $r$

from  $x$ . If  $Y$  is a subset of  $X$ , the *ball of  $Y$  with radius  $r$* , denoted by  $B_r(Y)$ , is defined by

$$B_r(Y) = \bigcup_{y \in Y} B_r(y).$$

For  $x \in X$ , we set  $B(x) = B_1(x)$  and call this set the *ball of  $x$* : in other words, the ball of  $x$  consists of  $x$  and its neighbours; for  $Y \subseteq X$ , we set  $B(Y) = B_1(Y)$  and call this set the *ball of  $Y$* .

Two distinct subsets of  $X$  are said to be *separated* if they have distinct balls with radius  $r$ . For a given integer  $\ell \geq 1$ , the graph  $G$  is said to be  $(r, \leq \ell)$ -*twin-free* if any two distinct subsets of at most  $\ell$  vertices are separated. In an  $(r, \leq \ell)$ -twin-free graph, for any subset  $V$  of  $X$ , there is at most one subset  $Y$  of  $X$ , with  $|Y| \leq \ell$ , such that  $B_r(Y) = V$ : the subsets of at most  $\ell$  vertices are characterized by their balls with radius  $r$ . In this case, it is also said that  $G$  is  $(r, \leq \ell)$ -*identifiable* or  $(r, \leq \ell)$ -*distinguishable*, or that  $G$  admits an  $(r, \leq \ell)$ -*identifying code*. See, among many others, [7]–[11] and [13] for results on these codes.

Graphs admitting  $(r, \leq 1)$ -identifying codes, i.e.,  $(r, \leq 1)$ -twin-free graphs, have particular structural properties (see for instance [1, 4] and [5]; see [12] for references upon these codes). In particular, it was proved in [1] that a connected  $(r, \leq 1)$ -twin-free graph with at least two vertices always contains as an induced subgraph the path  $P_{2r+1}$  on  $2r + 1$  vertices; since  $P_{2r+1}$  itself is  $(r, \leq 1)$ -twin-free, it is therefore the smallest  $(r, \leq 1)$ -twin-free graph.

Several results have been published about  $(r, \leq \ell)$ -identifying codes in various graphs (see [7]–[11] and [13]), but little is known about the structure of these graphs. Using, for  $i \geq 3$ , the notation  $\mathcal{C}_i$  (respectively,  $\mathcal{C}_{\geq i}$ ) for a cycle of length  $i$  (respectively, at least  $i$ ), it is easily seen that the cycles  $\mathcal{C}_{\geq 7}$  are  $(1, \leq 2)$ -twin-free and that the smallest  $(1, \leq 2)$ -twin-free graph is the cycle  $\mathcal{C}_7$ . Hence it seems natural to wonder whether a cycle  $\mathcal{C}_k$  with  $k \geq 7$  is contained in any  $(1, \leq 2)$ -twin-free graph.

Thus we shall restrict ourselves to the case  $r = 1$ ,  $\ell = 2$  and prove in this article that an undirected connected  $(1, \leq 2)$ -twin-free graph of order at least 2, contains an *elementary* cycle (not going through a vertex twice) with length at least 7.

We now give some basic definitions for a graph  $G = (X, E)$  (see [2, 3] or [6] for more). A *subgraph* of  $G$  is a graph  $G' = (X', E')$ , where  $X' \subseteq X$

and

$$E' \subseteq \{\{u, v\} \in E : u \in X', v \in X'\}.$$

Such a subgraph is said to be *induced by*  $X'$  if

$$E' = \{\{u, v\} \in E : u \in X', v \in X'\}.$$

A *cut-vertex* of  $G$  is a vertex  $u \in X$  such that the subgraph induced by  $X \setminus \{u\}$  has more connected components than  $G$ . A *cut-edge* of  $G$  is an edge  $e \in E$  such that the subgraph  $(X, E \setminus \{e\})$  has more connected components than  $G$ . If  $G$  is connected, the deletion of a cut-vertex or of a cut-edge makes  $G$  disconnected. More generally, a  *$h$ -connected* graph,  $h \geq 1$ , is a graph  $G$  such that the minimum number of vertices to be deleted in order to disconnect  $G$ , or to reduce it to a singleton, is at least  $h$ . A  *$h$ -connected component* of  $G$  is an induced subgraph which is  $h$ -connected and maximal (for inclusion) in  $G$ .

A *block* of  $G$  is a maximal induced subgraph with no cut-vertex, and a *bridge* is an induced subgraph consisting of two adjacent vertices, linked by an edge which is a cut-edge in  $G$ .

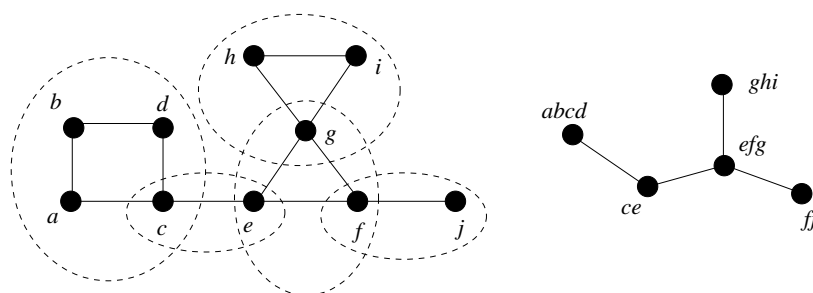
Throughout this article, the paths and cycles will be elementary, and  $G = (X, E)$  will be an undirected, simple graph of order at least 2. Moreover, we shall assume that  $G$  is connected: if not, the result would be obtained by choosing any connected component of  $G$ , with at least 2 vertices.

## 2. CHOOSING A *leaf-block* OF $G$

The blocks of  $G$  are 2-connected components or bridges. The graph given in the left part of Figure 1 contains 5 blocks:  $\{a, b, c, d\}$ ,  $\{c, e\}$ ,  $\{g, h, i\}$ ,  $\{e, f, g\}$ , and  $\{f, j\}$ , which are surrounded with dotted lines. Two blocks of  $G$  either do not intersect, or intersect on a cut-vertex of  $G$ . Define the graph  $G'$  whose vertices are the blocks of  $G$  and whose edges link blocks having a nonempty intersection:  $G'$  is a tree. Now a block of  $G$  which is a leaf in  $G'$  is called a *leaf-block* of  $G$ . For instance, the graph  $G$  in Figure 1 has 3 leaf-blocks.

We give the following definition:

**Definition 1.** Let  $G = (X, E)$  be an undirected connected graph,  $Y \subset X$ ,  $y \in Y$ , and  $s \in X \setminus Y$ . A  $(G, s, Y, y)$ -*path* is a path in  $G$  whose ends are  $s$  and  $t \in Y \setminus \{y\}$ , and whose vertices other than  $t$  are in  $X \setminus Y$ .

Figure 1. One example for the graphs  $G$  and  $G'$ .

We shall use the following proposition repeatedly.

**Proposition 1.** *Let  $G = (X, E)$  be an undirected connected graph,  $H$  a 2-connected component of  $G$ ,  $Y$  a subset of at least 2 vertices in  $H$ ,  $y$  a vertex in  $Y$  which is not a cut-vertex of  $G$ , and  $s$  a neighbour of  $y$  which is not in  $Y$ . Then  $s$  belongs to  $H$  and there is a  $(H, s, Y, y)$ -path.*

**Proof.** Let  $G \setminus \{y\}$  be the induced subgraph obtained from  $G$  by withdrawing the vertex  $y$ . Since  $y$  is not a cut-vertex, the graph  $G \setminus \{y\}$  is still connected: there exists in  $G \setminus \{y\}$  a path between  $s$  and  $Y \setminus \{y\}$ , whose vertices, other than its end in  $Y \setminus \{y\}$ , are in  $X \setminus Y$ , i.e., a  $(G, s, Y, y)$ -path; if we concatenate this path with the edge  $\{s, y\}$ , we get a path  $P$  between  $y$  and  $t$ , which are two distinct vertices in the 2-connected component  $H$ . Therefore, the union of  $H$  and  $P$  is still 2-connected, and, by the maximality of  $H$  as an induced 2-connected subgraph,  $P$  is a path in  $H$ . ■

Proposition 1 states that, if we wish to “leave” a subset  $Y$  of at least two vertices in a 2-connected component  $H$ , starting from a non cut-vertex  $y$ , then we stay inside  $H$  and we “come back” inside  $Y$ , on a vertex other than  $y$ .

**From now on and throughout this article, we assume that  $G$  is  $(1, \leq 2)$ -twin-free.**

Note that  $G$  cannot have vertices with degree 1: if  $x$  has degree 1 and  $y$  is its unique neighbour, then the sets  $\{y\}$  and  $\{x, y\}$  are not separated; actually, this is part of a more general result on  $(1, \leq \ell)$ -twin-free graphs, which have minimal degree at least  $\ell$  [11, Theorem 8]. Consequently, a leaf-block of  $G$  cannot be a bridge: all leaf-blocks of  $G$  are 2-connected components, and Proposition 1 can be applied to them. We denote by  $H$  one leaf-block of  $G$ . The graph  $H$  has at least one cycle.

Also, either  $H$  is the whole graph  $G$  and in this case has no cut-vertex, or  $H$  has one, and only one, cut-vertex of  $G$ ,  $\alpha$ . In the following, we keep the notation  $\alpha$  for the cut-vertex of  $G$  in the 2-connected component  $H$ , **if  $\alpha$  exists**.

### 3. THE LENGTH OF THE LONGEST CYCLE IN $H$ IS NOT 6

Lemma 1 will be used repeatedly to show Lemmas 2–4, which state that if  $H$  admits certain subgraphs, then, under certain conditions, a  $\mathcal{C}_{\geq 7}$  is a subgraph of  $H$ . Lemma 5 concludes this section, establishing that the length of the longest cycle in  $H$  is not 6.

**Lemma 1.** *We assume that the longest cycle in  $H$  has length 6. If the graph  $L$  given in Figure 2 is a subgraph of  $H$ , with  $x \neq \alpha$  and  $y \neq \alpha$ , then  $t$  is adjacent to either  $x$  or  $y$ , and  $x$  and  $y$  have no neighbours in  $G$  other than  $z, u$ , and, for exactly one of them,  $t$ .*

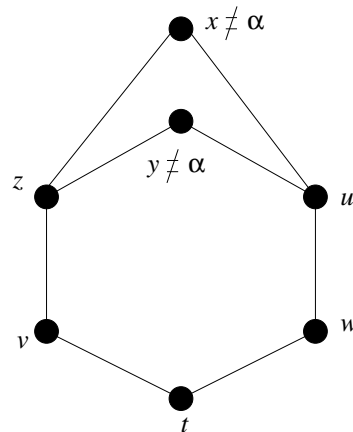


Figure 2. The graph  $L$  in Lemma 1.

**Proof.** We assume that  $H$  contains no  $\mathcal{C}_{\geq 7}$  and that  $L$  is a subgraph of  $H$ , with  $x \neq \alpha$  and  $y \neq \alpha$ . Let  $Y$  be the set of the 7 vertices in  $L$ .

First, we show that the neighbours, in  $G$ , of  $x$  and  $y$  belong to  $\{z, u, t\}$ . Assume on the contrary that  $x$  has a neighbour  $s \in X \setminus \{z, u, t\}$ .

If  $s$  belongs to  $Y$ , then  $s = y$ ,  $s = v$ , or  $s = w$ .

If  $s \notin Y$ , then, since  $x$  is not the cut-vertex, we can use Proposition 1: the vertex  $s$  belongs to  $H$  and there is a  $(H, s, Y, x)$ -path.

So, whether  $s \in Y$  or not, there is a path  $P$  of length at least 1 linking  $x$  and  $Y \setminus \{x\}$ , other than the edges  $\{x, z\}$ ,  $\{x, u\}$  and  $\{x, t\}$ , and whose vertices, but its two ends, do not belong to  $Y$ ; now we examine the different possible cases, represented in Figure 3.

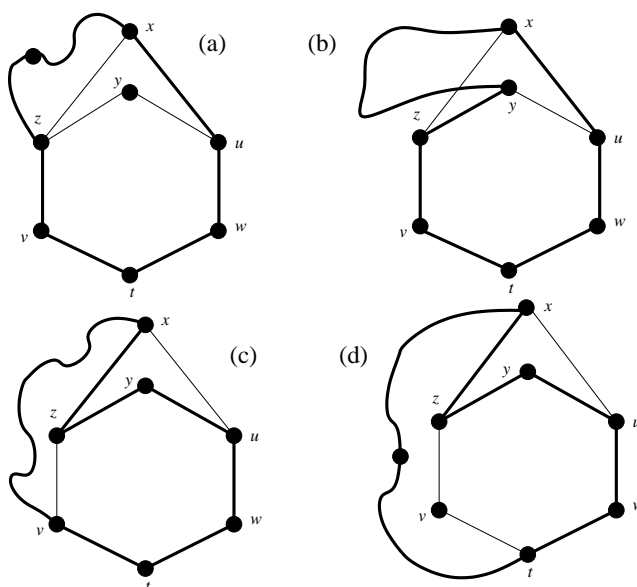


Figure 3. Illustrations for the proof of Lemma 1.

- (a) If  $P$  links  $x$  and  $z$ ,  $P$  has length at least 2; by concatenating it with the path  $z, v, t, w, u, x$ , we obtain a  $\mathcal{C}_{\geq 7}$ , given in bold in Figure 3(a); this case is impossible, as is the case when  $P$  links  $x$  and  $u$ .
- (b) If  $P$  links  $x$  and  $y$ , this path concatenated with the path  $y, z, v, t, w, u, x$  yields a  $\mathcal{C}_{\geq 7}$ : this case is impossible.
- (c) If  $P$  links  $x$  and  $v$ , this path concatenated with the path  $v, t, w, u, y, z, x$  yields a  $\mathcal{C}_{\geq 7}$ . Similarly,  $P$  cannot link  $x$  and  $w$ .
- (d) Finally, if  $P$  links  $x$  and  $t$ , then  $P$  has length at least 2 and by concatenating it with the path  $t, w, u, y, z, x$ , we get a  $\mathcal{C}_{\geq 7}$ , still a contradiction.

None of the above cases is possible, the neighbours of  $x$  are in  $\{z, u, t\}$  and the same is true for  $y$ . Furthermore, we have:  $B(\{z, x\}) \supset \{x, y, z, u\}$  and  $B(\{z, y\}) \supset \{x, y, z, u\}$ . In order to separate the sets  $\{z, x\}$  and  $\{z, y\}$ , it is

necessary to use  $t$ , and so, one, and only one, vertex in  $\{x, y\}$  is linked to  $t$ , which ends the proof of Lemma 1. ■

**Lemma 2.** *If the graph  $L$  given in Figure 2 is a subgraph of  $H$ , with  $x \neq \alpha$  and  $y \neq \alpha$ , then  $\mathcal{C}_{\geq 7}$  is a subgraph of  $H$ .*

**Proof.** We assume that no  $\mathcal{C}_{\geq 7}$  is a subgraph of  $H$ , that  $L$  is a subgraph of  $H$ , and that  $x \neq \alpha$ ,  $y \neq \alpha$ . We still denote by  $Y$  the set of the 7 vertices in  $L$ .

One can assume that, if  $\alpha \notin Y$ , then the path  $z, \alpha, t$  does not exist: indeed, if the path  $z, \alpha, t$  exists with  $\alpha \notin Y$ , then we delete in  $L$  the path  $z, v, t$  and replace it with the path  $z, \alpha, t$ , and  $\alpha$  is renamed as  $v$ . Similarly, one can assume that, if  $\alpha \notin Y$ , then the path  $u, \alpha, t$  does not exist.

If  $\alpha = z$  or  $\alpha = w$ , we rename the vertices, exchanging the names  $z$  and  $u$  as well as  $v$  and  $w$ , and so we can assume, without loss of generality, that  $\alpha \neq z$  and  $\alpha \neq w$ .

The graph  $L$  we shall consider from now on has the following properties.

- $L$  corresponds to Figure 2,
- $x \neq \alpha$ ,  $y \neq \alpha$ ,  $z \neq \alpha$ , and  $w \neq \alpha$ ,
- if the path  $z, \alpha, t$  exists, then  $\alpha$  belongs to  $Y$ ,
- if the path  $u, \alpha, t$  exists, then  $\alpha$  belongs to  $Y$ .

Using Lemma 1, we can moreover assume that  $y$  is linked to  $t$ , and we then know that  $x$  and  $y$  have no neighbours in  $G$  other than those in Figure 4. The graph represented in Figure 4 is a subgraph of  $H$ .

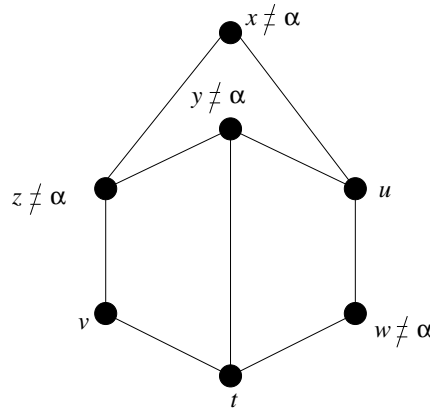


Figure 4. The graph  $L$ , with the edge  $\{y, t\}$ .

In order to prove Lemma 2, we proceed step by step, with intermediate results, from 1 to 7.

1. *The vertex  $w$  has no neighbour outside  $Y$ .*

Assume on the contrary that  $w$  has a neighbour  $s \notin Y$  (see Figure 5); since  $w \neq \alpha$ , there is a  $(H, s, Y, w)$ -path  $P$ . By Lemma 1,  $x$  and  $y$  have their neighbours in  $Y$ , so  $P$  cannot end in  $x$  or  $y$ . It cannot end in  $u$  or  $t$  either, since this would yield a  $\mathcal{C}_{\geq 7}$ , represented in bold in Figure 5(a) when  $P$  ends in  $u$ . If  $P$  ends in  $v$ , then we have a  $\mathcal{C}_{\geq 8}$ , and if it ends in  $z$ , then we have a  $\mathcal{C}_{\geq 7}$ : the path  $P$  cannot end in any vertex of  $Y$ . Consequently,  $w$  has no neighbour outside  $Y$ .

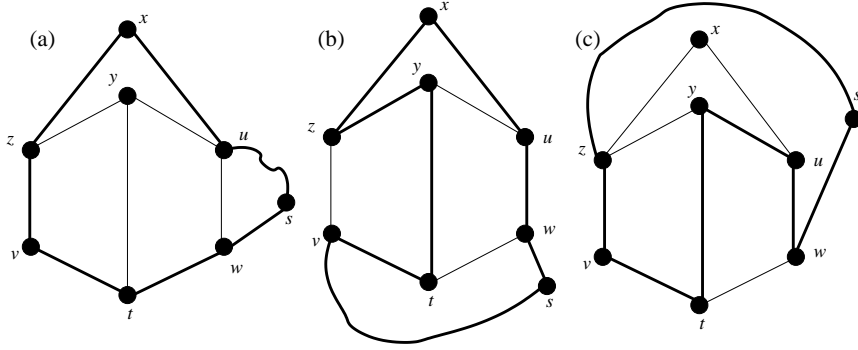


Figure 5. Lemma 2, illustrations for Result 1.

2. *If  $v \neq \alpha$ , then  $v$  has no neighbour outside  $Y$ .*

This result is obtained in exactly the same way as Result 1.

3. *There is no vertex outside  $Y$ , different from  $\alpha$  and adjacent to both  $z$  and  $u$ .*

Assume on the contrary that there exists  $s \notin Y$ , with  $s \neq \alpha$  and  $s$  adjacent to  $z$  and  $u$  (see Figure 6); by Lemma 1, since  $x$  is not adjacent to  $t$  and neither  $x$  nor  $s$  is the cut-vertex  $\alpha$ ,  $s$  is adjacent to  $t$ ; but now  $s \neq \alpha$ ,  $y \neq \alpha$ , and both  $s$  and  $y$  are adjacent to  $t$ : this contradicts Lemma 1.

4. *If  $v \neq \alpha$  and if  $z$  has a neighbour  $s \notin Y$ , then  $s = \alpha$  and the path  $z, \alpha, u$  exists.*

We assume that  $v \neq \alpha$  and that  $z$  has a neighbour  $s \notin Y$ . We recall that  $z \neq \alpha$ , so that by Proposition 1, there is a  $(H, s, Y, z)$ -path,  $P$ .

The path  $P$  cannot end in  $x$ ,  $y$ , or  $v$ , otherwise we would have a  $\mathcal{C}_{\geq 7}$ . On the same grounds, it cannot end in  $w$  either, cf. Figure 5(c).



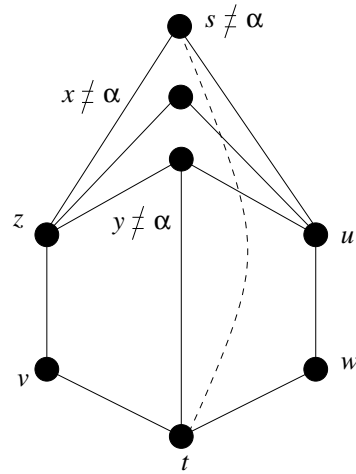
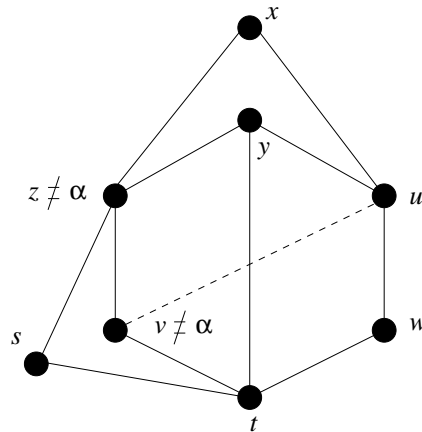


Figure 6. Lemma 2, illustration for Result 3.

Assume now that  $P$  ends in  $t$ ; necessarily,  $P$  has length 1 ( $P = \{s, t\}$ ), otherwise there would be a  $\mathcal{C}_{\geq 7}$ ; but  $L$  has been chosen so that, if the path  $z, \alpha, t$  exists, then  $\alpha \in Y$ : we can conclude that  $s \neq \alpha$ ; by Lemma 1, applied to  $s$  and  $v$ , either  $v$  or  $s$  is adjacent to  $u$ , and  $s$  and  $v$  have no neighbours outside  $\{z, t, u\}$ . We are going to show that  $v$  cannot be adjacent to  $u$ ; assume on the contrary that  $\{v, u\}$  exists. Since  $y$  has no neighbour outside  $\{z, u, t\}$ , we have (see Figure 7):

$$B(\{t, y\}) = B(\{t, v\}) = \{y, z, t, u, v\} \cup B(t).$$


 Figure 7. Lemma 2, illustration for Result 4, when  $P$  ends in  $t$ .

The sets  $\{t, y\}$  and  $\{t, v\}$  are not separated, and therefore  $v$  is not adjacent to  $u$ . In a similar way, if it is  $s$  which is adjacent to  $u$ , then the sets  $\{t, y\}$  and  $\{t, s\}$  are not separated. So neither  $v$  nor  $s$  can be adjacent to  $u$  and we have just proved that  $P$  cannot end in  $t$ .

There remains the possibility that  $P$  ends in  $u$ . Then, as previously,  $P$  has necessarily length 1, and we have the path  $z, s, u$ . Result 3 shows that  $s = \alpha$ , which ends the proof of Result 4.

5. *If  $u \neq \alpha$  and if  $u$  has a neighbour  $s \notin Y$ , then  $s = \alpha$  and the path  $u, \alpha, z$  exists.*

We assume that  $u \neq \alpha$  and have assumed previously that  $w \neq \alpha$ . The proof of Result 4 used the assumptions  $z \neq \alpha$ ,  $v \neq \alpha$ ; we can rerun this proof and obtain Result 5, symmetrically.

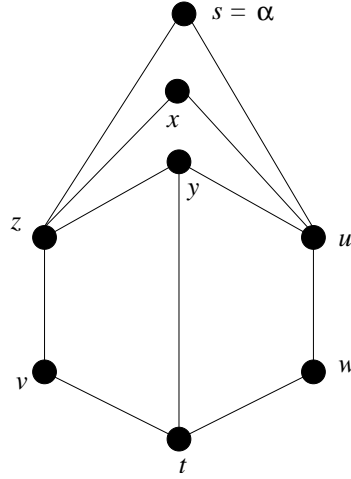


Figure 8. Lemma 2, illustration for Result 6.

6.  $\alpha = u$  or  $\alpha = v$ .

Assume that  $\alpha \neq u$ ,  $\alpha \neq v$ . By Results 1 and 2,  $v$  and  $w$  have no neighbours outside  $Y$ ; by Results 4 and 5,  $z$  and  $u$  can possibly have only one neighbour outside  $Y$ , that is  $\alpha$ , which they share in this case (see Figure 8). We have:

$$B(\{w, z\}) = B(\{v, u\}) = Y \text{ or } B(\{w, z\}) = B(\{v, u\}) = Y \cup \{\alpha\}.$$

The pairs  $\{w, z\}$  and  $\{v, u\}$  are not separated, so  $\alpha = u$  or  $\alpha = v$ .

7. *The sets  $\{x, t\}$  and  $\{z, w\}$  are not separated.*

By the previous result,  $t \neq \alpha$ . We have:

$$B(\{x, t\}) \cap Y = B(\{z, w\}) \cap Y = Y.$$

Remember that  $x$ ,  $y$ , and  $w$  have no neighbours outside  $Y$  (Lemma 1 and Result 1). To separate the pairs  $\{x, t\}$  and  $\{z, w\}$ ,  $t$  or  $z$  must have a neighbour outside  $Y$  which separates them.

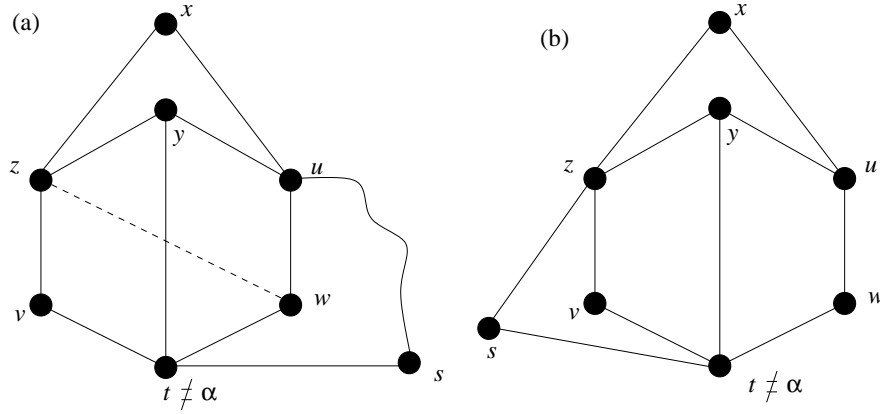


Figure 9. Lemma 2, illustrations for Result 7.

Assume first that  $t$  has a neighbour  $s \notin Y$  which separates  $\{x, t\}$  and  $\{z, w\}$ ; by Proposition 1 and since  $t$  is not the cut-vertex, there is a  $(H, s, Y, t)$ -path  $P$ , which can end neither in  $v$  nor  $w$ , because this would give a  $\mathcal{C}_{\geq 7}$ ; it cannot end in  $x$  or  $y$  either, because these vertices have no neighbours outside  $Y$ . Assume now that  $P$  ends in  $u$ , see Figure 9(a); this means that  $P$  is the path  $u, s, t$  (otherwise, existence of a  $\mathcal{C}_{\geq 7}$ ), and, using Result 6 (or the hypotheses on  $L$ ),  $s \neq \alpha$ . By Lemma 1 applied to  $w$  and  $s$ , either  $w$  or  $s$  is adjacent to  $z$ . Assume first that it is  $w$ . We have:

$$B(\{t, y\}) = B(\{t, w\}) = \{y, z, t, u, v, w\} \cup B(t).$$

Since  $y$  and  $w$  have no neighbours outside  $Y$ , only  $x$  could separate  $\{t, y\}$  and  $\{t, w\}$ , but we already know that the only neighbours of  $x$  in  $G$  are  $z$  and  $u$ : the sets  $\{t, y\}$  and  $\{t, w\}$  cannot be separated, and  $w$  is not adjacent to  $z$ . Similarly, if it is  $s$  which is adjacent to  $z$ , then the sets  $\{t, y\}$  and  $\{t, s\}$  are not separated. We have just proved that  $P$  cannot end in  $u$ , and the only possibility left is that it ends in  $z$ , in which case it has length 1, see

Figure 9(b), where  $s$  and  $z$  are neighbours. This however contradicts the choice of  $s$ , which was supposed to separate  $\{x, t\}$  and  $\{z, w\}$ .

Assume now that  $z$  has a neighbour  $s \notin Y$ , which separates  $\{x, t\}$  and  $\{z, w\}$ ; by Proposition 1, and because  $z \neq \alpha$ , there is a  $(H, s, Y, z)$ -path  $P$ , which cannot end in  $v$ ,  $x$ , or  $y$ , otherwise there would be a  $\mathcal{C}_{\geq 7}$ ; using Result 1,  $P$  cannot end in  $w$  either. If  $P$  ends in  $u$ , then it has length 1 and, since  $s \neq \alpha$ , this contradicts Result 3. Therefore,  $P$  ends in  $t$ , and it has length 1:  $s$  and  $t$  are neighbours, which again contradicts the choice of  $s$ .

The sets  $\{x, t\}$  and  $\{z, w\}$  cannot be separated.

The assumption that no  $\mathcal{C}_{\geq 7}$  is a subgraph of  $H$  led to a contradiction, and Lemma 2 is proved. ■

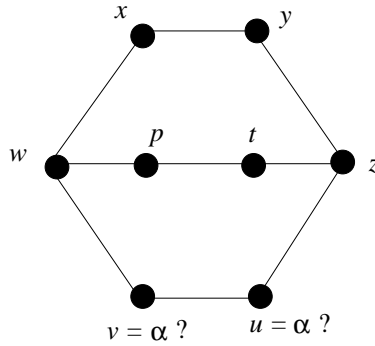


Figure 10. The graph  $K$  in Lemma 3.

**Lemma 3.** Consider the graph  $K$  given in Figure 10 and assume that, if  $\alpha$  exists, then  $\alpha = u$  or  $\alpha = v$ . If  $K$  is a subgraph of  $H$ , then  $\mathcal{C}_{\geq 7}$  is a subgraph of  $H$ .

**Proof.** Denote by  $Y$  the set of the 8 vertices in  $K$  and assume that we are in the conditions of Lemma 3. Since  $G$  is  $(1, \leq 2)$ -twin-free, the sets  $\{x, t\}$  and  $\{y, p\}$  are separated. By symmetry between  $\{x, y\}$  and  $\{p, t\}$ , then between  $x$  and  $y$ , it suffices to assume that  $x$  has a neighbour not in  $B(\{y, p\})$ . Now  $B(\{y, p\}) \supseteq \{x, y, z, p, t, w\}$ , and we have the following possibilities:

- $x$  is adjacent to  $s \in X \setminus Y$ ,  $s \neq \alpha$ . Since  $x \neq \alpha$ , there is a  $(H, s, Y, x)$ -path  $P$ . If  $P$  ends in  $w$ ,  $y$ ,  $p$ ,  $t$ ,  $v$ , or  $u$ , then we have a  $\mathcal{C}_{\geq 7}$ ; and if  $P$  ends in  $z$ , then either we directly obtain a  $\mathcal{C}_{\geq 7}$ , or  $P$  has length 1, which means that the edges  $\{x, s\}$  and  $\{s, z\}$  exist, with  $y \neq \alpha$ ,  $s \neq \alpha$ , and Lemma 2 can be applied.

- $\{x, v\}$  is an edge or  $\{x, u\}$  is an edge. In both cases, there is a  $\mathcal{C}_{\geq 7}$ .

In all the above cases, there is a  $\mathcal{C}_{\geq 7}$ , and Lemma 3 is proved.  $\blacksquare$

**Lemma 4.** *Consider the graph  $K'$  given in Figure 11 and assume that, if  $\alpha$  exists, then  $\alpha = u$  or  $\alpha = v$ . If  $K'$  is a subgraph of  $H$ , then  $\mathcal{C}_{\geq 7}$  is a subgraph of  $H$ .*

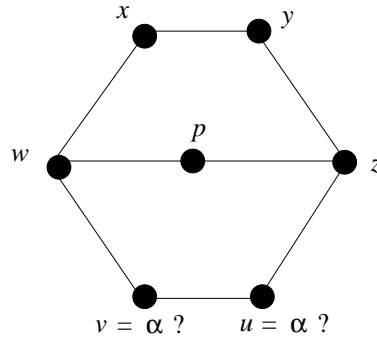


Figure 11. The graph  $K'$  in Lemma 4.

**Proof.** Denote by  $Y$  the set of the 7 vertices in  $K'$  and assume that we are in the conditions of Lemma 4. Since  $G$  is  $(1, \leq 2)$ -twin-free, the sets  $\{p, x\}$  and  $\{p, y\}$ , whose balls both contain  $x, y, z, w$ , and  $p$ , are separated; without loss of generality, we can assume that  $x$  has a neighbour not in  $B(\{p, y\})$ . Then we have the following possibilities:

- (a)  $x$  is adjacent to  $s \in X \setminus Y$ ,  $s \neq \alpha$ . Since  $x \neq \alpha$ , there is a  $(H, s, Y, x)$ -path  $P$ . If  $P$  ends in  $w, y, p, v$ , or  $u$ , then there is a  $\mathcal{C}_{\geq 7}$ ; and if  $P$  ends in  $z$ , then either we have a  $\mathcal{C}_{\geq 7}$  directly, or  $P$  has length 1, and we can apply Lemma 2, see the proof of Lemma 3.
- (b)  $\{x, u\}$  is an edge; then there is a  $\mathcal{C}_{\geq 7}$ .
- (c)  $\{x, v\}$  is an edge, see Figure 12; the sets  $\{z, x\}$  and  $\{z, w\}$ , whose balls contain  $Y$ , being separated,  $w$  or  $x$  must have a neighbour not in  $Y$ . If it is  $x$ , we can use case (a) above. Therefore we study the vertex  $w$ , a neighbour  $s \in X \setminus Y$  of  $w$  which is adjacent neither to  $x$  nor to  $z$ , and a  $(H, s, Y, w)$ -path  $P$ . If  $P$  yields a path of length 3 between  $w$  and  $z$  with only its ends,  $w$  and  $z$ , in  $Y$ , we apply Lemma 3; all other cases directly give a  $\mathcal{C}_{\geq 7}$ .

In all possible cases, we are led to the existence of a  $\mathcal{C}_{\geq 7}$ : Lemma 4 is proved. ■

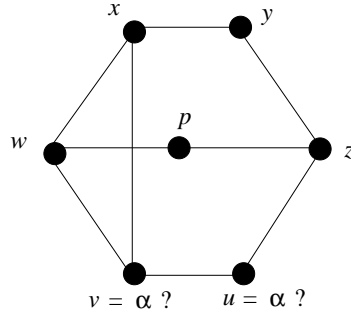


Figure 12. Illustration for the proof of Lemma 4, with the edge  $\{x, v\}$ .

We can now prove the following result.

**Lemma 5.** *The length of the longest cycle in  $H$  is not 6.*

**Proof.** Assume on the contrary that the longest cycle in  $H$  has length 6. If  $H$  admits a  $\mathcal{C}_6$  containing  $\alpha$ , we choose this cycle, otherwise we pick any  $\mathcal{C}_6$ , whose vertices we name  $a, b, c, d, e$ , and  $f$ , and we set  $Y = \{a, b, c, d, e, f\}$ . If the cycle contains  $\alpha$ , we assume that  $\alpha = f$  (see Figure 13). Lemmas 2, 3, and 4 as well as the nonexistence of a  $\mathcal{C}_{\geq 7}$  show that the only paths with length at least 2 with their ends in  $Y$  and their other vertices outside  $Y$  are:

- a possible path of length 2 between  $a$  and  $e$ ;
- a possible path of length 2 or 3 between  $c$  and  $f$ .

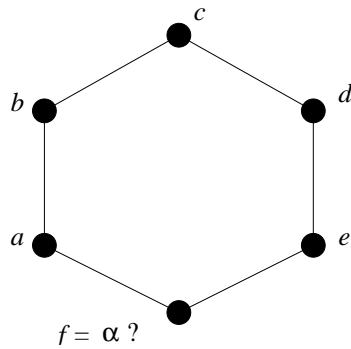


Figure 13. The length-6 cycle for Lemma 5.

Indeed, if a path links two consecutive vertices of the cycle, it gives a  $\mathcal{C}_{\geq 7}$ ; if it links two vertices at distance 2, other than  $a$  and  $e$ , either there is a  $\mathcal{C}_{\geq 7}$  or Lemma 2 applies; if it links two opposite vertices, other than  $c$  and  $f$ , either it gives a  $\mathcal{C}_{\geq 7}$ , or Lemma 3 or 4 applies; finally, if it has length at least 4 between  $c$  and  $f$ , then there is a  $\mathcal{C}_{\geq 7}$  in  $H$ .

Now the balls of the sets  $\{a, d\}$  and  $\{b, e\}$  contain  $Y$ ; these sets are not separated, since we have just seen that  $b$  and  $d$  have no neighbour outside  $Y$ , and that  $a$  and  $e$  either have no neighbour outside  $Y$ , or have exactly one neighbour outside  $Y$ , which they share. ■

#### 4. THE LENGTH OF THE LONGEST CYCLE IN $H$ IS NOT 5

**Lemma 6.** *If the graph  $M$  given in Figure 14 is a subgraph of  $H$ , with  $x \neq \alpha$  and  $y \neq \alpha$ , then  $\mathcal{C}_{\geq 6}$  is a subgraph of  $H$ .*

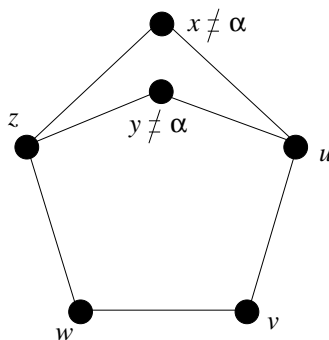


Figure 14. The graph  $M$  in Lemma 6.

**Proof.** Assume that  $M$  is a subgraph of  $H$ , with  $x \neq \alpha$ ,  $y \neq \alpha$ . The sets  $\{z, x\}$  and  $\{z, y\}$  being separated,  $x$  or  $y$  must have a neighbour  $s$  performing the separation. Assume, without loss of generality, that it is  $x$ . If there is an edge between  $x$  and  $v$  or  $w$ , we have a  $\mathcal{C}_{\geq 6}$ ; if not,  $x$  has a neighbour  $s$  outside  $M$ . Since  $x \neq \alpha$ , there is a  $(H, s, M, x)$ -path which in all cases will yield a  $\mathcal{C}_{\geq 6}$ . ■

**Lemma 7.** *The length of the longest cycle in  $H$  is not 5.*

**Proof.** Assume on the contrary that the longest cycle in  $H$  has length 5. If  $H$  admits a  $\mathcal{C}_5$  containing  $\alpha$ , we choose this cycle, otherwise we pick any  $\mathcal{C}_5$ ,

whose vertices we name  $a, b, c, d$ , and  $e$ , and we set  $Y = \{a, b, c, d, e\}$ . If the cycle contains  $\alpha$ , we assume that  $\alpha = e$  (see Figure 15).

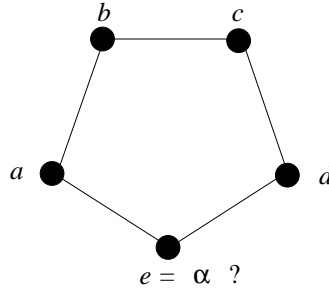


Figure 15. The length-5 cycle for Lemma 7.

As previously, the nonexistence of a  $\mathcal{C}_{\geq 6}$  and Lemma 6 show that the only path with length at least 2 whose ends are in  $Y$  and other vertices are not in  $Y$ , is a path of length 2 between  $a$  and  $d$ . This however does not separate the sets  $\{a, c\}$  and  $\{b, d\}$ , which, together with the fact that  $a, c, b, d$  are not the cut-vertex, ends the proof of Lemma 7. ■

## 5. THE LENGTH OF THE LONGEST CYCLE IN $H$ IS NOT 4 OR 3

**Lemma 8.** *The length of the longest cycle in  $H$  is not 4.*

**Proof.** Assume on the contrary that the longest cycle in  $H$  has length 4. Pick such a cycle, name its vertices  $a, b, c, d$  and assume, without loss of generality, that the cut-vertex is not  $a, b$ , or  $c$  (see Figure 16).

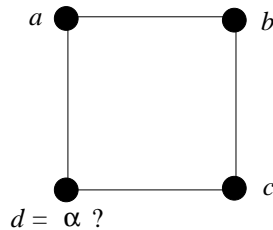


Figure 16. The length-4 cycle for Lemma 8.



The sets  $\{b, a\}$  and  $\{b, c\}$  being separated, there is a path of length at least 2 whose first end is  $a$  or  $c$ , whose second end, different from the first one, is on the cycle, and whose other vertices are not on the cycle. The only possibility, in order not to have a  $\mathcal{C}_{\geq 5}$ , is a path  $a, s, c$  where  $s$  does not belong to the cycle, but then  $s$  does not separate the sets  $\{b, a\}$  and  $\{b, c\}$ , which proves Lemma 8. ■

**Lemma 9.** *The length of the longest cycle in  $H$  is not 3.*

**Proof.** Assume on the contrary that the longest cycle in  $H$  has length 3. Pick such a cycle, name its vertices  $a, b, c$  and assume, without loss of generality, that the cut-vertex is not  $a$  or  $b$ . Then it is impossible to separate the sets  $\{c, a\}$  and  $\{c, b\}$  without creating a  $\mathcal{C}_{\geq 4}$ . ■

## 6. EXISTENCE OF A CYCLE OF LENGTH AT LEAST 7

**Theorem 1.** *Any undirected connected  $(1, \leq 2)$ -twin-free graph of order at least 2 admits an elementary cycle of length at least 7 as a subgraph.*

**Proof.** We have seen before Section 3 that the graph  $H$  admits a cycle; by Lemmas 5, 7–9, its longest cycle cannot have length 6, 5, 4, or 3: the longest cycle in  $H$ , hence the longest cycle in  $G$ , has length at least 7. ■

## 7. CONCLUSION: REMARKS AND OPEN ISSUES

We already mentioned in the introduction the parallel between the result we just proved and the fact that any connected  $(r, \leq 1)$ -twin-free graph of order at least 2 admits the path with  $2r + 1$  vertices as an *induced* subgraph [1]. We could wonder whether our result for  $(1, \leq 2)$ -twin-free graphs could be extended to the existence of an *induced* cycle with length at least seven. But considering the two graphs in Figure 17, one can see, in a straightforward if not clever way, that they are  $(1, \leq 2)$ -twin-free and have no chordless  $\mathcal{C}_{\geq 7}$  as an induced subgraph. Thus in Theorem 1, one cannot add the property “as an induced subgraph”. Also observe that the shortest possible cycle,  $\mathcal{C}_3$ , can be contained in a  $(1, \leq 2)$ -twin-free graph, as shown, for instance, by the second graph in Figure 17.

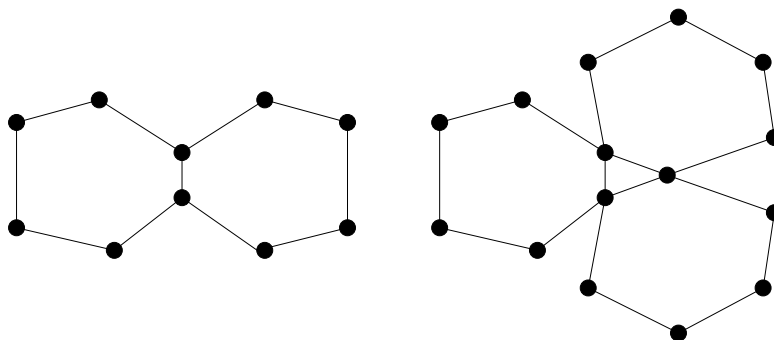


Figure 17. Two  $(1, \leq 2)$ -twin-free graphs with no chordless  $\mathcal{C}_{\geq 7}$  as induced subgraph.

Next, we state the following conjecture:

**Conjecture 1.** For all  $r \geq 2$ , the smallest connected  $(r, \leq 2)$ -twin-free graph with at least two vertices is the cycle on  $4r + 3$  vertices and all connected  $(r, \leq 2)$ -twin-free graphs with at least two vertices contain a cycle of length at least  $4r + 3$ .

For  $\ell = 3$ , T. Laihonon gives in [9] an example of a connected  $(1, \leq 3)$ -twin-free cubic graph with 16 vertices. It is, as far as we know, the smallest example of a nontrivial  $(1, \leq 3)$ -twin-free graph, but it remains unknown if these graphs always contain particular subgraphs. We do not dare for now to conjecture on this issue.

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