Discussiones Mathematicae
Graph Theory 30 (2010) 591–609

ON THE EXISTENCE OF A CYCLE OF LENGTH AT LEAST 7 IN A $(1, \leq 2)$-TWIN-FREE GRAPH

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Abstract

We consider a simple, undirected graph G. The ball of a subset Y of vertices in G is the set of vertices in G at distance at most one from a vertex in Y. Assuming that the balls of all subsets of at most two vertices in G are distinct, we prove that G admits a cycle with length at least 7.

Keywords: undirected graph, twin subsets, identifiable graph, distinguishable graph, identifying code, maximum length cycle.

2010 Mathematics Subject Classification: 05C38, 05C75.

1. Introduction

We consider a finite, undirected, simple graph $G = (X, E)$, where X is the vertex set and E the edge set.

If r is a positive integer and x a vertex in G, the *ball of x with radius r*, denoted by $B_r(x)$, is the set of vertices in G which are within distance r

from x. If Y is a subset of X, the *ball of Y with radius* r, denoted by $B_r(Y)$, is defined by

$$B_r(Y) = \bigcup_{y \in Y} B_r(y).$$

For $x \in X$, we set $B(x) = B_1(x)$ and call this set the *ball of x*: in other words, the ball of x consists of x and its neighbours; for $Y \subseteq X$, we set $B(Y) = B_1(Y)$ and call this set the *ball of Y*.

Two distinct subsets of X are said to be *separated* if they have distinct balls with radius r. For a given integer $\ell \geq 1$, the graph G is said to be $(r, \leq \ell)$*-twin-free* if any two distinct subsets of at most ℓ vertices are separated. In an $(r, \leq \ell)$-twin-free graph, for any subset V of X, there is at most one subset Y of X, with $|Y| \leq \ell$, such that $B_r(Y) = V$: the subsets of at most ℓ vertices are characterized by their balls with radius r. In this case, it is also said that G is $(r, \leq \ell)$*-identifiable* or $(r, \leq \ell)$*-distinguishable*, or that G admits an $(r, \leq \ell)$*-identifying code*. See, among many others, [7]–[11] and [13] for results on these codes.

Graphs admitting $(r, \leq 1)$-identifying codes, i.e., $(r, \leq 1)$-twin-free graphs, have particular structural properties (see for instance [1, 4] and [5]; see [12] for references upon these codes). In particular, it was proved in [1] that a connected $(r, \leq 1)$-twin-free graph with at least two vertices always contains as an induced subgraph the path P_{2r+1} on $2r + 1$ vertices; since P_{2r+1} itself is $(r, \leq 1)$-twin-free, it is therefore the smallest $(r, \leq 1)$-twin-free graph.

Several results have been published about $(r, \leq \ell)$-identifying codes in various graphs (see [7]–[11] and [13]), but little is known about the structure of these graphs. Using, for $i \geq 3$, the notation $\mathcal{C}_i$ (respectively, $\mathcal{C}_{\geq i}$) for a cycle of length i (respectively, at least i), it is easily seen that the cycles $\mathcal{C}_{\geq 7}$ are $(1, \leq 2)$-twin-free and that the smallest $(1, \leq 2)$-twin-free graph is the cycle $\mathcal{C}_7$. Hence it seems natural to wonder whether a cycle $\mathcal{C}_k$ with $k \geq 7$ is contained in any $(1, \leq 2)$-twin-free graph.

Thus we shall restrict ourselves to the case $r = 1$, $\ell = 2$ and prove in this article that an undirected connected $(1, \leq 2)$-twin-free graph of order at least 2, contains an *elementary* cycle (not going through a vertex twice) with length at least 7.

We now give some basic definitions for a graph $G = (X, E)$ (see [2, 3] or [6] for more). A *subgraph* of G is a graph $G' = (X', E')$, where $X' \subseteq X$

and

$$E' \subseteq \{\{u, v\} \in E : u \in X', v \in X'\}.$$

Such a subgraph is said to be *induced by* X' if

$$E' = \{\{u, v\} \in E : u \in X', v \in X'\}.$$

A *cut-vertex* of G is a vertex $u \in X$ such that the subgraph induced by $X \setminus \{u\}$ has more connected components than G. A *cut-edge* of G is an edge $e \in E$ such that the subgraph $(X, E \setminus \{e\})$ has more connected components than G. If G is connected, the deletion of a cut-vertex or of a cut-edge makes G disconnected. More generally, a *h-connected* graph, $h \geq 1$, is a graph G such that the minimum number of vertices to be deleted in order to disconnect G, or to reduce it to a singleton, is at least h. A *h-connected component* of G is an induced subgraph which is h-connected and maximal (for inclusion) in G.

A *block* of G is a maximal induced subgraph with no cut-vertex, and a *bridge* is an induced subgraph consisting of two adjacent vertices, linked by an edge which is a cut-edge in G.

Throughout this article, the paths and cycles will be elementary, and $G = (X, E)$ will be an undirected, simple graph of order at least 2. Moreover, we shall assume that G is connected: if not, the result would be obtained by choosing any connected component of G, with at least 2 vertices.

2. CHOOSING A *leaf-block* OF G

The blocks of G are 2-connected components or bridges. The graph given in the left part of Figure 1 contains 5 blocks: $\{a, b, c, d\}$, $\{c, e\}$, $\{g, h, i\}$, $\{e, f, g\}$, and $\{f, j\}$, which are surrounded with dotted lines. Two blocks of G either do not intersect, or intersect on a cut-vertex of G. Define the graph G' whose vertices are the blocks of G and whose edges link blocks having a nonempty intersection: G' is a tree. Now a block of G which is a leaf in G' is called a *leaf-block* of G. For instance, the graph G in Figure 1 has 3 leaf-blocks.

We give the following definition:

Definition 1. Let $G = (X, E)$ be an undirected connected graph, $Y \subset X$, $y \in Y$, and $s \in X \setminus Y$. A (G, s, Y, y)-*path* is a path in G whose ends are s and $t \in Y \setminus \{y\}$, and whose vertices other than t are in $X \setminus Y$.

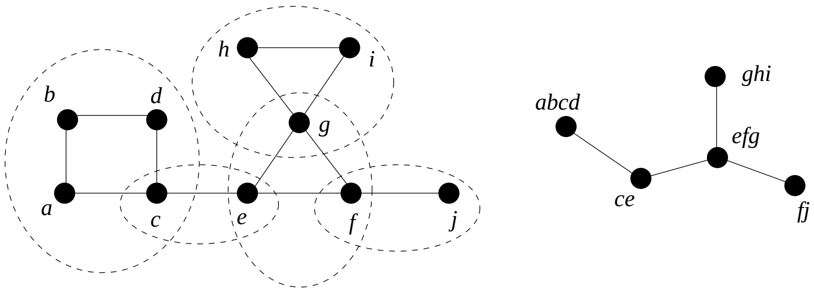

Figure 1. One example for the graphs G and G'.

We shall use the following proposition repeatedly.

Proposition 1. *Let $G = (X, E)$ be an undirected connected graph, H a 2-connected component of G, Y a subset of at least 2 vertices in H, y a vertex in Y which is not a cut-vertex of G, and s a neighbour of y which is not in Y. Then s belongs to H and there is a (H, s, Y, y)-path.*

Proof. Let $G \setminus \{y\}$ be the induced subgraph obtained from G by withdrawing the vertex y. Since y is not a cut-vertex, the graph $G \setminus \{y\}$ is still connected: there exists in $G \setminus \{y\}$ a path between s and $Y \setminus \{y\}$, whose vertices, other than its end in $Y \setminus \{y\}$, are in $X \setminus Y$, i.e., a (G, s, Y, y)-path; if we concatenate this path with the edge $\{s, y\}$, we get a path P between y and t, which are two distinct vertices in the 2-connected component H. Therefore, the union of H and P is still 2-connected, and, by the maximality of H as an induced 2-connected subgraph, P is a path in H. ■

Proposition 1 states that, if we wish to "leave"a subset Y of at least two vertices in a 2-connected component H, starting from a non cut-vertex y, then we stay inside H and we "come back" inside Y, on a vertex other than y.

From now on and throughout this article, we assume that G is $(1, \leq 2)$-twin-free.

Note that G cannot have vertices with degree 1: if x has degree 1 and y is its unique neighbour, then the sets $\{y\}$ and $\{x, y\}$ are not separated; actually, this is part of a more general result on $(1, \leq \ell)$-twin-free graphs, which have minimal degree at least ℓ [11, Theorem 8]. Consequently, a leaf-block of G cannot be a bridge: all leaf-blocks of G are 2-connected components, and Proposition 1 can be applied to them. We denote by H one leaf-block of G. The graph H has at least one cycle.

Also, either H is the whole graph G and in this case has no cut-vertex, or H has one, and only one, cut-vertex of G, α. In the following, we keep the notation α for the cut-vertex of G in the 2-connected component H, **if** α **exists**.

3. THE LENGTH OF THE LONGEST CYCLE IN H IS NOT 6

Lemma 1 will be used repeatedly to show Lemmas 2–4, which state that if H admits certain subgraphs, then, under certain conditions, a $\mathcal{C}_{\geq 7}$ is a subgraph of H. Lemma 5 concludes this section, establishing that the length of the longest cycle in H is not 6.

Lemma 1. *We assume that the longest cycle in H has length* 6. *If the graph L given in Figure* 2 *is a subgraph of H, with $x \neq \alpha$ and $y \neq \alpha$, then t is adjacent to either x or y, and x and y have no neighbours in G other than z, u, and, for exactly one of them, t.*

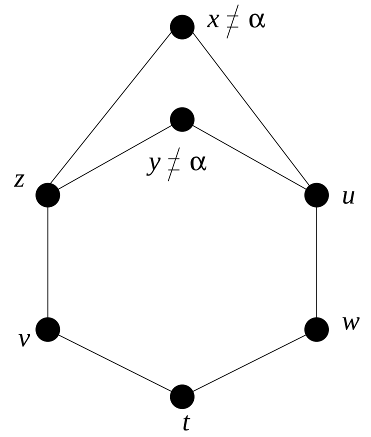

Figure 2. The graph L in Lemma 1.

Proof. We assume that H contains no $\mathcal{C}_{\geq 7}$ and that L is a subgraph of H, with $x \neq \alpha$ and $y \neq \alpha$. Let Y be the set of the 7 vertices in L.

First, we show that the neighbours, in G, of x and y belong to $\{z, u, t\}$. Assume on the contrary that x has a neighbour $s \in X \setminus \{z, u, t\}$.

If s belongs to Y, then $s = y$, $s = v$, or $s = w$.

If $s \notin Y$, then, since x is not the cut-vertex, we can use Proposition 1: the vertex s belongs to H and there is a (H, s, Y, x)-path.

So, whether $s \in Y$ or not, there is a path P of length at least 1 linking x and $Y \setminus \{x\}$, other than the edges $\{x,z\}$, $\{x,u\}$ and $\{x,t\}$, and whose vertices, but its two ends, do not belong to Y; now we examine the different possible cases, represented in Figure 3.

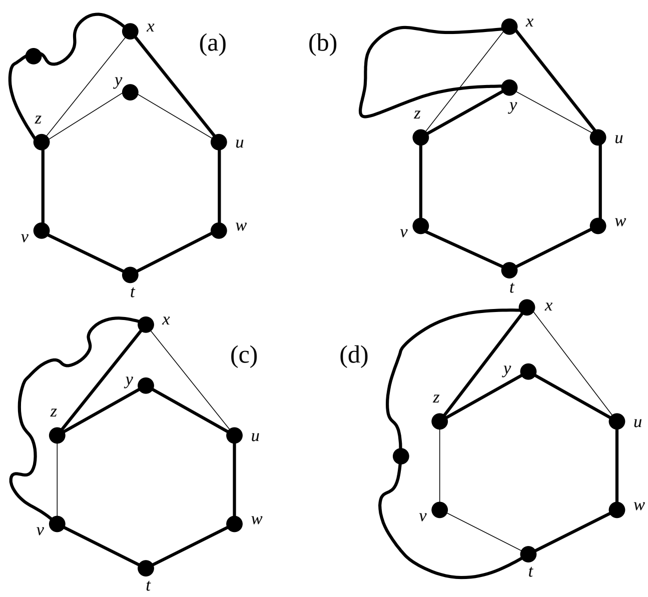

Figure 3. Illustrations for the proof of Lemma 1.

- (a) If P links x and z, P has length at least 2; by concatenating it with the path z, v, t, w, u, x, we obtain a $\mathcal{C}_{\geq 7}$, given in bold in Figure 3(a); this case is impossible, as is the case when P links x and u.
- (b) If P links x and y, this path concatenated with the path y, z, v, t, w, u, x yields a $\mathcal{C}_{\geq 7}$: this case is impossible.
- (c) If P links x and v, this path concatenated with the path v, t, w, u, y, z, x yields a $\mathcal{C}_{\geq 7}$. Similarly, P cannot link x and w.
- (d) Finally, if P links x and t, then P has length at least 2 and by concatenating it with the path t, w, u, y, z, x, we get a $\mathcal{C}_{\geq 7}$, still a contradiction.

None of the above cases is possible, the neighbours of x are in $\{z, u, t\}$ and the same is true for y. Furthermore, we have: $B(\{z,x\}) \supset \{x,y,z,u\}$ and $B(\{z,y\}) \supset \{x,y,z,u\}$. In order to separate the sets $\{z,x\}$ and $\{z,y\}$, it is necessary to use t, and so, one, and only one, vertex in $\{x, y\}$ is linked to t, which ends the proof of Lemma 1. ■

Lemma 2. *If the graph L given in Figure 2 is a subgraph of H, with $x \neq \alpha$ and $y \neq \alpha$, then $\mathcal{C}_{\geq 7}$ is a subgraph of H.*

Proof. We assume that no $\mathcal{C}_{\geq 7}$ is a subgraph of H, that L is a subgraph of H, and that $x \neq \alpha$, $y \neq \alpha$. We still denote by Y the set of the 7 vertices in L.

One can assume that, if $\alpha \notin Y$, then the path z, α, t does not exist: indeed, if the path z, α, t exists with $\alpha \notin Y$, then we delete in L the path z, v, t and replace it with the path z, α, t, and α is renamed as v. Similarly, one can assume that, if $\alpha \notin Y$, then the path u, α, t does not exist.

If $\alpha = z$ or $\alpha = w$, we rename the vertices, exchanging the names z and u as well as v and w, and so we can assume, without loss of generality, that $\alpha \neq z$ and $\alpha \neq w$.

The graph L we shall consider from now on has the following properties.

- L corresponds to Figure 2,
- $x \neq \alpha$, $y \neq \alpha$, $z \neq \alpha$, and $w \neq \alpha$,
- if the path z, α, t exists, then α belongs to Y,
- if the path u, α, t exists, then α belongs to Y.

Using Lemma 1, we can moreover assume that y is linked to t, and we then know that x and y have no neighbours in G other than those in Figure 4. The graph represented in Figure 4 is a subgraph of H.

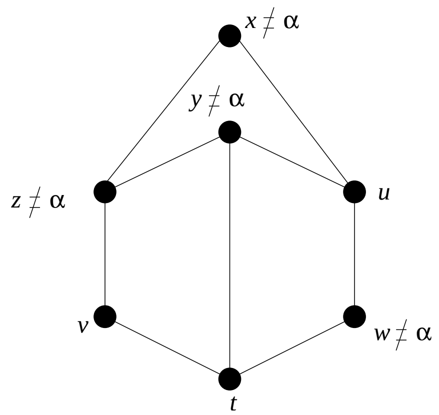

Figure 4. The graph L, with the edge $\{y, t\}$.

In order to prove Lemma 2, we proceed step by step, with intermediate results, from 1 to 7.

1. *The vertex w has no neighbour outside Y.*
Assume on the contrary that w has a neighbour $s \notin Y$ (see Figure 5); since $w \neq \alpha$, there is a (H, s, Y, w)-path P. By Lemma 1, x and y have their neighbours in Y, so P cannot end in x or y. It cannot end in u or t either, since this would yield a $\mathcal{C}_{\geq 7}$, represented in bold in Figure 5(a) when P ends in u. If P ends in v, then we have a $\mathcal{C}_{\geq 8}$, and if it ends in z, then we have a $\mathcal{C}_{\geq 7}$: the path P cannot end in any vertex of Y. Consequently, w has no neighbour outside Y.

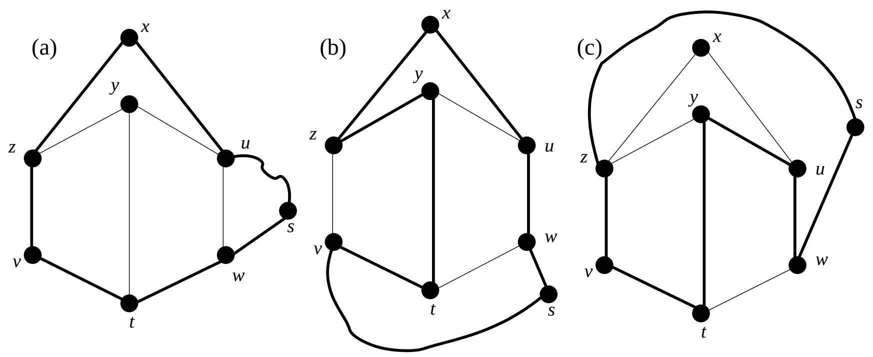

Figure 5. Lemma 2, illustrations for Result 1.

2. *If $v \neq \alpha$, then v has no neighbour outside Y.*
This result is obtained in exactly the same way as Result 1.

3. *There is no vertex outside Y, different from α and adjacent to both z and u.*

Assume on the contrary that there exists $s \notin Y$, with $s \neq \alpha$ and s adjacent to z and u (see Figure 6); by Lemma 1, since x is not adjacent to t and neither x nor s is the cut-vertex α, s is adjacent to t; but now $s \neq \alpha$, $y \neq \alpha$, and both s and y are adjacent to t: this contradicts Lemma 1.

4. *If $v \neq \alpha$ and if z has a neighbour $s \notin Y$, then $s = \alpha$ and the path z, α, u exists.*

We assume that $v \neq \alpha$ and that z has a neighbour $s \notin Y$. We recall that $z \neq \alpha$, so that by Proposition 1, there is a (H, s, Y, z)-path, P.

The path P cannot end in x, y, or v, otherwise we would have a $\mathcal{C}_{\geq 7}$. On the same grounds, it cannot end in w either, cf. Figure 5(c).

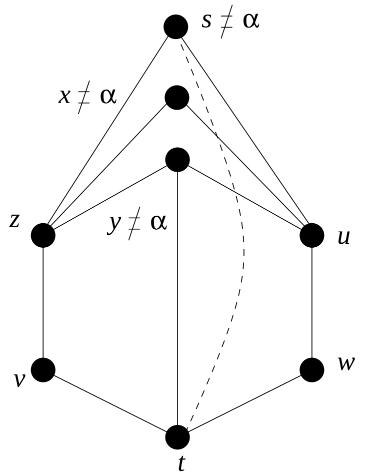

Figure 6. Lemma 2, illustration for Result 3.

Assume now that P ends in t; necessarily, P has length 1 ($P = \{s, t\}$), otherwise there would be a $\mathcal{C}_{\geq 7}$; but L has been chosen so that, if the path z, α, t exists, then $\alpha \in Y$: we can conclude that $s \neq \alpha$; by Lemma 1, applied to s and v, either v or s is adjacent to u, and s and v have no neighbours outside $\{z, t, u\}$. We are going to show that v cannot be adjacent to u; assume on the contrary that $\{v, u\}$ exists. Since y has no neighbour outside $\{z, u, t\}$, we have (see Figure 7):

$$B(\{t, y\}) = B(\{t, v\}) = \{y, z, t, u, v\} \cup B(t).$$

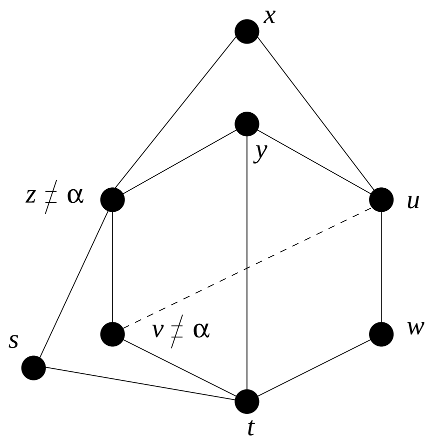

Figure 7. Lemma 2, illustration for Result 4, when P ends in t.

The sets $\{t, y\}$ and $\{t, v\}$ are not separated, and therefore v is not adjacent to u. In a similar way, if it is s which is adjacent to u, then the sets $\{t, y\}$ and $\{t, s\}$ are not separated. So neither v nor s can be adjacent to u and we have just proved that P cannot end in t.

There remains the possibility that P ends in u. Then, as previously, P has necessarily length 1, and we have the path z, s, u. Result 3 shows that $s = \alpha$, which ends the proof of Result 4.

5. *If $u \neq \alpha$ and if u has a neighbour $s \notin Y$, then $s = \alpha$ and the path u, α, z exists.*

We assume that $u \neq \alpha$ and have assumed previously that $w \neq \alpha$. The proof of Result 4 used the assumptions $z \neq \alpha$, $v \neq \alpha$; we can rerun this proof and obtain Result 5, symmetrically.

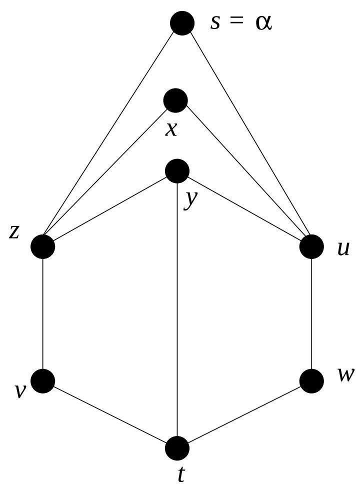

Figure 8. Lemma 2, illustration for Result 6.

6. *$\alpha = u$ or $\alpha = v$.*
Assume that $\alpha \neq u$, $\alpha \neq v$. By Results 1 and 2, v and w have no neighbours outside Y; by Results 4 and 5, z and u can possibly have only one neighbour outside Y, that is α, which they share in this case (see Figure 8). We have:

$$B(\{w, z\}) = B(\{v, u\}) = Y \text{ or } B(\{w, z\}) = B(\{v, u\}) = Y \cup \{\alpha\}.$$

The pairs $\{w, z\}$ and $\{v, u\}$ are not separated, so $\alpha = u$ or $\alpha = v$.

7. *The sets $\{x, t\}$ and $\{z, w\}$ are not separated.*

By the previous result, $t \neq \alpha$. We have:

$$B(\{x,t\}) \cap Y = B(\{z,w\}) \cap Y = Y.$$

Remember that x, y, and w have no neighbours outside Y (Lemma 1 and Result 1). To separate the pairs $\{x,t\}$ and $\{z,w\}$, t or z must have a neighbour outside Y which separates them.

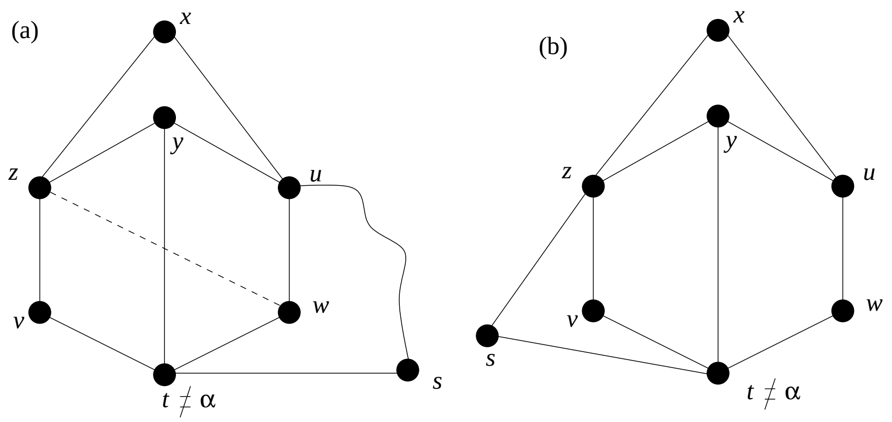

Figure 9. Lemma 2, illustrations for Result 7.

Assume first that t has a neighbour $s \notin Y$ which separates $\{x,t\}$ and $\{z,w\}$; by Proposition 1 and since t is not the cut-vertex, there is a (H,s,Y,t)-path P, which can end neither in v nor w, because this would give a $\mathcal{C}_{\geq 7}$; it cannot end in x or y either, because these vertices have no neighbours outside Y. Assume now that P ends in u, see Figure 9(a); this means that P is the path u,s,t (otherwise, existence of a $\mathcal{C}_{\geq 7}$), and, using Result 6 (or the hypotheses on L), $s \neq \alpha$. By Lemma 1 applied to w and s, either w or s is adjacent to z. Assume first that it is w. We have:

$$B(\{t,y\}) = B(\{t,w\}) = \{y,z,t,u,v,w\} \cup B(t).$$

Since y and w have no neighbours outside Y, only x could separate $\{t,y\}$ and $\{t,w\}$, but we already know that the only neighbours of x in G are z and u: the sets $\{t,y\}$ and $\{t,w\}$ cannot be separated, and w is not adjacent to z. Similarly, if it is s which is adjacent to z, then the sets $\{t,y\}$ and $\{t,s\}$ are not separated. We have just proved that P cannot end in u, and the only possibility left is that it ends in z, in which case it has length 1, see

Figure 9(b), where s and z are neighbours. This however contradicts the choice of s, which was supposed to separate $\{x, t\}$ and $\{z, w\}$.

Assume now that z has a neighbour $s \notin Y$, which separates $\{x, t\}$ and $\{z, w\}$; by Proposition 1, and because $z \neq \alpha$, there is a (H, s, Y, z)-path P, which cannot end in v, x, or y, otherwise there would be a $\mathcal{C}_{\geq 7}$; using Result 1, P cannot end in w either. If P ends in u, then it has length 1 and, since $s \neq \alpha$, this contradicts Result 3. Therefore, P ends in t, and it has length 1: s and t are neighbours, which again contradicts the choice of s.

The sets $\{x, t\}$ and $\{z, w\}$ cannot be separated.

The assumption that no $\mathcal{C}_{\geq 7}$ is a subgraph of H led to a contradiction, and Lemma 2 is proved. ■

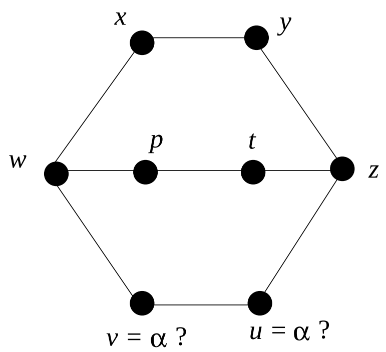

Figure 10. The graph K in Lemma 3.

Lemma 3. *Consider the graph K given in Figure* 10 *and assume that, if α exists, then $\alpha = u$ or $\alpha = v$. If K is a subgraph of H, then $\mathcal{C}_{\geq 7}$ is a subgraph of H.*

Proof. Denote by Y the set of the 8 vertices in K and assume that we are in the conditions of Lemma 3. Since G is $(1, \leq 2)$-twin-free, the sets $\{x, t\}$ and $\{y, p\}$ are separated. By symmetry between $\{x, y\}$ and $\{p, t\}$, then between x and y, it suffices to assume that x has a neighbour not in $B(\{y, p\})$. Now $B(\{y, p\}) \supseteq \{x, y, z, p, t, w\}$, and we have the following possibilities:

- x is adjacent to $s \in X \setminus Y$, $s \neq \alpha$. Since $x \neq \alpha$, there is a (H, s, Y, x)-path P. If P ends in w, y, p, t, v, or u, then we have a $\mathcal{C}_{\geq 7}$; and if P ends in z, then either we directly obtain a $\mathcal{C}_{\geq 7}$, or P has length 1, which means that the edges $\{x, s\}$ and $\{s, z\}$ exist, with $y \neq \alpha, s \neq \alpha$, and Lemma 2 can be applied.

- $\{x, v\}$ is an edge or $\{x, u\}$ is an edge. In both cases, there is a $\mathcal{C}_{\geq 7}$.

In all the above cases, there is a $\mathcal{C}_{\geq 7}$, and Lemma 3 is proved. ■

Lemma 4. *Consider the graph K' given in Figure* 11 *and assume that, if α exists, then $\alpha = u$ or $\alpha = v$. If K' is a subgraph of H, then $\mathcal{C}_{\geq 7}$ is a subgraph of H.*

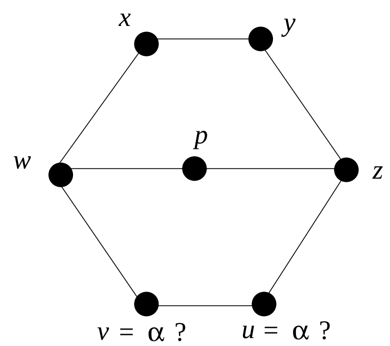

Figure 11. The graph K' in Lemma 4.

Proof. Denote by Y the set of the 7 vertices in K' and assume that we are in the conditions of Lemma 4. Since G is $(1, \leq 2)$-twin-free, the sets $\{p, x\}$ and $\{p, y\}$, whose balls both contain x, y, z, w, and p, are separated; without loss of generality, we can assume that x has a neighbour not in $B(\{p, y\})$. Then we have the following possibilities:

- (a) x is adjacent to $s \in X \setminus Y$, $s \neq \alpha$. Since $x \neq \alpha$, there is a (H, s, Y, x)-path P. If P ends in w, y, p, v, or u, then there is a $\mathcal{C}_{\geq 7}$; and if P ends in z, then either we have a $\mathcal{C}_{\geq 7}$ directly, or P has length 1, and we can apply Lemma 2, see the proof of Lemma 3.
- (b) $\{x, u\}$ is an edge; then there is a $\mathcal{C}_{\geq 7}$.
- (c) $\{x, v\}$ is an edge, see Figure 12; the sets $\{z, x\}$ and $\{z, w\}$, whose balls contain Y, being separated, w or x must have a neighbour not in Y. If it is x, we can use case (a) above. Therefore we study the vertex w, a neighbour $s \in X \setminus Y$ of w which is adjacent neither to x nor to z, and a (H, s, Y, w)-path P. If P yields a path of length 3 between w and z with only its ends, w and z, in Y, we apply Lemma 3; all other cases directly give a $\mathcal{C}_{\geq 7}$.

In all possible cases, we are led to the existence of a $\mathcal{C}_{\geq 7}$: Lemma 4 is proved. ■

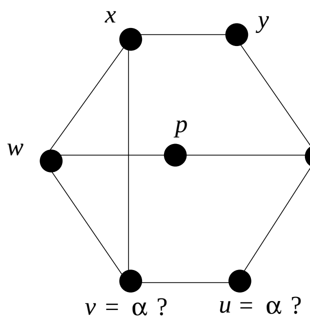

Figure 12. Illustration for the proof of Lemma 4, with the edge $\{x, v\}$.

We can now prove the following result.

Lemma 5. *The length of the longest cycle in H is not* 6.

Proof. Assume on the contrary that the longest cycle in H has length 6. If H admits a $\mathcal{C}_6$ containing α, we choose this cycle, otherwise we pick any $\mathcal{C}_6$, whose vertices we name a, b, c, d, e, and f, and we set $Y = \{a, b, c, d, e, f\}$. If the cycle contains α, we assume that $\alpha = f$ (see Figure 13). Lemmas 2, 3, and 4 as well as the nonexistence of a $\mathcal{C}_{\geq 7}$ show that the only paths with length at least 2 with their ends in Y and their other vertices outside Y are:

- a possible path of length 2 between a and e;
- a possible path of length 2 or 3 between c and f.

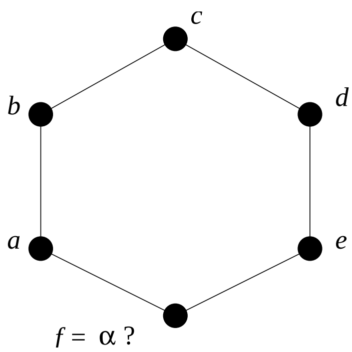

Figure 13. The length-6 cycle for Lemma 5.

Indeed, if a path links two consecutive vertices of the cycle, it gives a $\mathcal{C}_{\geq 7}$; if it links two vertices at distance 2, other than a and e, either there is a $\mathcal{C}_{\geq 7}$ or Lemma 2 applies; if it links two opposite vertices, other than c and f, either it gives a $\mathcal{C}_{\geq 7}$, or Lemma 3 or 4 applies; finally, if it has length at least 4 between c and f, then there is a $\mathcal{C}_{\geq 7}$ in H.

Now the balls of the sets $\{a, d\}$ and $\{b, e\}$ contain Y; these sets are not separated, since we have just seen that b and d have no neighbour outside Y, and that a and e either have no neighbour outside Y, or have exactly one neighbour outside Y, which they share. ■

4. The Length of the Longest Cycle in H is Not 5

Lemma 6. *If the graph M given in Figure 14 is a subgraph of H, with $x \neq \alpha$ and $y \neq \alpha$, then $\mathcal{C}_{\geq 6}$ is a subgraph of H.*

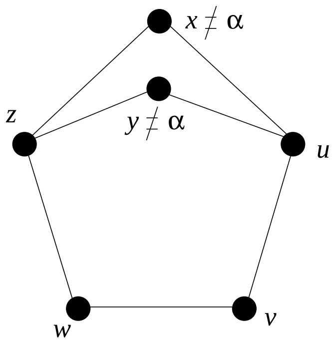

Figure 14. The graph M in Lemma 6.

Proof. Assume that M is a subgraph of H, with $x \neq \alpha$, $y \neq \alpha$. The sets $\{z, x\}$ and $\{z, y\}$ being separated, x or y must have a neighbour s performing the separation. Assume, without loss of generality, that it is x. If there is an edge between x and v or w, we have a $\mathcal{C}_{\geq 6}$; if not, x has a neighbour s outside M. Since $x \neq \alpha$, there is a (H, s, M, x)-path which in all cases will yield a $\mathcal{C}_{\geq 6}$. ■

Lemma 7. *The length of the longest cycle in H is not 5.*

Proof. Assume on the contrary that the longest cycle in H has length 5. If H admits a $\mathcal{C}_5$ containing α, we choose this cycle, otherwise we pick any $\mathcal{C}_5$, whose vertices we name a, b, c, d, and e, and we set $Y = \{a, b, c, d, e\}$. If the cycle contains α, we assume that $\alpha = e$ (see Figure 15).

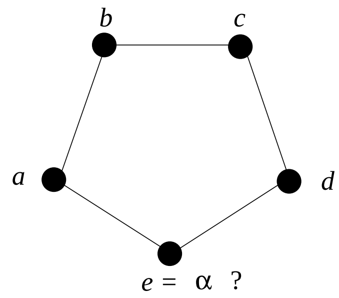

Figure 15. The length-5 cycle for Lemma 7.

As previously, the nonexistence of a $\mathcal{C}_{\geq 6}$ and Lemma 6 show that the only path with length at least 2 whose ends are in Y and other vertices are not in Y, is a path of length 2 between a and d. This however does not separate the sets $\{a, c\}$ and $\{b, d\}$, which, together with the fact that a, c, b, d are not the cut-vertex, ends the proof of Lemma 7. ∎

5. The Length of the Longest Cycle in H is Not 4 or 3

Lemma 8. *The length of the longest cycle in H is not* 4.

Proof. Assume on the contrary that the longest cycle in H has length 4. Pick such a cycle, name its vertices a, b, c, d and assume, without loss of generality, that the cut-vertex is not a, b, or c (see Figure 16).

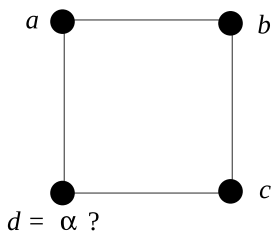

Figure 16. The length-4 cycle for Lemma 8.

The sets $\{b, a\}$ and $\{b, c\}$ being separated, there is a path of length at least 2 whose first end is a or c, whose second end, different from the first one, is on the cycle, and whose other vertices are not on the cycle. The only possibility, in order not to have a $\mathcal{C}_{\geq 5}$, is a path a, s, c where s does not belong to the cycle, but then s does not separate the sets $\{b, a\}$ and $\{b, c\}$, which proves Lemma 8. ■

Lemma 9. *The length of the longest cycle in H is not* 3.

Proof. Assume on the contrary that the longest cycle in H has length 3. Pick such a cycle, name its vertices a, b, c and assume, without loss of generality, that the cut-vertex is not a or b. Then it is impossible to separate the sets $\{c, a\}$ and $\{c, b\}$ without creating a $\mathcal{C}_{\geq 4}$. ■

6. EXISTENCE OF A CYCLE OF LENGTH AT LEAST 7

Theorem 1. *Any undirected connected $(1, \leq 2)$-twin-free graph of order at least* 2 *admits an elementary cycle of length at least* 7 *as a subgraph.*

Proof. We have seen before Section 3 that the graph H admits a cycle; by Lemmas 5, 7–9, its longest cycle cannot have length 6, 5, 4, or 3: the longest cycle in H, hence the longest cycle in G, has length at least 7. ■

7. CONCLUSION: REMARKS AND OPEN ISSUES

We already mentioned in the introduction the parallel between the result we just proved and the fact that any connected $(r, \leq 1)$-twin-free graph of order at least 2 admits the path with $2r + 1$ vertices as an *induced* subgraph [1]. We could wonder whether our result for $(1, \leq 2)$-twin-free graphs could be extended to the existence of an *induced* cycle with length at least seven. But considering the two graphs in Figure 17, one can see, in a straightforward if not clever way, that they are $(1, \leq 2)$-twin-free and have no chordless $\mathcal{C}_{\geq 7}$ as an induced subgraph. Thus in Theorem 1, one cannot add the property "as an induced subgraph". Also observe that the shortest possible cycle, $\mathcal{C}_3$, can be contained in a $(1, \leq 2)$-twin-free graph, as shown, for instance, by the second graph in Figure 17.

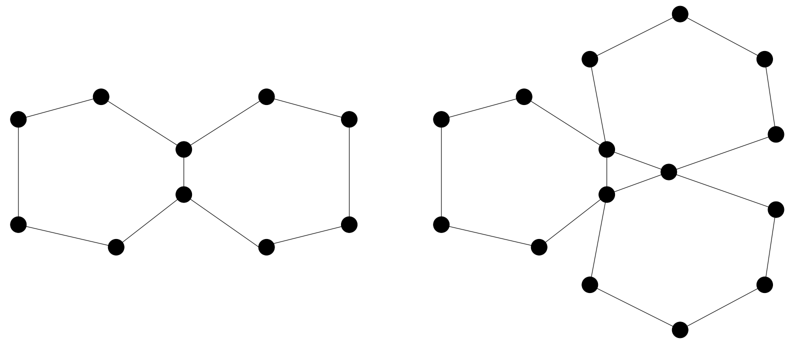

Figure 17. Two $(1, \leq 2)$-twin-free graphs with no chordless $\mathcal{C}_{\geq 7}$ as induced subgraph.

Next, we state the following conjecture:

Conjecture 1. For all $r \geq 2$, the smallest connected $(r, \leq 2)$-twin-free graph with at least two vertices is the cycle on $4r+3$ vertices and all connected $(r, \leq 2)$-twin-free graphs with at least two vertices contain a cycle of length at least $4r+3$.

For $\ell = 3$, T. Laihonen gives in [9] an example of a connected $(1, \leq 3)$-twin-free cubic graph with 16 vertices. It is, as far as we know, the smallest example of a nontrivial $(1, \leq 3)$-twin-free graph, but is remains unknown if these graphs always contain particular subgraphs. We do not dare for now to conjecture on this issue.

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Received 27 July 2009
Revised 14 December 2009
Accepted 14 December 2009