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CANCELLATION OF DIRECT PRODUCTS OF DIGRAPHS

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Abstract

We investigate expressions of form $A \times C \cong B \times C$ involving direct products of digraphs. Lovász gave exact conditions on C for which it necessarily follows that $A \cong B$. We are here concerned with a different aspect of cancellation. We describe exact conditions on A for which it necessarily follows that $A \cong B$. In the process, we do the following: Given an arbitrary digraph A and a digraph C that admits a homomorphism onto an arc, we classify all digraphs B for which $A \times C \cong B \times C$.

Keywords: graph direct product, graph product cancellation, digraphs.

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The article [2] describes the exact conditions a graph A must meet in order that any expression $A \times C \cong B \times C$ necessarily implies $A \cong B$. It also classifies—given graphs A and C—all graphs B for which $A \times C \cong B \times C$. This paper generalizes these results to digraphs. Thus, given that a graph is just a symmetric digraph, the current article implies the results of [2] but is considerably more general. We begin by recalling the relevant definitions and notation.

1. Notation and Notions

For us, a *digraph* A is simply a binary relation $E(A)$ on a vertex set $V(A)$, that is a subset $E(A) \subseteq V(A) \times V(A)$. For brevity, an ordered pair $(a, a') \in E(A)$ is denoted aa', and is visualized as an arrow pointing from a to a'. Elements of $E(A)$ are called *arcs*. A reflexive arc aa is called a *loop*, and is drawn as a closed curve beginning and ending at a. (We normally do not embellish such a closed curve with an arrowhead.)

We denote by $\overrightarrow{K_2}$ the digraph with two vertices 0 and 1 and a single arc 01. (The digraph $\overrightarrow{K_2}$ plays a large role in this discussion.) Given a positive integer n, we denote by $\overrightarrow{C_n}$ the digraph whose vertices are $\{0, 1, 2, \ldots, n-1\}$ and whose edges are $\{01, 12, 23, \ldots, (n-1)0\}$. By convention we agree that $\overrightarrow{C_1}$ consists of a single vertex with a loop. Some of these digraphs are illustrated in Figure 1. Notice that $\overrightarrow{C_1}$ and $\overrightarrow{C_2}$ are symmetric (as relations) but $\overrightarrow{K_2}$ and $\overrightarrow{C_n}$ $(n > 2)$ are not symmetric.

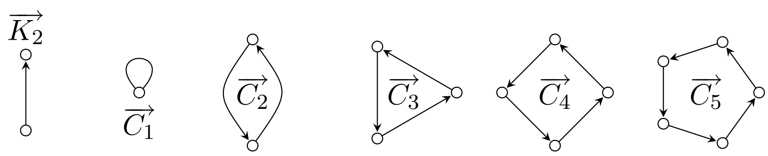

Figure 1. Examples of digraphs.

A symmetric digraph A (i.e., one satisfying $aa' \in E(A)$ if and only if $a'a \in E(A)$ for all $a, a' \in V(A)$) is called a *graph*. In drawing graphs it is common to represent the two arcs aa' and $a'a$ as a single undirected edge joining a and a'. As usual, K_n denotes the complete graph on n vertices. By K_n^* we mean the graph obtained from K_n by adding a loop at each vertex.

If A and B are digraphs, then $A + B$ denotes the disjoint union of A and B. If n is a natural number, then nA denotes the digraph formed from n disjoint copies of A.

The *direct product* of two digraphs A and B is the digraph $A \times B$ whose vertex set is the Cartesian product $V(A) \times V(B)$ and whose arcs are the pairs $(a, b)(a', b')$ with $aa' \in E(A)$ and $bb' \in E(B)$. A *homomorphism* from digraph A to digraph B is a map $\varphi : V(A) \to V(B)$ with the property that $aa' \in E(A)$ implies $\varphi(a)\varphi(a') \in E(B)$. We assume the reader to be familiar with direct products and homomorphisms. For standard references see [4, 3].

2. Cancellation Laws of Lovász

In his classic paper [5], Lovász defines a digraph C to be a *zero divisor* if there exist non-isomorphic digraphs A and B for which $A \times C \cong B \times C$. For example, Figure 2 shows that $\overrightarrow{C_3}$ is a zero divisor: If $A = \overrightarrow{C_3}$ and $B = 3\overrightarrow{C_1}$, then clearly $A \ncong B$, yet $A \times \overrightarrow{C_3} \cong B \times \overrightarrow{C_3}$. (Both products are isomorphic to three copies of $\overrightarrow{C_3}$.)

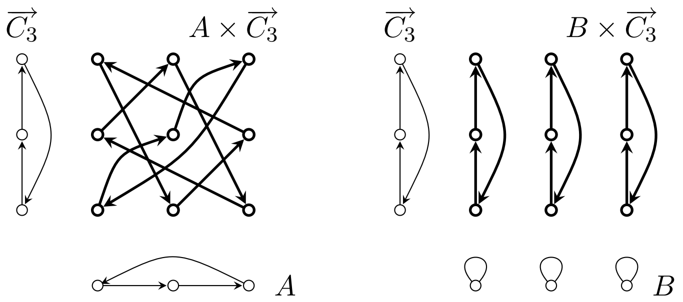

Figure 2. Example of a zero divisor.

Here is the main result concerning zero divisors.

Theorem 1 (Lovász [5], Theorem 8)**.** *A digraph C is a zero divisor if and only if there is a homomorphism $\varphi : C \to \overrightarrow{C_{p_1}} + \overrightarrow{C_{p_2}} + \overrightarrow{C_{p_3}} + \cdots + \overrightarrow{C_{p_k}}$ for prime numbers $p_1, p_2, \ldots, p_k$.*

Thus, for instance, $\overrightarrow{K_2}$ is a zero divisor. Also any $\overrightarrow{C_n}$ with $n > 1$ is a zero divisor. (Even if n is not prime, there is an $\frac{n}{p}$-fold homomorphic cover $\varphi : \overrightarrow{C_n} \to \overrightarrow{C_p}$ for any prime divisor p of n.) Theorem 1 becomes quite simple if C is a graph with at least one edge, for in this case no homomorphism φ can carry a (symmetric) edge of C to an (asymmetric) arc of $\overrightarrow{C_n}$. As $\overrightarrow{C_n}$ is symmetric only for $n = 2$, it follows that a graph C is a zero divisor if an only if there is a homomorphism $\varphi : C \to \overrightarrow{C_2}$, that is if and only if C is bipartite.

Corollary 1. *A graph C with at least one edge is a zero divisor if and only if C is bipartite.*

Theorem 1 and its corollary can be regarded as cancellation laws for the direct product. They give exact conditions on C (namely the absence of a homomorphism $\varphi : C \to \overrightarrow{C_{p_1}} + \overrightarrow{C_{p_2}} + \cdots + \overrightarrow{C_{p_k}}$) under which $A \times C \cong B \times C$ necessarily implies $A \cong B$.

However, this does not fully resolve every question concerning cancellation. One might ask what conditions on A (or B) might guarantee that $A \times C \cong B \times C$ always implies $A \cong B$. For example, if $A = \overrightarrow{C_1}$, then $A \times C \cong B \times C$ implies $A \cong B$ whether or not C meets the hypothesis of Theorem 1. It is reasonable to ask what other digraphs A have this property. The answer to that question is the purpose of this paper. Along the way we will describe a means of classifying—given digraphs A and C—all digraphs B for which $A \times C \cong B \times C$. Our methods will require the following theorems due to Lovász.

Theorem 2 (Lovász [5], Theorem 6)**.** *Let A, B, C and D be digraphs. If $A \times C \cong B \times C$ and there is a homomorphism from D to C, then $A \times D \cong B \times D$.*

Theorem 3 (Lovász [5], Theorem 7)**.** *Let A, B and C be digraphs. If $A \times C \cong B \times C$, then there is an isomorphism from $A \times C$ to $B \times C$ of the form $(a, c) \mapsto (\psi(a, c), c)$ for some homomorphism $\psi : A \times C \to B$.*

3. An Arc as a Factor

This section addresses the equation $A \times C \cong B \times C$ where the common factor C is the single arc $\overrightarrow{K_2}$. Given a digraph A we describe the structure of all digraphs B having the property that $A \times \overrightarrow{K_2} \cong B \times \overrightarrow{K_2}$. From this we will obtain necessary conditions on B for which $A \times C \cong B \times C$ (where C is arbitrary).

Given a digraph A, we denote the set of permutations of $V(A)$ as $\mathrm{Perm}(V(A))$. The following definition is central to the remainder of this paper.

Definition 1. Given a digraph A and a permutation $\pi \in \mathit{Perm}(V(A))$, we define a digraph A^π as $V(A^\pi) = V(A)$ and $E(A^\pi) = \{a\pi(a') : aa' \in E(A)\}$. Thus $aa' \in E(A)$ if and only if $a\pi(a') \in E(A^\pi)$, and $aa' \in E(A^\pi)$ if and only if $a\pi^{-1}(a') \in E(A)$.

Figure 3 shows examples. The upper-left displays a digraph A on vertices $\{0, 1, 2\}$, and the digraphs A^π for each of the six permutations of $V(G)$ are shown. (Note that $A^{\mathrm{id}} = A$.) In this case the A^π are six pairwise nonisomorphic graphs.

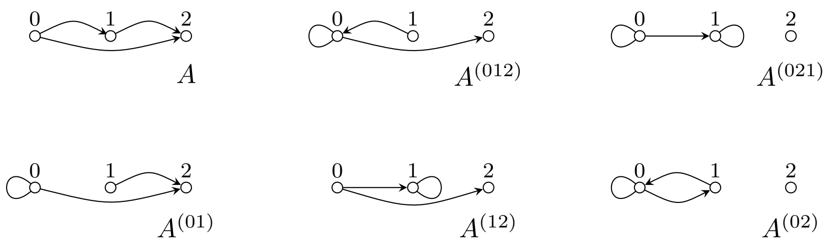

Figure 3. Examples of A and A^π.

For another example, take $A = \overrightarrow{C_3}$ and let $\pi = (021)$ permute the vertices cyclicly in the direction opposite to the arcs. Then $A^\pi = 3\overrightarrow{C_1}$. (Digraphs A and $B = A^\pi$ appear as factors in Figure 2.)

It is perhaps a startling fact that even though $A \ncong A^\pi$ in general, it is nonetheless true that $A \times \overrightarrow{K_2} \cong A^\pi \times \overrightarrow{K_2}$ for any π. Before proving this, observe that it is true for the digraphs A and A^π from Figure 3: Figure 4 confirms that all the products $A^\pi \times \overrightarrow{K_2}$ are isomorphic.

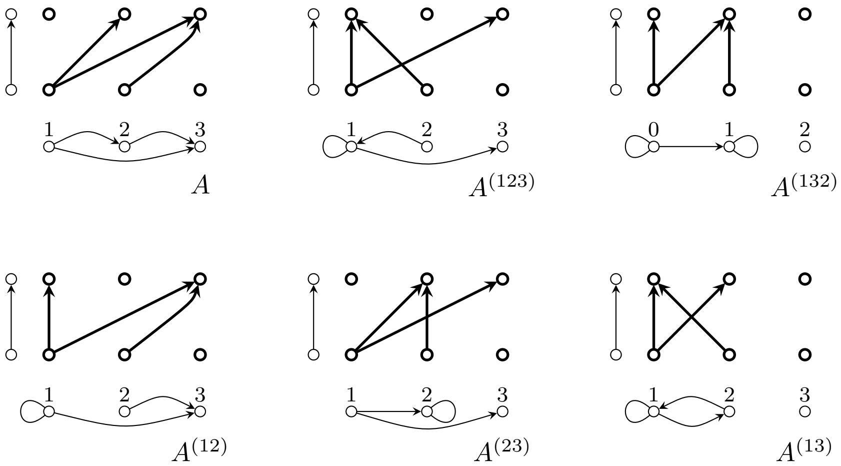

Figure 4. Products with A^π as a factor.

In fact, a somewhat stronger statement can be proved.

Proposition 1. *If A and B are digraphs, then $A \times \overrightarrow{K_2} \cong B \times \overrightarrow{K_2}$ if and only if $B \cong A^\pi$ for some $\pi \in \mathrm{Perm}(V(A))$.*

Proof. Suppose $B \cong A^\pi$ for some $\pi \in \mathrm{Perm}(V(A))$. In showing $A \times \overrightarrow{K_2} \cong B \times \overrightarrow{K_2}$, there is no harm in assuming further that $B = A^\pi$. Thus $V(B) = V(A)$ and $E(B) = \{a\pi(a') : aa' \in E(A)\}$. Define a map $\varphi : V(A \times \overrightarrow{K_2}) \to V(B \times \overrightarrow{K_2})$ as follows.

$$\varphi(a,\varepsilon) = \begin{cases} (\pi(a),\varepsilon) & \text{if } \varepsilon = 1, \\ (a,\varepsilon) & \text{if } \varepsilon = 0. \end{cases}$$

Now, φ is clearly bijective. Consider a typical arc of $A \times \overrightarrow{K_2}$, which necessarily has the form $(a,0)(a',1)$ for some $aa' \in E(A)$. Observe that $\varphi(a,0)\varphi(a',1) = (a,0)(\pi(a'),1)$ is an arc of $A^\pi \times \overrightarrow{K_2}$, so φ is a homomorphism. On the other hand, if $(a,0)(a',1) \in A^\pi \times \overrightarrow{K_2}$, then $aa' \in E(A^\pi)$, so $a\pi^{-1}(a') \in E(A)$. Thus $(a,0)(\pi^{-1}(a'),1) \in E(A \times \overrightarrow{K_2})$ and $\varphi(a,0)\varphi(\pi^{-1}(a'),1) = (a,0)(a',1)$. We therefore have an isomorphism $\varphi : A \times \overrightarrow{K_2} \to B \times \overrightarrow{K_2}$.

Conversely, suppose there exists an isomorphism $\varphi : A \times \overrightarrow{K_2} \to B \times \overrightarrow{K_2}$. We will produce a π for which $B \cong A^\pi$. By Theorem 3 we may assume that φ has the form $\varphi(a,\varepsilon) = (\psi(a,\varepsilon),\varepsilon)$ for some map $\psi : A \times \overrightarrow{K_2} \to B$. (Actually this can be deduced quickly in the present simple case where the common factor C is $\overrightarrow{K_2}$: The vertices of $A \times \overrightarrow{K_2}$ with positive out-degree all belong to $V(A) \times \{0\}$, so φ necessarily sends them to $V(B) \times \{0\}$. Likewise vertices of $A \times \overrightarrow{K_2}$ with positive in-degree all belong to $V(A) \times \{1\}$, so φ necessarily sends them to $V(B) \times \{1\}$. Thus if (a,ε) is a non-isolated vertex of $A \times \overrightarrow{K_2}$, then φ does not alter its second coordinate. One quickly confirms that the action of φ on the isolated vertices can be modified if necessary so that it does not alter the second coordinates.)

Now consider maps $\mu_0, \mu_1 : V(A) \to V(B)$ defined as $\mu_0(a) = \psi(a,0)$ and $\mu_1(a) = \psi(a,1)$. It is straightforward to verify that bijectivity of φ implies that μ_0 and μ_1 are bijections too. Set $\pi = \mu_0^{-1}\mu_1$, so $\pi \in \mathrm{Perm}(V(A))$. The proof is completed by showing that the bijection $\mu_0 : V(A^\pi) \to V(B)$ is an isomorphism. For this, consider the following chain of equivalences.

$$
\begin{aligned}
aa' \in E(A^\pi) &\iff a\,\pi^{-1}(a') \in E(A) && \text{(definition of } A^\pi)\\
&\iff a\,\mu_1^{-1}\mu_0(a') \in E(A) && \text{(definition of } \pi)\\
&\iff (a,0)(\mu_1^{-1}\mu_0(a'),1) \in E(A\times \overrightarrow{K_2}) && \text{(definition of } \times)\\
&\iff \varphi(a,0)\varphi(\mu_1^{-1}\mu_0(a'),1) \in E(B\times \overrightarrow{K_2}) && (\varphi \text{ is isomorphism})\\
&\iff (\psi(a,0),0)(\psi(\mu_1^{-1}\mu_0(a'),1),1) \in E(B\times \overrightarrow{K_2}) && \text{(property of } \varphi)\\
&\iff (\mu_0(a),0)(\mu_0(a'),1) \in E(B\times \overrightarrow{K_2}) && \text{(definition of } \mu_0,\mu_1)\\
&\iff \mu_0(a)\mu_0(a') \in E(B). && \text{(definition of } \times)
\end{aligned}
$$

Thus $\mu_0 : A^\pi \to B$ is an isomorphism, and the proof is complete. ■

Digraph $\overrightarrow{K_2}$ in Proposition 1 can be replaced with the class of digraphs that admit homomorphisms onto $\overrightarrow{K_2}$.

Corollary 2. *Suppose A, B and C are digraphs, and there is a surjective homomorphism $C \to \overrightarrow{K_2}$. Then $A\times C \cong B\times C$ if and only if $B \cong A^\pi$ for some $\pi \in \mathrm{Perm}(V(A))$.*

Proof. Suppose $A\times C \cong B\times C$. Since C has at least one arc, there is a homomorphism $\overrightarrow{K_2} \to C$. Therefore Theorem 2 implies $A\times\overrightarrow{K_2} \cong B\times\overrightarrow{K_2}$. Proposition 1 now guarantees a permutation $\pi \in \mathrm{Perm}(V(A))$ for which $B \cong A^\pi$.

Conversely suppose $B \cong A^\pi$, so $A\times\overrightarrow{K_2} \cong B\times\overrightarrow{K_2}$ by Proposition 1. Since there is a homomorphism $C \to \overrightarrow{K_2}$, Theorem 2 implies $A\times C \cong B\times C$. ■

We can not expect to relax the conditions on C in Corollary 2. The reason is that the existence of the homomorphism $C \to \overrightarrow{K_2}$ means that C is a zero divisor. Without such a homomorphsm C might not be a zero divisor, and then we could only have $A\times C \cong A^\pi \times C$ in the event that $A \cong A^\pi$. Still, one direction of Corollary 2 can be generalized to an arbitrary C, as follows.

Corollary 3. *Suppose A, B and C are digraphs and C has at least one arc. If $A\times C \cong B\times C$, then $B \cong A^\pi$ for some $\pi \in \mathrm{Perm}(V(A))$.*

Proof. (First paragraph of the proof of Corollary 2.) ■

Taken together, Corollaries 2 and 3 tell us that the digraphs C which admit homomorphisms onto $\overrightarrow{K_2}$ are the most "egregious" of all zero divisors. Corollary 2 implies that if C is such a digraph, then for an arbitrary digraph A, there are potentially $|V(A)|!$ different digraphs B for which $A \times C \cong B \times C$. By contrast, Corollary 3 suggests that if C does not admit such a homomorphism, then there are perhaps fewer such digraphs B.

We now continue our investigation of equations $A \times C \cong B \times C$ where C is a zero divisor which admits a homomorphism onto $\overrightarrow{K_2}$. For a given digraph A, Corollary 2 completely describes the structure of all digraphs B for which $A \times C \cong B \times C$. However, it does not adequately enumerate them, for it is possible that $A^\pi \cong A^\mu$ for different permutations $\pi, \mu \in \mathrm{Perm}(V(A))$. In order to resolve such redundancy, we introduce a factorial operations on digraphs.

4. The Digraph Factorial

A version of the following definition was introduced in [2]. It is here modified slightly and adapted to digraphs.

Definition 2. Given a digraph A, the *factorial* of A is another digraph, denoted as $A!$, and defined as follows. The vertex set of $A!$ is $V(A!) = Perm(V(A))$. Given permutations $\alpha, \beta \in V(A!)$, there is an arc from α to β provided that $aa' \in E(A)$ if and only if $\alpha(a)\beta(a') \in E(A)$, for all $a, a' \in V(A)$. We denote an arc from α to β as $(\alpha)(\beta)$ to avoid confusion with composition.

We remark in passing that $A!$ is a subgraph of the digraph exponential A^A. (See Section 2.4 of [3].) Observe that the definition implies that there is a loop at a vertex $\alpha \in E(A!)$ if and only if α is an automorphism of A. In particular any $A!$ has a loop at the identity permutation id.

As our first example, consider the factorial of the graph K_n^*, the complete (symmetric) graph with a loop at each vertex. Here we have both $aa' \in E(A)$ and $\alpha(a)\beta(a') \in E(A)$ for all possible $a, a' \in V(K_n^*)$ and all possible permutations α and β. Consequently any two α, β form an arc of $K_n^*!$. Thus $K_n^*! \cong K_{n!}^*$, which factors as

$$K_n^*! \cong K_n^* \times K_{n-1}^* \times K_{n-2}^* \times \cdots \times K_3^* \times K_2^* \times K_1^*$$

and explains our choice of the word "factorial" for this operation.

Figure 5 shows further examples of digraph factorials. Each part (a), (b) and (c), shows a digraph A on vertex set $\{0, 1, 2\}$, with its factorial on the right.

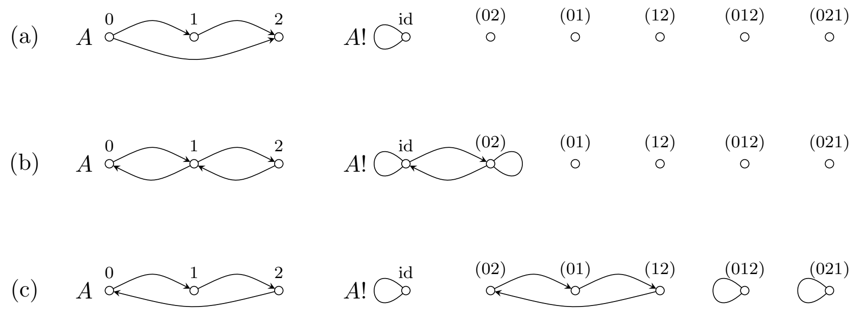

Figure 5. Examples of A and $A!$.

For a given graph A we define a relation $\sim$ on $V(A!)$ by declaring $\mu \sim \lambda$ if and only if there is an arc $(\alpha)(\beta) \in E(A!)$ for which $\mu = \alpha^{-1}\lambda\beta$. This is an equivalence relation as follows. It is reflexive since $\mu = \mathrm{id}^{-1}\mu\mathrm{id}$ and $(\mathrm{id})(\mathrm{id}) \in E(A!)$. It is symmetric as follows. Suppose $\mu \sim \lambda$ and take $(\alpha)(\beta) \in E(A!)$ with $\mu = \alpha^{-1}\lambda\beta$, so $\lambda = (\alpha^{-1})^{-1}\mu\beta^{-1}$. Then $\lambda \sim \mu$ provided $(\alpha^{-1})(\beta^{-1}) \in E(A!)$. But this is clear: From $(\alpha)(\beta) \in E(A!)$ we get $aa' \in E(A)$ if and only if $\alpha(a)\beta(a') \in E(A)$ for all pairs $a, a' \in V(A)$. Substituting a and a' with $\alpha^{-1}(a)$ and $\beta^{-1}(a')$, produces $\alpha^{-1}(a)\beta^{-1}(a') \in E(A)$ if and only if $aa' \in E(A)$ for all pairs $a, a' \in V(A)$, which means $(\alpha^{-1})(\beta^{-1}) \in E(A!)$. Finally we check transitivity. Suppose $\mu \sim \lambda$ and $\lambda \sim \pi$, so $A!$ has arcs $(\alpha)(\beta)$ and $(\gamma)(\delta)$ with $\mu = \alpha^{-1}\lambda\beta$ and $\lambda = \gamma^{-1}\pi\delta$. Thus $\mu = (\gamma\alpha)^{-1}\pi(\delta\beta)$. It is immediate that $(\gamma\alpha)(\delta\beta) \in E(A!)$, so $\lambda \sim \pi$.

Proposition 2. *Suppose A is a digraph and $\mu, \lambda \in \mathrm{Perm}(V(A))$. Then $A^\mu \cong A^\lambda$ if and only if $\mu \sim \lambda$.*

Proof. Suppose $\mu \sim \lambda$, so there is an arc $(\alpha)(\beta) \in E(A!)$ for which $\mu = \alpha^{-1}\lambda\beta$. The following chain of equivalences shows that $\alpha : A^\mu \to A^\lambda$ is an isomorphism.

$$\begin{aligned}
aa' \in E(A^\mu) &\iff a\,\mu^{-1}(a') \in E(A) && \text{(definition of } A^\mu) \\
&\iff \alpha(a)\beta\mu^{-1}(a') \in E(A) && \text{(since } (\alpha)(\beta) \in E(A!)) \\
&\iff \alpha(a)\lambda\beta\mu^{-1}(a') \in E(A^\lambda) && \text{(definition of } A^\lambda) \\
&\iff \alpha(a)\alpha\alpha^{-1}\lambda\beta\mu^{-1}(a') \in E(A^\lambda) && \\
&\iff \alpha(a)\alpha(a') \in E(A^\lambda). && (\alpha^{-1}\lambda\beta = \mu)
\end{aligned}$$

Conversely, suppose $\varphi : A^{\mu} \to A^{\lambda}$ is an isomorphism. Observe $\mu = (\varphi^{-1})\lambda(\lambda^{-1}\varphi\mu)$, so we will have $\mu \sim \lambda$ as soon as we can show $(\varphi)(\lambda^{-1}\varphi\mu) \in E(A!)$. For this, consider the following reasoning.

$$\begin{aligned} aa' \in E(A) &\iff a\,\mu(a') \in E(A^{\mu}) && \text{(definition of } A^{\mu}\text{)} \\ &\iff \varphi(a)\varphi\mu(a') \in E(A^{\lambda}) && (\varphi \text{ is isomorphism}) \\ &\iff \varphi(a)\lambda^{-1}\varphi\mu(a') \in E(A). && \text{(definition of } A^{\lambda}\text{)} \end{aligned}$$

From the definition of $A!$ it now follows that $(\varphi)(\lambda^{-1}\varphi\mu) \in E(A!)$. ■

Combining Proposition 2 and Corollary 3 produces the following theorem concerning zero divisors that admit homomorphisms onto $\overrightarrow{K_2}$. Given such a zero divisor C and a digraph A, it classifies all B for which $A \times C \cong B \times C$.

Theorem 4. *Suppose A and C are digraphs and there is a surjective homomorphism $C \to \overrightarrow{K_2}$. Let $\mu_1, \mu_2, \ldots, \mu_k \in V(A!)$ be representatives from the k equivalence classes of $\sim$. Then the digraphs B (up to isomorphism) for which $A \times C \cong B \times C$ are exactly $B = A^{\mu_1}, A^{\mu_2}, \ldots, A^{\mu_k}$.*

Let us look at several examples of this theorem. Consider the digraph A from Figure 5(a). Here $A!$ has only one arc (id)(id), so the equivalence class containing any permutation π consists only of the element $\mathrm{id}^{-1}\pi\,\mathrm{id} = \pi$. Thus there are six equivalence classes, each one containing a single permutation of $\{0, 1, 2\}$, and consequently six distinct digraphs $B = A^{\pi}$ for which $A \times C \cong B \times C$. These are listed in Figure 3.

Next consider A in Figure 5(b) The arcs of the factorial are (id)(id), (02)(id), (id)(02) and (02)(02), so it is not hard to work out the $\sim$ equivalence classes. The equivalence class containing id is

$$\{\mathrm{id}^{-1}\mathrm{id}\,\mathrm{id},\quad (02)^{-1}\mathrm{id}\,\mathrm{id},\quad \mathrm{id}^{-1}\mathrm{id}(02),\quad (02)^{-1}\mathrm{id}(02)\} = \{\mathrm{id}, (02)\},$$

and the class containing (01) is

$$\{\mathrm{id}^{-1}(01)\mathrm{id},\ (02)^{-1}(01)\mathrm{id},\ \mathrm{id}^{-1}(01)(02),\ (02)^{-1}(01)(02)\}$$

$$= \{(01), (012), (021), (12)\}.$$

These are the only equivalence classes. Taking representatives id and (01), Theorem 4 tells us that there are only two digraphs B for which $A \times C \cong B \times C$, and they are $B = A^{\mathrm{id}} = A$ and $B = A^{(01)}$, drawn in Figure 6.

Figure 6. Two graphs B guaranteed by Theorem 4.

Figure 7 shows $A^{\mathrm{id}} \times C$ and $A^{(01)} \times C$, for a particular graph C that admits a homomorphism $C \to \overrightarrow{K_2}$. It is clear that these products are both isomorphic to $A \times C$.

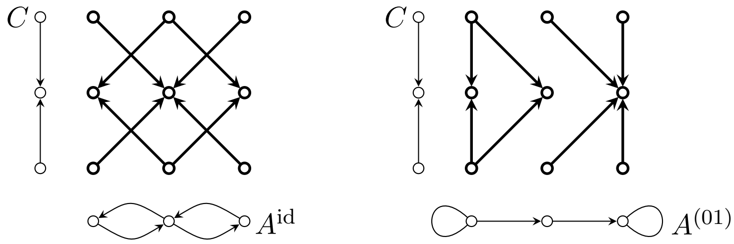

Figure 7. $A \times C \cong B \times C$.

5. Cancellation Digraphs

Let us call a digraph A a *cancellation digraph* if whenever $A \times C \cong B \times C$ for digraphs B and C (where C has at least on edge) it is necessarily true that $A \cong B$. For example, K_1^* is a cancellation digraph. The graph $A = A^{\mathrm{id}}$ from Figure 6 is not a cancellation digraph, for there is a different graph $B = A^{(01)}$ with $A \times C \cong B \times C$, as illustrated in Figure 7.

This section presents a characterization of cancellation digraphs.

Observe that if $A \times C \cong B \times C$, then Theorem 2 implies $A \times \overrightarrow{K_2} \cong B \times \overrightarrow{K_2}$. By Theorem 4, it must be the case that $B \cong A^\pi$ for some permutation π of $V(A)$. Further, it follows that A is a cancellation digraph if and only if $A^\pi \cong A$ for all permutations π, that is if and only if $\sim$ has only one equivalence class. This idea is repackaged in the next theorem.

Proposition 3. *Given a digraph A, let $\Phi : E(A!) \to V(A!)$ be defined as $\Phi((\alpha)(\beta)) = \alpha^{-1}\beta$. Then A is a cancellation digraph if and only if Φ is surjective.*

Proof. It suffices to prove that the image of Φ is $\{\pi \in V(A!) : \pi \sim \mathrm{id}\}$, for then Φ is surjective if and only if every $\pi \in V(A!)$ is equivalent to the identity, if and only if $A^\pi \cong A^{\mathrm{id}} = A$ for every $\pi \in V(A!)$.

Indeed, any element in the image of Φ is of form $\alpha^{-1}\beta$ for some arc $(\alpha)(\beta) \in E(A!)$. As $\alpha^{-1}\beta = \alpha^{-1}\mathrm{id}\beta$, we have $\alpha^{-1}\beta \sim \mathrm{id}$. Conversely, if $\mathrm{id} \sim \pi$ then $\pi = \alpha^{-1}\mathrm{id}\beta$ for some arc $(\alpha)(\beta) \in E(A!)$, so $\pi = \alpha^{-1}\beta$ is in the image of Φ. ∎

We now construct two families of cancellation digraphs. Given non-negative integers m and n, form a digraph V_m^n as follows. Begin with the disjoint union $K_n^* + \overline{K_m^*}$, where the bar represents complementation. (So $\overline{K_m^*}$ consists of m isolated vertices, without loops.) Finally establish arcs pointing from each vertex of K_n^* to every vertex of $\overline{K_m^*}$. The digraph on the left side of Figure 8 is V_3^2. We also construct a family of digraphs Λ_m^n by starting with $K_n^* + \overline{K_m^*}$ and establishing arcs pointing from each vertex of $\overline{K_m^*}$ to every vertex of K_n^*. The digraph on the right side of Figure 8 is Λ_3^2. Notice that $V_0^n = \Lambda_0^n = K_n^*$ and $V_m^0 = \Lambda_m^0 = \overline{K_m^*}$, but these are the only cases where $V_m^n \cong \Lambda_p^q$.

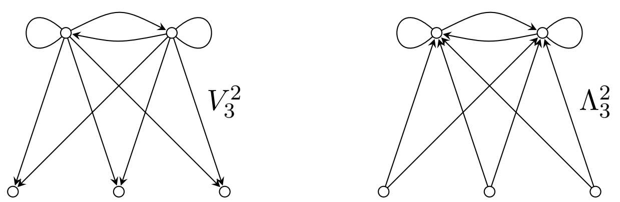

Figure 8. Examples of cancellation digraphs.

We can use Proposition 3 to show that V_m^n is a cancellation digraph. Our strategy is to show $(\mathrm{id})(\pi) \in E(V_m^n!)$ for every permutation $\pi \in V(V_m^n!)$, and then, as $\Phi((\mathrm{id})(\pi)) = \pi$, it follows that map Φ is surjective. Therefore we need to confirm that $aa' \in E(V_m^n) \iff \mathrm{id}(a)\pi(a') \in E(V_m^n)$ for each $a, a' \in V(V_m^n)$. Thus consider $a, a' \in V(V_m^n)$. Suppose $aa' \in E(V_m^n)$. By construction of V_m^n, it must be that $a \in V(K_n^*)$. But $a \in V(K_n^*)$ implies ab is an arc of V_m^n for any $b \in V(V_m^n)$, so $a\pi(a') = \mathrm{id}(a)\pi(a') \in E(V_m^n)$. Conversely, if $aa' \notin E(V_m^n)$ then it must be that $a \in V(\overline{K_m^*})$, and this means $ab \notin E(V_m^n)$ for any b, whence $\mathrm{id}(a)\pi(a') \notin E(V_m^n)$. It follows that $(\mathrm{id})(\pi) \in E(V_m^n!)$, so V_m^n is indeed a cancellation digraph.

Next we show that Λ_m^n is a cancellation digraph. Our strategy is the same as for V_m^n, except here we show $(\pi^{-1})(\mathrm{id}) \in E(\Lambda_m^n!)$ for any $\pi \in$

$V(\Lambda_m^n!)$. Thus consider $a, a' \in V(\Lambda_m^n)$. By construction of Λ_m^n we have $aa' \in E(\Lambda_m^n)$ implies $a' \in V(K_n^*)$. But $a' \in V(K_n^*)$ means ba' is an arc of Λ_m^n for any b, so $\pi^{-1}(a)\mathrm{id}(a') \in E(\Lambda_m^n)$. Conversely, if $aa' \notin E(\Lambda_m^n)$ then it must be that $a' \in V(\overline{K_m^*})$, and this means $ba' \notin E(\Lambda_m^n)$ for any b, whence $\pi^{-1}(a)\mathrm{id}(a') \notin E(\Lambda_m^n)$. It follows that $(\pi^{-1})(\mathrm{id}) \in E(\Lambda_m^n!)$, so Λ_m^n is a cancellation digraph.

Theorem 5. *A digraph is a cancellation digraph if and only if it is isomorphic to some V_m^n or Λ_m^n.*

Proof. We have already noted that V_m^n and Λ_m^n are cancellation digraphs. Now consider any cancellation digraph A. We will make two observations about its structure. Both observations rely on the following remark.

Remark. If $\tau \in V(A!)$ is the transposition that interchanges two vertices $a, a' \in V(A)$, then A and A^τ have the same number of loops at a and a'. (That is either A and A^τ each have loops at both a and a', or neither has a loop at a nor a', or each has exactly one loop at a or a'.) To see this, observe that if $b \in V(A) - \{a, a'\}$, then $bb \in E(A)$ if and only if $bb = b\tau(b) \in E(A^\tau)$, so A has a loop at such a b if and only if A^τ has a loop at b. But $A \cong A^\tau$ (because A is a cancellation digraph) so A and A^τ have the same number of loops. It follows that A and A^τ have the same number of loops at a and a'.

Now we make two observations concerning pairs of vertices $a, a' \in V(A)$.

1. Note $aa \in E(A)$ and $a'a' \in E(A)$ if and only if $aa' \in E(A)$ and $a'a \in E(A)$. To see this, first suppose A has loops at both a and a'. By the remark, $aa \in E(A^\tau)$ and $a'a' \in E(A^\tau)$, and this means $a\tau^{-1}(a) = aa' \in E(A)$ and $a'\tau^{-1}(a') = a'a \in E(A)$. Conversely, if $aa' \in E(A)$ and $a'a \in E(A)$ then $a\tau(a') = aa \in E(A^\tau)$ and $a'\tau(a) = a'a' \in E(A^\tau)$. The remark implies that A has loops at a and a'.
2. Note $aa \notin E(A)$ and $a'a' \notin E(A)$ if and only if $aa' \notin E(A)$ and $a'a \notin E(A)$. To see this, first suppose $aa \notin E(A)$ and $a'a' \notin E(A)$. By the remark, $aa \notin E(A^\tau)$ and $a'a' \notin E(A^\tau)$, and this means $a\tau^{-1}(a) = aa' \notin E(A)$ and $a'\tau^{-1}(a') = a'a \in E(A)$. Conversely, if $aa' \notin E(A)$ and $a'a \notin E(A)$ then $a\tau(a') = aa \notin E(A^\tau)$ and $a'\tau(a) = a'a' \notin E(A^\tau)$. The remark implies $aa \notin E(A)$ and $a'a' \notin E(A)$.

At this point we can see why $A \cong V_m^n$ or $A \cong \Lambda_m^n$. Let $L \subseteq V(A)$ be the vertices of A with loops and let $N \subseteq V(A)$ be the vertices without loops.

Put $n = |L|$ and $m = |N|$. By Observation 1, A has two arcs between any two vertices in L. By Observation 2, A has no arcs between any two vertices in N. Thus the subgraph $\langle L \rangle$ induced on L is K_n^*, and the subgraph $\langle N \rangle$ induced on N is $\overline{K_m^*}$.

Moreover Observations 1 and 2 taken together imply that there is exactly one arc between any two vertices $a \in L$ and $a' \in N$. We need to show that either all such arcs are directed from L to N (in which case $A \cong V_m^n$) or all are directed from N to L (in which case $A \cong \Lambda_m^n$). Suppose to the contrary that this is not the case. Then A must have one of the following induced subgraphs, where the upper vertices are in L and the lower are in N.

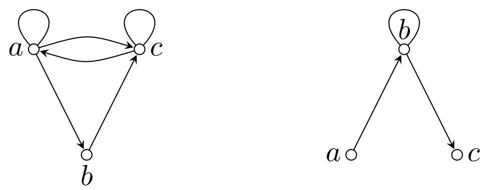

Figure 9. Forbidden configurations in a cancellation digraph.

In either case consider the cyclic permutation $\pi = (abc)$, and note that A^π has fewer loops on the vertices $\{a, b, c\}$ than does A. Since π does not alter vertices in $V(A) - \{a, b, c\}$, it follows that A has a loop at $x \in V(A) - \{a, b, c\}$ if and only if A^π has a loop at x. Thus A has more loops than A^π, so $A \ncong A^\pi$, contradicting the fact that A is a cancellation digraph. ■

6. Conclusion

In summary, our main results are as follows, where A is a fixed digraph and C is a digraph that has at least one edge.

1. **(Corollary 2)** If there is a surjective homomorphism $C \to \overrightarrow{K_2}$, then $A \times C \cong B \times C \iff B \cong A^\pi$ for some $\pi \in \mathrm{Perm}(V(A))$.
2. **(Corollary 3)** If C has at least one edge, then $A \times C \cong B \times C \implies B \cong A^\pi$ for some $\pi \in \mathrm{Perm}(V(A))$.
3. **(Proposition 2)** If there is a surjective homomorphism $C \to \overrightarrow{K_2}$, then the graphs B for which $A \times C \cong B \times C$ are precisely $B = A^{\mu_i}$, where $\mu_1, \mu_2, \ldots, \mu_k$ are representatives from the distinct $\sim$ equivalences classes of $V(A!)$.

4. **(Theorem 5)** A digraph A satisfies $A \times C \cong B \times C \Longrightarrow A \cong B$ for all C if and only if $A \cong \Lambda_m^n$ or $A \cong V_m^n$ for some m, n.

In conclusion we mention two areas that merit future study. Our first remark concerns Item 2, above. In the situations where C is a zero divisor but there is no homomorphism $C \to \overrightarrow{K_2}$, it remains to spell out precisely the permutations π for which $A \times C \cong B \times C \Longleftrightarrow B \cong A^\pi$. This will be the subject of a future paper.

The second remark concerns a more satisfactory adaptation of these ideas to graphs. In general, if A is a graph (symmetric digraph), then A^π need not be symmetric. In fact A^π will be a graph if and only if π satisfies $aa' \in E(A) \Longleftrightarrow \pi(a)\pi^{-1}(a') \in E(A)$ for all pairs $a, a' \in V(A)$. Such a π is called an *antiautomorphism* in [2], where analogues of the present Propositions 1 and 2, Corollaries 2 and 3, and Theorem 4 are derived for graphs. However, it is not known if there is a version of Theorem 5 for graphs.

Indeed the class of "cancellation graphs" appears to be far richer than the cancellation digraphs described by Theorem 5. The reason is that a graph C is a zero divisor only if there is a homomorphism $C \to \overrightarrow{C_2}$ (Corollary 1). Thus, roughly speaking, there are fewer zero divisors in the class of graphs than in the class of digraphs. Consequently we expect the implication $A \times C \cong B \times C \Longrightarrow A \cong B$ is less likely to fail in the class of graphs than in the class of digraphs, so there should be a wider variety of cancellation graphs than cancellation digraphs. Indeed, this is borne out in [1], which proves that a bipartite graph is a cancellation graph if and only if it has no automorphism that reverses the bipartition of one of its components. (In this sense, "most" bipartite graphs are cancellation graphs.) It would be interesting to find a similar characterization for nonbipartite cancellation graphs.

References

[1] R. Hammack, *A cancellation property for the direct product of graphs*, Discuss. Math. Graph Theory **28** (2008) 179–185.

[2] R. Hammack, *On direct product cancellation of graphs*, Discrete Math. **309** (2009) 2538–2543.

[3] P. Hell and J. Nešetřil, Graphs and Homomorphisms, Oxford Lecture Series in Mathematics (Oxford U. Press, 2004).

[4] W. Imrich and S. Klavžar, Product Graphs: Structure and recognition, Wiley-Interscience Series in Discrete Mathematics and Optimization (John Wiley and Sons, New York, 2000).

[5] L. Lovász, *On the cancellation law among finite relational structures*, Period. Math. Hungar. **1** (1971) 145–156.

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