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RAINBOW NUMBERS FOR SMALL STARS WITH ONE EDGE ADDED

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Abstract

A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. For a graph H and a positive integer n, the *anti-Ramsey number* $f(n,H)$ is the maximum number of colors in an edge-coloring of K_n with no rainbow copy of H. The *rainbow number* $rb(n,H)$ is the minimum number of colors such that any edge-coloring of K_n with $rb(n,H)$ number of colors contains a rainbow copy of H. Certainly $rb(n,H) = f(n,H) + 1$. Anti-Ramsey numbers were introduced by Erdös *et al.* [5] and studied in numerous papers.

We show that $rb(n, K_{1,4} + e) = n + 2$ in all nontrivial cases.

Keywords: rainbow number, anti-Ramsey number.

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1. Introduction

A subgraph of an edge-colored graph is *rainbow* if all of its edges have different colors. For a graph H and a positive integer n, the *anti-Ramsey number* $f(n,H)$ is the maximum number of colors in an edge-coloring of K_n with no rainbow copy of H. The *rainbow number* $rb(n,H)$ is the minimum number of colors such that any edge-coloring of K_n with $rb(n,H)$ number of colors contains a rainbow copy of H. Certainly $rb(n,H) = f(n,H) + 1$. Anti-Ramsey numbers were introduced by Erdös *et al.* [5]. They showed that these are closely related to Turán numbers. The *Turán number* $ex(n,\mathcal{H})$ of

$\mathcal{H}$ is the maximum number of edges of an n-vertex simple graph having no member of $\mathcal{H}$ as a subgraph.

For a given graph H let $\mathcal{H} = \{H - e : e \in E(H)\}$. Erdös *et al.* [5] showed $f(n, H) - ex(n, \mathcal{H}) = o(n^2)$, as $n \to \infty$. If $d = \min\{\chi(G) : G \in \mathcal{H}\} \geq 3$, then by an earlier result of Erdös and Simonovits [4], we have $ex(n, \mathcal{H}) = \frac{d-2}{d-1}\binom{n}{2} + o(n^2)$. So the above theorem determines rainbow numbers asymptotically in that case. If $d \leq 2$, however, we have $ex(n, \mathcal{H}) = o(n^2)$ and the above theorem says little about rainbow numbers. Therefore it was suggested by Erdös *et al.* [5] to study Ramsey numbers for graphs that contain an edge whose delection leaves a bipartite subgraph and put forward two conjectures about paths and cycles.

Simonovits and Sós proved the conjecture for paths determining $f(n, P_k)$ for n large enough [15]. As for the conjecture for cycles, they proved it for C_3 by themselves. For C_4 it was proved by Alon [1] and for C_5 and C_6 independently by Jiang and West [9] and by Schiermeyer [13] and completely solved by Montellano–Ballesteros and Neuman-Lara [12].

Moreover rainbow numbers were studied for complete bipartite graphs by Axenovich and Jiang [2], for trees by Jiang and West [10], for subdivided graphs by Jiang [7] and for complete graphs and matchings by Schiermeyer [14]. Recently cycles with an edge added were studied by Montellano-Ballesteros [11] and Gorgol [6].

The aim of the paper is to prove Theorem 4 which says that we need $n + 2$ colors to be sure that in any coloring of the edges of K_n with this number of colors we always obtain a rainbow $K_{1,4} + e$.

2. Preliminaries

Graphs considered below will always be simple. Throughout the paper we use the standard graph theory notation (see, e.g., [3]). In particular, $G \cup H$, K_n and $K_{1,r}$ stand, respectively, for disjoint sum of graphs G and H, the complete graph on n vertices and a star with r rays. A graph K_3 we call a triangle. For a graph G and its subgraph H by $G - H$ we mean a graph obtained from G by deleting all vertices of H. For a set S by $|S|$ we denote the cardinality of S.

We will need the following theorems.

Theorem 1 [5]. *$rb(n, K_3) = n$ for $n \geq 3$.*

Theorem 2 [8]. *Given positive integers n and r, where $r \leq n-2$, $rb(n, K_{1,r+1}) = \lfloor \frac{1}{2}n(r-1) \rfloor + \lfloor \frac{n}{n-r+1} \rfloor + \varepsilon$, where $\varepsilon = 1$ or 2 if n is odd, r is even and $\lfloor \frac{2n}{n-r+1} \rfloor$ is odd; $\varepsilon = 1$ otherwise.*

We will also need some lemmas to conduct the inductive proof of Theorem 4. Throughout all proofs we use the following notation. $C(G)$ is a set of colors used on the edges of a graph G; $C(G, H)$ is a set of colors used on the edges with one endvertex in the vertex-set of a graph G and the other in the vertex-set of a graph H; $C(v)$ is a set of colors used on the edges incident to a vertex v and $c(e)$ denotes the color of the edge e.

In a graph $K_{1,4} + e$ a vertex of degree 4 we call a center.

A vertex v is called *monochromatic* if $|C(v)| = 1$.

We call a color c *private* for a vertex v if all edges of color c are incident to v.

Claim 1. If a color c is private for two vertices v and w then an edge vw is the only edge of color c.

Proof. It follows immediately from the definition of a private vertex. □

Claim 2. An arbitrary color can be private for at most two vertices.

Proof. It is a straightforward consequence of Claim 1. □

For a fixed coloring of the edges of $K_n = K$ we construct a bipartite graph B with bipartition sets V and C as follows. Let $V = V(K)$ and $C = C(K)$. We put an edge between $v \in V$ and $c \in C$ if and only if c is private for v. Note that by Claim 2 each vertex from C has degree at most 2.

Note that by Claim 1, B cannot contain C_4.

Lemma 1. $rb(n, K_{1,4} + e) = n + 2$ *for* $n \in \{5, 6\}$.

Proof. The lower bound follows from Theorem 2. So we have to prove the upper bound. We color the edges of $K_n = K$ with $n + 2$ colors arbitrarily and show that there exists a rainbow $K_{1,4} + e$. By Theorem 2 there exists a rainbow $K_{1,4} = S$. The existence of a rainbow $K_{1,4} + e$ is obvious for $n \in \{5, 6\}$. ■

3. Main Results

Although the next theorem is proved in a more general case in [6], we state it here with a proof to make the paper self-contained.

Theorem 3. *$rb(n, K_{1,3}+e) = n$ for $n \geq 4$.*

Proof. By Theorem 1 we have $rb(n, K_{1,3}+e) \geq n$ for $n \geq 4$. So we have to prove the opposite inequality. It is easy to check it for $n \in \{4, 5, 6\}$. Therefore let $n \geq 7$. We color the edges of $K_n = K$ with n colors. By Theorem 1 there exists a rainbow triangle T with the set of colors $C(T)$. If the condition $C(T, K_n - T) \cap (C(K) - C(T)) \neq \emptyset$ holds then there exists a rainbow $K_{1,3}+e$. Otherwise $|V(K_n - T)| = n-3$ and $|C(K_n - T)| \geq n-3$ and we have a rainbow copy of $K_{1,3}+e$ in $K_n - T$ by induction. ■

Theorem 4. *$rb(n, K_{1,4}+e) = n+2$ for $n \geq 5$.*

Proof. The lower bound follows from Theorem 2. So we have to prove the upper bound. The proof will be conducted by induction with respect to n. For $n \in \{5, 6\}$ it is Lemma 1.

Therefore let $n \geq 7$. We color the edges of $K_n = K$ with $n+2$ colors arbitrarily and construct an appropriate bipartite graph B.

If there exists a vertex v such that $|C(K_n) - C(v)| \geq n+1$ then $K_n - v$ is a K_{n-1} colored with at least $n+1$ colors, so a rainbow $K_{1,4}+e$ exists by induction. A contradiction. Therefore for each vertex $v \in V(K_n)$ there exist at least two private colors. So each vertex from V has degree at least 2.

Before the next part of the proof we will show the following two facts.

Fact 1. If there is an isolated vertex $c_0 \in C$ in B then K contains a rainbow $K_{1,4}+e$.

***Proof of Fact* 1.** Assume there is an isolated vertex $c_0 \in C$ in B, but there is not any rainbow $K_{1,4}+e$ in K.

By Claim 2 each vertex from C has degree at most 2. By Claim 1 it means that in coloring of K the respective colors appear exactly once. Hence at least $n-1$ vertices from C have degree exactly 2.

For each $v \in V$ choose exactly two private colors at v, and consider a subgraph B' of B with $V(B') = V(B) - \{c_0\}$, but with an edge between $v \in V$ and $c \in C$ if c is one of the two chosen private colors at v. Thus

$E(B') \subset E(B)$, $|E(B')| = 2n \leq |E(B)| \leq 2n+2$ and the maximal degree in B' is 2.

The above degree conditions determine the structure of B'. Namely B' consists of a path with an odd number ≥ 1 of vertices, starting and ending in C, and of zero or more cycle components. Without loss of generality this path could be assumed to be $c_r v_r c_{r+1} \ldots v_n c_{n+1}$, where $1 \leq r \leq n+1$ ($r = n+1$ if and only if c_{n+1} is an isolated vertex in B').

Note that the graph on $V(K)$ consisting only of the edges colored in the three or two colors c_0, c_r, c_{n+1} is connected. Thus there is some vertex v_i with at least two or three colors occurring among its edges: it is easy to see that we may choose $v_i \notin \{v_r, v_n\}$, if $r \leq n$. Hence, in fact, either v_i belongs to a 6-cycle $v_{i-1}c_i v_i c_{i+1} v_{i+1} c'$ or to a path $v_{i-1}c_{i-1}v_i c_i v_{i+1} c_{i+1} v_{i+2}$ in B'. In either case, $|C(v_i)| \geq 4$. In the case of C_6, v_i would be a center of a rainbow $K_{1,4}+e$, where e would be colored with c'. Similarly in the case of the path, v_i would be a center of a rainbow $K_{1,4}+e$, where $e = v_i v_{i+2}$. Indeed, let $c(v_i v_{i+2}) = c''$ be the color of the edge $v_i v_{i+2}$. It can be c_0 or c_{n+1} (if $i+2 = n$) and by the choice of v_i there is another edge coming out of it of color from $\{c_0, c_r, c_{n+1}\} \setminus \{c''\}$. $\square$

Fact 2. If K contains two disjoint rainbow triangles, then K contains a rainbow $K_{1,4}+e$.

***Proof of Fact* 2.** Let T_1 and T_2 be these triangles. Note that if there is $c \in C(T_1) \cap C(T_2)$ then indeed c is not private at any vertex in K and thus is isolated in B, whence then Fact 1 applies. Therefore we have to consider the case when $T_1 \cup T_2$ is a rainbow $2K_3$.

Let $V(T_1) = \{x, y, z\}$, $V(T_2) = \{a, b, c\}$, $C(T_1) = \{c_1, c_2, c_3\}$ and $C(T_2) = \{c_4, c_5, c_6\}$.

If $|C(T_1) \cup C(T_2) \cup C(T_1, T_2)| \geq 8$ then Fact 2 follows from Lemma 1 so we can assume that $|C(T_1) \cup C(T_2) \cup C(T_1, T_2)| \leq 7$ which means that there can be at most one color in $C(T_1, T_2)$ not belonging to $C(T_1) \cup C(T_2)$. Let $K' = K - (T_1 \cup T_2)$.

Suppose that there exists an edge e between the triangles T_1 and T_2 of the color not belonging to $C(T_1) \cup C(T_2)$. Without loss of generality we can assume that $e = xa$ and $c(e) = c$. Let $C_R = C(K) - (C(T_1) \cup C(T_2) \cup \{c\})$. Note that either we have a rainbow $K_{1,4}+e$ or $c(xv) \in C(T_1) \cup \{c\}$ for all vertices $v \in V(K - T_1)$ and $c(av) \in C(T_2) \cup \{c\}$ for all vertices $v \in V(K - T_2)$. If there is at least one edge between $T_1 \cup T_2$ and K' of color from C_R, say yw, where $w \in V(K')$, then we obtain a rainbow $K_{1,4}+e$. It is the triangle

T_1 with edges ya and yw. Note that surely it is the case for $n = 9$ since $|C_R| = 4$ and $|E(K')| = 3$. If such an edge does not exist it means that $n \geq 10$, $C(T_1 \cup T_2, K') \subset (C(T_1) \cup C(T_2) \cup \{c\})$ and all colors from C_R are used on edges of K'. If there is a rainbow $K_{1,3} + e$ in K' then it gives a rainbow $K_{1,4} + e$ together with one edge coming to $T_1 \cup T_2$. Note that obviously it is the case for $n = 10$ and for $n \geq 11$ and $C(K') = C_R$ it follows from Theorem 3. If $|C(K')| > |C_R|$ for $n \geq 11$ we obtain a rainbow $K_{1,4} + e$ by induction.

Therefore we assume that $C(T_1, T_2) \subset C(T_1) \cup C(T_2)$. Let $C_R = C - (C(T_1) \cup C(T_2))$. If there is at least one edge between $T_1 \cup T_2$ and K' of color from C_R, say xw, where $w \in V(K')$, then all edges coming from x to T_2 are of colors from $C(T_1)$ otherwise we get a rainbow $K_{1,4} + e$. As a further consequence we get that either there is a rainbow $K_{1,4} + e$ in K or all edges coming out from T_2 are of colors from $C(T_1) \cup C(T_2)$. In the latter case the graph $K - T_2$ is colored with at least $n - 1$ colors so the induction completes the proof. Note that surely it is the case for $n \in \{7, 8, 9\}$. So we can assume that $n \geq 10$, $C(T_1 \cup T_2, K') \subset (C(T_1) \cup C(T_2))$ and all colors from C_R are used on edges of K'. Repeating the arguments from the previous part of the proof we prove the fact. □

Now we are ready to finish the proof of Theorem 4.

By Theorem 1 there exists a rainbow triangle T_1 with the vertex-set $\{x, y, z\}$ and the set of colors $C(T_1)$.

Let $K' = K - T_1$. Note that if K' contains a rainbow triangle then K contains a rainbow $K_{1,4} + e$ by Fact 2. Assume then it is not the case. Then $|C(K')| \leq n - 4$ by Theorem 1.

Let $C_R = C(K) - C(T_1)$. Note that if there is a vertex v in T_1 with $|C(v) \cap C_R| \geq 2$, then there is a rainbow $K_{1,4} + e$ with center v and containing T_1. Hence the converse can be assumed.

So we are to consider only the case $|C(K')| = n - 4$, $|C(v) \cap C_R| = 1$ and the colors $C(v) \cap C_R$ are distinct for each $v \in \{x, y, z\}$.

Then certainly $|C(T_1, K') \cap C_R| = 3$ and $C(K') \cap (C(T_1) \cup C(T_1, K')) = \emptyset$. Now either we have a rainbow $K_{1,4} + e$ or each edge between T_1 and K' of the color from C_R comes out from a different vertex of T_1. If such an edge of color c comes out, say from x, to a vertex a which is not monochromatic in K' then $c(ay) = c(xy)$ and $c(za) = c(xz)$ or we have a rainbow $K_{1,4} + e$. But in this case we also get a rainbow $K_{1,4} + e$. It is a rainbow triangle ayz with two edges coming out from the vertex a.

It is easy to note that there can be at most one monochromatic vertex in K'. If there would be at least two such vertices a and b then $C(a) \cap C(K') = C(b) \cap C(K') = \{c(ab)\}$ and so $K' - \{a, b\}$ would be K_{n-5} colored with $n-5$ colors, against the assumption for K'.

Hence the vertex a is monochromatic in K' and all edges between T_1 and K' of the three colors in $C(T_1, K') \cap C_R$ have a as an endpoint. Thus $|C(a)| = 4$ and there is a rainbow $K_{1,4} + e$ with the center a and containing an edge from T_1. ■

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