# RAINBOW NUMBERS FOR SMALL STARS WITH ONE EDGE ADDED 

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#### Abstract

A subgraph of an edge-colored graph is rainbow if all of its edges have different colors. For a graph $H$ and a positive integer $n$, the anti-Ramsey number $f(n, H)$ is the maximum number of colors in an edge-coloring of $K_{n}$ with no rainbow copy of $H$. The rainbow number $r b(n, H)$ is the minimum number of colors such that any edge-coloring of $K_{n}$ with $r b(n, H)$ number of colors contains a rainbow copy of $H$. Certainly $r b(n, H)=f(n, H)+1$. Anti-Ramsey numbers were introduced by Erdös et al. [5] and studied in numerous papers.


We show that $r b\left(n, K_{1,4}+e\right)=n+2$ in all nontrivial cases.
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## 1. Introduction

A subgraph of an edge-colored graph is rainbow if all of its edges have different colors. For a graph $H$ and a positive integer $n$, the anti-Ramsey number $f(n, H)$ is the maximum number of colors in an edge-coloring of $K_{n}$ with no rainbow copy of $H$. The rainbow number $r b(n, H)$ is the minimum number of colors such that any edge-coloring of $K_{n}$ with $r b(n, H)$ number of colors contains a rainbow copy of $H$. Certainly $r b(n, H)=f(n, H)+1$. AntiRamsey numbers were introduced by Erdös et al. [5]. They showed that these are closely related to Turán numbers. The Turán number ex $(n, \mathcal{H})$ of
$\mathcal{H}$ is the maximum number of edges of an $n$-vertex simple graph having no member of $\mathcal{H}$ as a subgraph.

For a given graph $H$ let $\mathcal{H}=\{H-e: e \in E(H)\}$. Erdös et al. [5] showed $f(n, H)-e x(n, \mathcal{H})=o\left(n^{2}\right)$, as $n \rightarrow \infty$. If $d=\min \{\chi(G): G \in \mathcal{H}\} \geq 3$, then by an earlier result of Erdös and Simonovits [4], we have $e x(n, \mathcal{H})=\frac{d-2}{d-1}\binom{n}{2}+$ $o\left(n^{2}\right)$. So the above theorem determines rainbow numbers asymptotically in that case. If $d \leq 2$, however, we have $e x(n, \mathcal{H})=o\left(n^{2}\right)$ and the above theorem says little about rainbow numbers. Therefore it was suggested by Erdös et al. [5] to study Ramsey numbers for graphs that contain an edge whose delection leaves a bipartite subgraph and put forward two conjectures about paths and cycles.

Simonovits and Sós proved the conjecture for paths determining $f\left(n, P_{k}\right)$ for $n$ large enough [15]. As for the conjecture for cycles, they proved it for $C_{3}$ by themselves. For $C_{4}$ it was proved by Alon [1] and for $C_{5}$ and $C_{6}$ independently by Jiang and West [9] and by Schiermeyer [13] and completely solved by Montellano-Ballesteros and Neuman-Lara [12].

Moreover rainbow numbers were studied for complete bipartite graphs by Axenovich and Jiang [2], for trees by Jiang and West [10], for subdivided graphs by Jiang $[7]$ and for complete graphs and matchings by Schiermeyer [14]. Recently cycles with an edge added were studied by MontellanoBallesteros [11] and Gorgol [6].

The aim of the paper is to prove Theorem 4 which says that we need $n+2$ colors to be sure that in any coloring of the edges of $K_{n}$ with this number of colors we always obtain a rainbow $K_{1,4}+e$.

## 2. Preliminaries

Graphs considered below will always be simple. Throughout the paper we use the standard graph theory notation (see, e.g., [3]). In particular, $G \cup H$, $K_{n}$ and $K_{1, r}$ stand, respectively, for disjoint sum of graphs $G$ and $H$, the complete graph on $n$ vertices and a star with $r$ rays. A graph $K_{3}$ we call a triangle. For a graph $G$ and its subgraph $H$ by $G-H$ we mean a graph obtained from $G$ by deleting all vertices of $H$. For a set $S$ by $|S|$ we denote the cardinality of $S$.

We will need the following theorems.
Theorem 1 [5]. $r b\left(n, K_{3}\right)=n$ for $n \geq 3$.

Theorem 2 [8]. Given positive integers $n$ and $r$, where $r \leq n-2$, $r b\left(n, K_{1, r+1}\right)=\left\lfloor\frac{1}{2} n(r-1)\right\rfloor+\left\lfloor\frac{n}{n-r+1}\right\rfloor+\varepsilon$, where $\varepsilon=1$ or 2 if $n$ is odd, $r$ is even and $\left\lfloor\frac{2 n}{n-r+1}\right\rfloor$ is odd; $\varepsilon=1$ otherwise.

We will also need some lemmas to conduct the inductive proof of Theorem 4. Throughout all proofs we use the following notation. $C(G)$ is a set of colors used on the edges of a graph $G ; C(G, H)$ is a set of colors used on the edges with one endvertex in the vertex-set of a graph $G$ and the other in the vertex-set of a graph $H ; C(v)$ is a set of colors used on the edges incident to a vertex $v$ and $c(e)$ denotes the color of the edge $e$.

In a graph $K_{1,4}+e$ a vertex of degree 4 we call a center.
A vertex $v$ is called monochromatic if $|C(v)|=1$.
We call a color $c$ private for a vertex $v$ if all edges of color $c$ are incident to $v$.

Claim 1. If a color $c$ is private for two vertices $v$ and $w$ then an edge $v w$ is the only edge of color $c$.

Proof. It follows immediately from the definition of a private vertex.

Claim 2. An arbitrary color can be private for at most two vertices.
Proof. It is a straightforward consequence of Claim 1.
For a fixed coloring of the edges of $K_{n}=K$ we construct a bipartite graph $B$ with bipartition sets $V$ and $C$ as follows. Let $V=V(K)$ and $C=C(K)$. We put an edge between $v \in V$ and $c \in C$ if and only if $c$ is private for $v$. Note that by Claim 2 each vertex from $C$ has degree at most 2 .

Note that by Claim 1, $B$ cannot contain $C_{4}$.
Lemma 1. $r b\left(n, K_{1,4}+e\right)=n+2$ for $n \in\{5,6\}$.
Proof. The lower bound follows from Theorem 2. So we have to prove the upper bound. We color the edges of $K_{n}=K$ with $n+2$ colors arbitrarily and show that there exists a rainbow $K_{1,4}+e$. By Theorem 2 there exists a rainbow $K_{1,4}=S$. The existence of a rainbow $K_{1,4}+e$ is obvious for $n \in\{5,6\}$.

## 3. Main Results

Although the next theorem is proved in a more general case in [6], we state it here with a proof to make the paper self-contained.

Theorem 3. $r b\left(n, K_{1,3}+e\right)=n$ for $n \geq 4$.
Proof. By Theorem 1 we have $r b\left(n, K_{1,3}+e\right) \geq n$ for $n \geq 4$. So we have to prove the opposite inequality. It is easy to check it for $n \in\{4,5,6\}$. Therefore let $n \geq 7$. We color the edges of $K_{n}=K$ with $n$ colors. By Theorem 1 there exists a rainbow triangle $T$ with the set of colors $C(T)$. If the condition $C\left(T, K_{n}-T\right) \cap(C(K)-C(T)) \neq \emptyset$ holds then there exists a rainbow $K_{1,3}+e$. Otherwise $\left|V\left(K_{n}-T\right)\right|=n-3$ and $\left|C\left(K_{n}-T\right)\right| \geq n-3$ and we have a rainbow copy of $K_{1,3}+e$ in $K_{n}-T$ by induction.

Theorem 4. $\operatorname{rb}\left(n, K_{1,4}+e\right)=n+2$ for $n \geq 5$.
Proof. The lower bound follows from Theorem 2. So we have to prove the upper bound. The proof will be conducted by induction with respect to $n$. For $n \in\{5,6\}$ it is Lemma 1 .

Therefore let $n \geq 7$. We color the edges of $K_{n}=K$ with $n+2$ colors arbitrarily and construct an appropriate bipartite graph $B$.

If there exists a vertex $v$ such that $\left|C\left(K_{n}\right)-C(v)\right| \geq n+1$ then $K_{n}-v$ is a $K_{n-1}$ colored with at least $n+1$ colors, so a rainbow $K_{1,4}+e$ exists by induction. A contradiction. Therefore for each vertex $v \in V\left(K_{n}\right)$ there exist at least two private colors. So each vertex from $V$ has degree at least 2 .

Before the next part of the proof we will show the following two facts.
Fact 1. If there is an isolated vertex $c_{0} \in C$ in $B$ then $K$ contains a rainbow $K_{1,4}+e$.

Proof of Fact 1. Assume there is an isolated vertex $c_{0} \in C$ in $B$, but there is not any rainbow $K_{1,4}+e$ in $K$.

By Claim 2 each vertex from $C$ has degree at most 2. By Claim 1 it means that in coloring of $K$ the respective colors appear exactly once. Hence at least $n-1$ vertices from $C$ have degree exactly 2 .

For each $v \in V$ choose exactly two private colors at $v$, and consider a subgraph $B^{\prime}$ of $B$ with $V\left(B^{\prime}\right)=V(B)-\left\{c_{0}\right\}$, but with an edge between $v \in V$ and $c \in C$ if $c$ is one of the two chosen private colors at $v$. Thus
$E\left(B^{\prime}\right) \subset E(B),\left|E\left(B^{\prime}\right)\right|=2 n \leq|E(B)| \leq 2 n+2$ and the maximal degree in $B^{\prime}$ is 2 .

The above degree conditions determine the structure of $B^{\prime}$. Namely $B^{\prime}$ consists of a path with an odd number $\geq 1$ of vertices, starting and ending in $C$, and of zero or more cycle components. Without loss of generality this path could be assumed to be $c_{r} v_{r} c_{r+1} \ldots v_{n} c_{n+1}$, where $1 \leq r \leq n+1$ ( $r=n+1$ if and only if $c_{n+1}$ is an isolated vertex in $B^{\prime}$ ).

Note that the graph on $V(K)$ consisting only of the edges colored in the three or two colors $c_{0}, c_{r}, c_{n+1}$ is connected. Thus there is some vertex $v_{i}$ with at least two or three colors occurring among its edges: it is easy to see that we may choose $v_{i} \notin\left\{v_{r}, v_{n}\right\}$, if $r \leq n$. Hence, in fact, either $v_{i}$ belongs to a 6 -cycle $v_{i-1} c_{i} v_{i} c_{i+1} v_{i+1} c^{\prime}$ or to a path $v_{i-1} c_{i-1} v_{i} c_{i} v_{i+1} c_{i+1} v_{i+2}$ in $B^{\prime}$. In either case, $\left|C\left(v_{i}\right)\right| \geq 4$. In the case of $C_{6}, v_{i}$ would be a center of a rainbow $K_{1,4}+e$, where $e$ would be colored with $c^{\prime}$. Similarly in the case of the path, $v_{i}$ would be a center of a rainbow $K_{1,4}+e$, where $e=v_{i} v_{i+2}$. Indeed, let $c\left(v_{i} v_{i+2}\right)=c^{\prime \prime}$ be the color of the edge $v_{i} v_{i+2}$. It can be $c_{0}$ or $c_{n+1}$ (if $i+2=n$ ) and by the choice of $v_{i}$ there is another edge coming out of it of color from $\left\{c_{0}, c_{r}, c_{n+1}\right\} \backslash\left\{c^{\prime \prime}\right\}$.

Fact 2. If $K$ contains two disjoint rainbow triangles, then $K$ contains a rainbow $K_{1,4}+e$.

Proof of Fact 2. Let $T_{1}$ and $T_{2}$ be these triangles. Note that if there is $c \in C\left(T_{1}\right) \cap C\left(T_{2}\right)$ then indeed $c$ is not private at any vertex in $K$ and thus is isolated in $B$, whence then Fact 1 applies. Therefore we have to consider the case when $T_{1} \cup T_{2}$ is a rainbow $2 K_{3}$.

Let $V\left(T_{1}\right)=\{x, y, z\}, V\left(T_{2}\right)=\{a, b, c\}, C\left(T_{1}\right)=\left\{c_{1}, c_{2}, c_{3}\right\}$ and $C\left(T_{2}\right)=\left\{c_{4}, c_{5}, c_{6}\right\}$.

If $\left|C\left(T_{1}\right) \cup C\left(T_{2}\right) \cup C\left(T_{1}, T_{2}\right)\right| \geq 8$ then Fact 2 follows from Lemma 1 so we can assume that $\left|C\left(T_{1}\right) \cup C\left(T_{2}\right) \cup C\left(T_{1}, T_{2}\right)\right| \leq 7$ which means that there can be at most one color in $C\left(T_{1}, T_{2}\right)$ not belonging to $C\left(T_{1}\right) \cup C\left(T_{2}\right)$. Let $K^{\prime}=K-\left(T_{1} \cup T_{2}\right)$.

Suppose that there exists an edge $e$ between the triangles $T_{1}$ and $T_{2}$ of the color not belonging to $C\left(T_{1}\right) \cup C\left(T_{2}\right)$. Without loss of generality we can assume that $e=x a$ and $c(e)=c$. Let $C_{R}=C(K)-\left(C\left(T_{1}\right) \cup C\left(T_{2}\right) \cup\{c\}\right)$. Note that either we have a rainbow $K_{1,4}+e$ or $c(x v) \in C\left(T_{1}\right) \cup\{c\}$ for all vertices $v \in V\left(K-T_{1}\right)$ and $c(a v) \in C\left(T_{2}\right) \cup\{c\}$ for all vertices $v \in V\left(K-T_{2}\right)$. If there is at least one edge between $T_{1} \cup T_{2}$ and $K^{\prime}$ of color from $C_{R}$, say $y w$, where $w \in V\left(K^{\prime}\right)$, then we obtain a rainbow $K_{1,4}+e$. It is the triangle
$T_{1}$ with edges $y a$ and $y w$. Note that surely it is the case for $n=9$ since $\left|C_{R}\right|=4$ and $\left|E\left(K^{\prime}\right)\right|=3$. If such an edge does not exist it means that $n \geq 10, C\left(T_{1} \cup T_{2}, K^{\prime}\right) \subset\left(C\left(T_{1}\right) \cup C\left(T_{2}\right) \cup\{c\}\right)$ and all colors from $C_{R}$ are used on edges of $K^{\prime}$. If there is a rainbow $K_{1,3}+e$ in $K^{\prime}$ then it gives a rainbow $K_{1,4}+e$ together with one edge coming to $T_{1} \cup T_{2}$. Note that obviously it is the case for $n=10$ and for $n \geq 11$ and $C\left(K^{\prime}\right)=C_{R}$ it follows from Theorem 3. If $\left|C\left(K^{\prime}\right)\right|>\left|C_{R}\right|$ for $n \geq 11$ we obtain a rainbow $K_{1,4}+e$ by induction.

Therefore we assume that $C\left(T_{1}, T_{2}\right) \subset C\left(T_{1}\right) \cup C\left(T_{2}\right)$. Let $C_{R}=C-$ $\left(C\left(T_{1}\right) \cup C\left(T_{2}\right)\right)$. If there is at least one edge between $T_{1} \cup T_{2}$ and $K^{\prime}$ of color from $C_{R}$, say $x w$, where $w \in V\left(K^{\prime}\right)$, then all edges coming from $x$ to $T_{2}$ are of colors from $C\left(T_{1}\right)$ otherwise we get a rainbow $K_{1,4}+e$. As a further consequence we get that either there is a rainbow $K_{1,4}+e$ in $K$ or all edges coming out from $T_{2}$ are of colors from $C\left(T_{1}\right) \cup C\left(T_{2}\right)$. In the latter case the graph $K-T_{2}$ is colored with at least $n-1$ colors so the induction completes the proof. Note that surely it is the case for $n \in\{7,8,9\}$. So we can assume that $n \geq 10, C\left(T_{1} \cup T_{2}, K^{\prime}\right) \subset\left(C\left(T_{1}\right) \cup C\left(T_{2}\right)\right)$ and all colors from $C_{R}$ are used on edges of $K^{\prime}$. Repeating the arguments from the previous part of the proof we prove the fact.

Now we are ready to finish the proof of Theorem 4.
By Theorem 1 there exists a rainbow triangle $T_{1}$ with the vertex-set $\{x, y, z\}$ and the set of colors $C\left(T_{1}\right)$.

Let $K^{\prime}=K-T_{1}$. Note that if $K^{\prime}$ contains a rainbow triangle then $K$ contains a rainbow $K_{1,4}+e$ by Fact 2. Assume then it is not the case. Then $\left|C\left(K^{\prime}\right)\right| \leq n-4$ by Theorem 1.

Let $C_{R}=C(K)-C\left(T_{1}\right)$. Note that if there is a vertex $v$ in $T_{1}$ with $\left|C(v) \cap C_{R}\right| \geq 2$, then there is a rainbow $K_{1,4}+e$ with center $v$ and containing $T_{1}$. Hence the converse can be assumed.

So we are to consider only the case $\left|C\left(K^{\prime}\right)\right|=n-4,\left|C(v) \cap C_{R}\right|=1$ and the colors $C(v) \cap C_{R}$ are distinct for each $v \in\{x, y, z\}$.

Then certainly $\left|C\left(T_{1}, K^{\prime}\right) \cap C_{R}\right|=3$ and $C\left(K^{\prime}\right) \cap\left(C\left(T_{1}\right) \cup C\left(T_{1}, K^{\prime}\right)\right)=\emptyset$. Now either we have a rainbow $K_{1,4}+e$ or each edge between $T_{1}$ and $K^{\prime}$ of the color from $C_{R}$ comes out from a different vertex of $T_{1}$. If such an edge of color $c$ comes out, say from $x$, to a vertex $a$ which is not monochromatic in $K^{\prime}$ then $c(a y)=c(x y)$ and $c(z a)=c(x z)$ or we have a rainbow $K_{1,4}+e$. But in this case we also get a rainbow $K_{1,4}+e$. It is a rainbow triangle ayz with two edges coming out from the vertex $a$.

It is easy to note that there can be at most one monochromatic vertex in $K^{\prime}$. If there would be at least two such vertices $a$ and $b$ then $C(a) \cap C\left(K^{\prime}\right)=$ $C(b) \cap C\left(K^{\prime}\right)=\{c(a b)\}$ and so $K^{\prime}-\{a, b\}$ would be $K_{n-5}$ colored with $n-5$ colors, against the assumption for $K^{\prime}$.

Hence the vertex $a$ is monochromatic in $K^{\prime}$ and all edges between $T_{1}$ and $K^{\prime}$ of the three colors in $C\left(T_{1}, K^{\prime}\right) \cap C_{R}$ have $a$ as an endpoint. Thus $|C(a)|=4$ and there is a rainbow $K_{1,4}+e$ with the center $a$ and containing an edge from $T_{1}$.

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