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MONOCHROMATIC PATHS AND MONOCHROMATIC SETS OF ARCS IN QUASI-TRANSITIVE DIGRAPHS

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Abstract

Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. We call the digraph D an m-coloured digraph if each arc of D is coloured by an element of $\{1, 2, \ldots, m\}$ where $m \geq 1$. A directed path is called monochromatic if all of its arcs are coloured alike. A set N of vertices of D is called a kernel by monochromatic paths if there is no monochromatic path between two vertices of N and if for every vertex v not in N there is a monochromatic path from v to some vertex in N. A digraph D is called a quasi-transitive digraph if $(u, v) \in A(D)$ and $(v, w) \in A(D)$ implies $(u, w) \in A(D)$ or $(w, u) \in A(D)$. We prove that if D is an m-coloured quasi-transitive digraph such that for every vertex u of D the set of arcs that have u as initial end point is monochromatic and D contains no C_3 (the 3-coloured directed cycle of length 3), then D has a kernel by monochromatic paths.

Keywords: m-coloured quasi-transitive digraph, kernel by monochromatic paths.

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1. Introduction

For general concepts we refer the reader to [3]. A *kernel* N of a digraph D is an independent set of vertices of D such that for every $w \in V(D) \setminus N$ there exists an arc from w to N. A digraph D is called *kernel perfect* digraph when every induced subdigraph of D has a kernel. We call the digraph D an *m-coloured* digraph if each arc of D is coloured by an element of $\{1, 2, \ldots, m\}$ where $m \geq 1$. A path is called *monochromatic* if all of its arcs are coloured alike. If C is a path of D we denote its length by $\ell(C)$. A set N of vertices of D is called a *kernel by monochromatic paths* if for every pair of vertices of N there is no monochromatic path between them and for every vertex v not in N there is a monochromatic path from v to some vertex in N. The *closure of* D, denoted by $\mathfrak{C}(D)$, is the m-coloured digraph defined as follows: $V(\mathfrak{C}(D)) = V(D)$ and $A(\mathfrak{C}(D))$ is the set of the ordered pairs (u, v) of distinct vertices of D such that there is a monochromatic uv-path. Notice that for any digraph D, $\mathfrak{C}(\mathfrak{C}(D)) \cong \mathfrak{C}(D)$. The problem of the existence of a kernel in a given digraph has been studied by several authors in particular Richardson [19, 20]; Duchet and Meyniel [6]; Duchet [4, 5]; Galeana-Sánchez and V. Neumann-Lara [9, 10]. The concept of kernel by monochromatic paths is a generalization of the concept of kernel and it was introduced by Galeana-Sánchez [7]. In that work she obtained some sufficient conditions for the existence of a kernel by monochromatic paths in an m-coloured tournament. More information about m-coloured digraphs can be found in [7, 8, 21, 23, 24]. Another interesting generalization is the concept of (k, l)-kernel introduced by M. Kwaśnik [17]. Other results about (k, l)-kernels have been developed by M. Kucharska [15]; M. Kucharska and M. Kwaśnik [16]; M. Kwaśnik [18]; and A. Włoch and I. Włoch [22].

A digraph D is called *quasi-transitive* if $(u, v) \in A(D)$ and $(v, w) \in A(D)$ implies $(u, w) \in A(D)$ or $(w, u) \in A(D)$. The concept of quasi-transitive digraph was introduced by Ghouilá-Houri [13] and has been studied by several authors for example Bang-Jensen and Huang [1, 2]. Ghouilá-Houri [13] proved that an undirected graph can be oriented as a quasi-transitive digraph if and only if it can be oriented as a transitive digraph, these graphs are namely *comparability graphs*. More information about comparability graphs can be found in [12, 14].

In [11] H. Galena-Sánchez and R. Rojas-Monroy proved that if D is a digraph such that $D = D_1 \cup D_2$, where D_i is a quasi-transitive digraph which contains no asymmetrical infinite outward path (in D_i) for $i \in \{1, 2\}$; and every directed cycle of length 3 contained in D has at least two symmetrical arcs, then D has a kernel.

For a vertex u in an m-coloured digraph D we denote by $A^+(u)$ the set of arcs that have u as initial end point. And we denote by C_3 the directed cycle of length 3 whose arcs are coloured with three distinct colours.

In this paper, we prove that if D is an m-coloured quasi-transitive digraph such that for every vertex u of D, $A^+(u)$ is monochromatic (all of its elements have the same colour) and D contains no C_3, then D has a kernel by monochromatic paths.

We will need the following results.

Theorem 1.1 ([7]). *D has a kernel by monochromatic paths if and only if $\mathfrak{C}(D)$ has a kernel.*

Theorem 1.2 (Duchet [4]). *If D is a digraph such that every directed cycle has at least one symmetrical arc, then D is a kernel-perfect digraph.*

We use the following notations where D denotes an m-coloured digraph; given $u \neq v \in V(D)$, $u \to v$ means $(u,v) \in A(D)$, $u \xrightarrow{i} v$ means that the arc (u,v) of D is coloured by $i \in \{1,\ldots,m\}$, $u \not\to v$ means $(u,v) \notin A(D)$, $u \Rightarrow v$ means that there exists a monochromatic path from u to v and $u \not\Rightarrow v$ means that there is no monochromatic path from u to v. Given $u \in V(D)$, $N^+(u) = \{v \in V(D) : u \to v\}$, $N^-(u) = \{v \in V(D) : v \to u\}$ and $c(u) = i$ means that all the arcs of $A^+(u)$ are coloured by i where $i \in \{1,\ldots,m\}$ (if $A^+(u) = \emptyset$, then $c(u) = 1$). Given $u \neq v \in V(D)$ such that $u \Rightarrow v$, $l(u,v)$ denotes the minimal length of a monochromatic path from u to v.

2. Monochromatic Paths

We will establish some previous lemmas in order to prove the main theorem.

Lemma 2.1. *Let D be an m-coloured quasi-transitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic and let $T = (u = u_0, u_1, \ldots, u_n = v)$ be a monochromatic uv-path of minimum length contained in D. Then $u_i \not\to u_j$ for every $i,j \in \{0,\ldots,n\}$ with $j > i+1$. In particular, for every $i \in \{0,\ldots,n-2\}$, $u_{i+2} \to u_i$.*

Proof. The proof is straightforward. ■

Lemma 2.2. *Let D be an m-coloured quasi-transitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic and let $T = (u = u_0, u_1, \ldots, u_n = v)$ be a monochromatic uv-path of minimum length contained in D. Then $u_j \to u_i$ for every $i, j \in \{0, \ldots, n\}$ with $j > i + 1$, unless $|V(T)| = 4$, in which case the arc (u_3, u_0) may be absent.*

Proof. If $| V(T) |= 3$, the result follows from Lemma 2.1.

When $| V(T) |= 4$, let $T = (u_0, u_1, u_2, u_3)$ be a monochromatic u_0u_3-path. By Lemma 2.1 we have $u_3 \to u_1$ and $u_2 \to u_0$, and the arc (u_3, u_0) may be absent.

Now, we proceed by induction on $|V(T)|$.

Suppose that $| V(T) |= 5$. Let $T = (u_0, u_1, u_2, u_3, u_4)$ be a monochromatic u_0u_4-path of minimum length, then from Lemma 2.1 and since D is a quasi-transitive digraph we have that $u_4 \to u_2$, $u_3 \to u_1$, $u_2 \to u_0$ and $u_4 \to u_0$. Also, since $u_4 \to u_0$, $u_0 \to u_1$ and D is a quasi-transitive digraph then $u_4 \to u_1$ or $u_1 \to u_4$. Lemma 2.1 implies that $u_1 \not\to u_4$, then $u_4 \to u_1$. Since $u_3 \to u_4$, $u_4 \to u_0$ and D is a quasi-transitive digraph then $u_3 \to u_0$ or $u_0 \to u_3$. If $u_0 \to u_3$, we have a contradiction with Lemma 2.1. Then $u_3 \to u_0$. We conclude $u_j \to u_i$ for every $i, j \in \{0, 1, 2, 3, 4\}$ with $j > i + 1$.

Let $T = (u_0, u_1, \ldots, u_n)$ be a monochromatic path of minimum length n with $n \geq 6$.

Let $T_1 = (u_0, u_1, \ldots, u_{n-1})$ and $T_2 = (u_1, \ldots, u_n)$ then $\ell(T_1) \geq 5$ and $\ell(T_2) \geq 5$, by the inductive hypothesis T_1 and T_2 satisfy that $u_j \to u_i$ for every $j > i + 1$. Now, we need to prove that $u_n \to u_0$. Since $u_2 \to u_0$ and $u_n \to u_2$, and D is a quasi-transitive digraph then $u_0 \to u_n$ or $u_n \to u_0$. By Lemma 2.1 $u_0 \not\to u_n$, thus $u_n \to u_0$. ■

Lemma 2.3. *Let D be an m-coloured quasi-transitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. Given $u \neq v \in V(D)$ such that $v \not\to u$, if $u \Rightarrow v$, then one and only one of the following conditions is satisfied:*

1. *$u \to v$.*
2. *$u \not\to v$ and there exists a monochromatic path $(u = u_0, u_1, u_2, u_3 = v)$ of length 3 such that $u_2 \to u_0$ and $u_3 \to u_1$. Moreover, there exists no path of length 2 between u and v.*

Proof. Clearly the Lemma holds when $l(u, v) = 1$. So, assume that $l(u, v) \geq 2$.

If $l(u,v) \geq 4$, it follows from Lemma 2.2 that $v \to u$, contradicting the hypothesis. Hence $l(u,v) \leq 3$. When $l(u,v) = 3$, let $(u = u_0, u_1, u_2, u_3 = v)$ be a monochromatic uv-path of minimum length, Lemma 2.1 implies that $u_2 \to u_0$ and $u_3 \to u_1$.

Now, if T' is a path of length 2 from u to v or from v to u, since D is a quasi-transitive digraph then $u \to v$ or $v \to u$. The hypothesis implies that $v \not\to u$, then $u \to v$ contradicting the assumption $l(u,v) \geq 2$. We conclude that there is no path of length 2 between u and v. ■

3. The Main Result

Lemma 3.1. *Let D be an m-coloured quasi-transitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. Given distinct vertices u, v, w of D, if $u \Rightarrow v$, $v \not\Rightarrow u$, $v \Rightarrow w$ and $w \not\Rightarrow v$, then $w \to u$ or $u \Rightarrow w$.*

Proof. Since $u \Rightarrow v$ and $v \not\to u$, it follows from Lemma 2.3 that $l(u,v) = 1$ or 3. Similarly $l(v,w) = 1$ or 3. Assume that $u \not\to w$ and $w \not\to u$. Since D is quasi-transitive, we obtain that $N^+(u) \cap N^-(w) = N^+(w) \cap N^-(u) = \emptyset$.

Clearly $u \Rightarrow w$ when $c(u) = c(v)$. So assume that $c(u) \neq c(v)$. To begin we show that $l(u,v) = 3$. Otherwise $l(u,v) = 1$, that is, $u \to v$. As $v \notin N^+(u) \cap N^-(w)$, $v \not\to w$. Hence $l(v,w) = 3$ and there are vertices $v = v_0, v_1, v_2, v_3 = w$ of D such that $v \overset{c(v)}{\longrightarrow} v_1 \overset{c(v)}{\longrightarrow} v_2 \overset{c(v)}{\longrightarrow} w$. If $v_1 \to u$ (respectively, $v_2 \to u$), then we would have $v \Rightarrow u$ by considering $v \overset{c(v)}{\longrightarrow} v_1 \overset{c(v)}{\longrightarrow} u$ (respectively, $v \overset{c(v)}{\longrightarrow} v_1 \overset{c(v)}{\longrightarrow} v_2 \overset{c(v)}{\longrightarrow} u$). Thus $v_1 \not\to u$ and $v_2 \not\to u$.

As $u \to v \to v_1$ and $v_1 \not\to u$, we obtain $u \to v_1$ because D is quasi-transitive. Therefore $u \to v_1 \to v_2$. Since D is quasi-transitive and since $v_2 \not\to u$, we have $u \to v_2$ and we would obtain $v_2 \in N^+(u) \cap N^-(w)$. Consequently, $l(u,v) = 3$ and there are vertices $u = u_0, u_1, u_2, u_3 = v$ of D such that $u \overset{c(u)}{\longrightarrow} u_1 \overset{c(u)}{\longrightarrow} u_2 \overset{c(u)}{\longrightarrow} v$. As $l(u,v) = 3$, we get $u_2 \to u$.

Now, assume that $l(v,w) = 1$, that is, $v \to w$. As $u_2 \to v \to w$, we have $u_2 \to w$ or $w \to u_2$ because D is quasi-transitive. If $w \to u_2$, the we would obtain $u_2 \in N^+(w) \cap N^-(u)$. Thus $u_2 \to w$ and hence $u \Rightarrow w$ by considering $u \overset{c(u)}{\longrightarrow} u_1 \overset{c(u)}{\longrightarrow} u_2 \overset{c(u)}{\longrightarrow} w$.

Lastly, assume that $l(v,w) = 3$ and consider vertices $v = v_0, v_1, v_2, v_3 = w$ of D such that $v \overset{c(v)}{\longrightarrow} v_1 \overset{c(v)}{\longrightarrow} v_2 \overset{c(v)}{\longrightarrow} w$. We still have $v_1 \not\to u$ and $v_2 \not\to u$ because $v \not\Rightarrow u$. Since D is quasi-transitive and since $u_2 \to v \to v_1$,

$u_2 \to v_1$ or $v_1 \to u_2$. We prove that $u_2 \to v_1$. Otherwise $v_1 \to u_2$ and hence $v_1 \to u_2 \to u$. As D is quasi-transitive and as $v_1 \not\to u$, we get $u \to v_1$ and so $u \to v_1 \to v_2$. Since D is quasi-transitive and since $v_2 \not\to u$, we would obtain $u \to v_2$ so that $v_2 \in N^+(u) \cap N^-(w)$. It follows that $u_2 \to v_1$. We have $u_2 \to v_1 \to v_2$. As D is quasi-transitive, $u_2 \to v_2$ or $v_2 \to u_2$. We show that $u_2 \to v_2$. Otherwise $v_2 \to u_2$ and hence $v_2 \to u_2 \to u$. Since D is quasi-transitive and since $v_2 \not\to u$, we would get $u \to v_2$ so that $v_2 \in N^+(u) \cap N^-(w)$. Consequently $u_2 \to v_2$ and so $u_2 \to v_2 \to w$. As D is quasi-transitive, we have $u_2 \to w$ or $w \to u_2$. If $w \to u_2$, then we would have $u_2 \in N^+(w) \cap N^-(u)$. Thus $u_2 \to w$ and $u \Rightarrow w$ by considering $u \xrightarrow{c(u)} u_1 \xrightarrow{c(u)} u_2 \xrightarrow{c(u)} w$. ■

Proposition 3.2. *Let D be an m-coloured quasi-transitive digraph containing no C_3 and such that $A^+(u)$ is monochromatic for every $u \in V(D)$. Given distinct vertices u, v, w of D, if $u \Rightarrow v$, $v \not\Rightarrow u$, $v \Rightarrow w$ and $w \not\Rightarrow v$ and $c(u) \neq c(v)$, then $u \Rightarrow w$ and $w \not\Rightarrow u$.*

Proof. By the previous lemma, it suffices to establish that $w \not\Rightarrow u$. Suppose, for a contradiction, that $w \Rightarrow u$. There are vertices $w = w_0, \ldots, w_p = u$ such that $w_q \xrightarrow{c(w)} w_{q+1}$ for $0 \leq q \leq p-1$. Clearly $c(w) \notin \{c(u), c(v)\}$ because $v \not\Rightarrow u$ and $w \not\Rightarrow v$. As observed at the beginning of the preceding proof, $l(u,v) = 1$ or 3 and $l(v,w) = 1$ or 3.

Suppose that $l(u,v) = 1$, that is, $u \to v$. As D is quasi-transitive and $w_{p-1} \to u \to v$, we have $w_{p-1} \to v$ or $v \to w_{p-1}$. If $w_{p-1} \to v$, then $w \Rightarrow v$ by considering the monochromatic path $(w = w_0, \ldots, w_{p-1}, v)$. If $v \to w_{p-1}$, then $u \xrightarrow{c(u)} v \xrightarrow{c(v)} w_{p-1} \xrightarrow{c(w)} u$ and D would contain C_3. Thus $u \not\to v$ and $l(u,v) = 3$. There are vertices $u = u_0, u_1, u_2, u_3 = v$ of D such that $u \xrightarrow{c(u)} u_1 \xrightarrow{c(u)} u_2 \xrightarrow{c(u)} v$. Since D is quasi-transitive and since $u \not\to v$ and $v \not\to u$, we obtain that $N^+(u) \cap N^-(v) = N^+(v) \cap N^-(u) = \emptyset$.

Suppose that $l(v,w) = 1$, that is, $v \to w$. We get $v \to w_0$ and $v \not\to w_p$. Consider the largest $q \in \{0, \ldots, p-1\}$ such that $v \to w_q$. As D is quasi-transitive and as $v \to w_q \to w_{q+1}$, we have $v \to w_{q+1}$ or $w_{q+1} \to v$. By the maximality of q, $v \not\to w_{q+1}$ and hence $w_{q+1} \to v$. Since $u \not\to v$ then $q+1 < p$. Therefore $w \Rightarrow v$ by considering the monochromatic path $(w = w_0, \ldots, w_{q+1}, v)$. Consequently $v \not\to w$ and $l(v,w) = 3$. There are vertices $v = v_0, v_1, v_2, v_3 = w$ of D such that $v \xrightarrow{c(v)} v_1 \xrightarrow{c(v)} v_2 \xrightarrow{c(v)} w$. Since $v \not\Rightarrow u$, we have $v_1 \not\to u$ and $v_2 \not\to u$. It follows that $N^+(u) \cap N^-(v_2) = \emptyset$. Otherwise there is $x \in V(D)$ such that $u \to x \to v_2$. As D is quasi-transitive and $v_2 \not\to u$, we have $u \to v_2$. Since $l(v, w) = 3$, we have $v_2 \to v$ and we would get $v_2 \in N^+(u) \cap N^-(v)$. Moreover $(N^+(v_2) \cap N^-(u)) \cap \{w_0, \ldots, w_{p-1}\} = \emptyset$. Otherwise there is $i \in \{0, \ldots, p-1\}$ such that $w_i \in N^+(v_2) \cap N^-(u)$. Thus $v_2 \xrightarrow{c(v)} w_i \xrightarrow{c(w)} u \xrightarrow{c(u)} v_2$ and D would contain C_3.

As $v_1 \to v_2$, we have $u \not\to v_1$ because $N^+(u) \cap N^-(v_2) = \emptyset$. Since $v_1 \not\to u$ and D is quasitransitive, we obtain that $N^+(u) \cap N^-(v_1) = \emptyset$. As $l(v, w) = 3$, $w = v_3 \to v_1$ and hence $u \not\to w$. We have also $w \not\to u$ because $v_2 \to w$ and $(N^+(v_2) \cap N^-(u)) \cap \{w_0, \ldots, w_{p-1}\} = \emptyset$. By Lemma 2.3, $l(w, u) = 3$.

We have $v_2 \to w \to w_1$. Since D is quasi-transitive, $v_2 \to w_1$ or $w_1 \to v_2$. As $l(w, u) = 3$, $u \to w_1$ and hence $w_1 \not\to v_2$ because $N^+(u) \cap N^-(v_2) = \emptyset$. Therefore $v_2 \to w_1$. So we get $v_2 \to w_1 \to w_2$. Since D is quasi-transitive, $v_2 \to w_2$ or $w_2 \to v_2$. But $v_2 \not\to w_2$ because $w_2 \to u$ and $(N^+(v_2) \cap N^-(u)) \cap \{w_0, w_1, w_2\} = \emptyset$. Consequently $w_2 \to v_2$. As $l(v, w) = 3$, $v_2 \to v$. Finally, we obtain $w_2 \to v_2 \to v$. Since D is quasi-transitive, $w_2 \to v$ or $v \to w_2$. As $w_2 \to u$ and $N^+(v) \cap N^-(u) = \emptyset$, we have $v \not\to w_2$ and hence $w_2 \to v$. We would obtain $w \Rightarrow v$ by considering $w \xrightarrow{c(w)} w_1 \xrightarrow{c(w)} w_2 \xrightarrow{c(w)} v$. ■

Theorem 3.3. *Let D be an m-coloured quasi-transitive digraph containing no C_3 and such that $A^+(u)$ is monochromatic for every $u \in V(D)$. Then $\mathfrak{C}(D)$ is a kernel-perfect digraph.*

Proof. We will prove that each cycle in $\mathfrak{C}(D)$ possesses at least one symmetrical arc. Thus the assertion in Theorem 3.3 will follow from Theorem 1.2.

Suppose, for a contradiction, that there exists a cycle in $\mathfrak{C}(D)$ which has no symmetrical arc. Let $C = (u_0, u_1, \ldots, u_n = u_0)$ be one of minimum length. Note that $n \geq 2$. Thus for each $i \in \{0, 1, \ldots, n-1\}$ we have $u_i \Rightarrow u_{i+1}$ and $u_{i+1} \not\Rightarrow u_i$. Since C has no symmetrical arc (in $\mathfrak{C}(D)$), we may assume w.l.o.g. that $c(u_0) \neq c(u_1)$. The Proposition 2.3 implies that $u_0 \Rightarrow u_2$ and $u_2 \not\Rightarrow u_0$. So, $n \geq 3$. It follows that $(u_0, u_2, u_3, \ldots, u_n = u_0)$ is a cycle in $\mathfrak{C}(D)$ which has no symmetrical arc and its length is less than $\ell(C)$, contradicting our assumption about C. ■

The following is an immediate consequence of Theorems 1.1 and 3.3.

Corollary 3.4. *Let D be an m-coloured quasi-transitive digraph containing no C_3 and such that $A^+(u)$ is monochromatic for every $u \in V(D)$. Then D has a kernel by monochromatic paths.*

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