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ARITHMETIC LABELINGS AND GEOMETRIC LABELINGS OF COUNTABLE GRAPHS

Gurusamy Rengasamy Vijayakumar

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba, Mumbai 400 005, India

e-mail: vijay@math.tifr.res.in

Abstract

An injective map from the vertex set of a graph G—its order may not be finite—to the set of all natural numbers is called an *arithmetic* (a *geometric*) *labeling* of G if the map from the edge set which assigns to each edge the sum (product) of the numbers assigned to its ends by the former map, is injective and the range of the latter map forms an arithmetic (a geometric) progression. A graph is called *arithmetic* (*geometric*) if it admits an arithmetic (a geometric) labeling. In this article, we show that the two notions just mentioned are equivalent—i.e., a graph is arithmetic if and only if it is geometric.

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All graphs considered in this article are countable and simple. The set of all positive integers is denoted by $\mathbb{N}$ and $\mathcal{N} = \{0\} \cup \mathbb{N}$. For basic information about graph theory we rely on [7]. Let $G = (V, E)$ be a graph and f be any map from V to $\mathbb{N}$; we associate with f two maps from E to $\mathbb{N}$, denoted by f^+ and $f^\times$: for all $uv \in E$, $f^+(uv) = f(u) + f(v)$ and $f^\times(uv) = f(u) \times f(v)$. If f and f^+ are injective and the elements of $f^+(E)$ form an arithmetic progression—i.e., if this set can be written as $\{k+nd : n \in \mathcal{N} \text{ and } n < |E|\}$—then f is called an *arithmetic labeling* of G. (For information about this labeling for finite graphs, see [1, 2].) If f and $f^\times$ are injective and the elements of $f^\times(E)$ form a geometric progression—i.e., if this set can be written as $\{ar^n : n \in \mathbb{N} \text{ and } n < |E|\}$ where r may not be an integer—then f is called a *geometric labeling* of G. (For details about this labeling for finite graphs, we refer the reader to [4, 5]; in this connection, see [3] also.) If a graph admits an arithmetic (a geometric) labeling then it is called *arithmetic* (*geometric*). If σ is an arithmetic labeling of a graph G, then the map: $v \mapsto 2^{\sigma(v)}$ where $v \in V(G)$ is a geometric labeling of G. Thus, if a graph is arithmetic then it is geometric also. The objective of this article is to prove the converse; to this end, we begin with a simple result.

Lemma 1. *Let f be a geometric labeling of a graph $G = (V, E)$ and $f^{\times}(E) = \{ar^n : n \in \mathbb{N}$ and $n < |E|\}$. Let H be a component of G and p, q be two vertices of H. If the distance between p and q is odd, then for some $n \in \mathbb{Z}$, $f(p)f(q) = ar^n$; otherwise for some $n \in \mathbb{Z}$, $f(q) = f(p)r^n$. If H is not a bipartite graph, then for each $x \in V(H)$, there is some $n \in \mathbb{Z}$ such that $(f(x))^2 = ar^n$.*

Proof. If $x \in N(p)$, then for some $n \in \mathbb{Z}$, $f(p)f(x) = ar^n$; if w is a vertex of H such that $d(w, p) = 2$, then it is easy to verify that $f(w) = f(p)r^n$ for some $n \in \mathbb{Z}$. Thus continuing, we obtain the following: If $d(p, q)$ is odd, then for some $n \in \mathbb{Z}$, $f(p)f(q) = ar^n$; otherwise for some $n \in \mathbb{Z}$, $f(q) = f(p)r^n$.

Now suppose that H is not bipartite. Then either $A := \{x \in V(H) : d(x, p) \text{ is odd}\}$ or $V(H) \setminus A$ is not an independent set of H; therefore either in A or in $V(H) \setminus A$, there exists a pair of adjacent vertices u, v. By the hypothesis, for some $k \in \mathbb{Z}$, $f(u)f(v) = ar^k$. If $u, v \in A$, then by what has been derived above, for some $m, n \in \mathbb{Z}$, $f(p)f(u) = ar^m$ and $f(p)f(v) = ar^n$ whence by the preceding three equalities, it follows that $(f(p))^2 = ar^\ell$ for some $\ell \in \mathbb{Z}$; if $u, v \in (V(H) \setminus A)$ then for some $m, n \in \mathbb{Z}$, $f(u) = f(p)r^m$ and $f(v) = f(p)r^n$ whence for some $\ell \in \mathbb{Z}$, $(f(p))^2 = ar^\ell$. Since p is arbitrary, the second part of the conclusion also holds. ■

Proposition 2. *Let $G = (V, E)$ be a graph such that the number of its nontrivial components which are bipartite is finite. If G is geometric, then it is arithmetic.*

Proof. We can assume that G has no isolated vertices. Suppose that f is a geometric labeling of G. Let U be the union of the vertex sets of all non-bipartite components and (A_i, B_i), $i = 1, 2, \ldots, n-1$ be the bipartitions of the remaining components. (see Definition 1.2.17 of [7].) Let $f^{\times}(E) = \{ar^k : k \in \mathbb{N} \text{ and } k < |E|\}$. For each $i \in \{1, 2, \ldots, n-1\}$, choose a vertex $x_i \in A_i$.

Let v be a vertex in U. By Lemma 1, for some $k \in \mathbb{Z}$, $(f(v))^2 = ar^k$; therefore $2\log f(v) - \log a$ is an integer, viz., k. (In this proof, 'r' is the base for every logarithm.) For any $i \in \{1, 2, \ldots, n-1\}$, let x and y be vertices in A_i and B_i, respectively. Then by Lemma 1, for some $j, k \in \mathbb{Z}$, $f(x) = f(x_i)r^j$ and $f(y)f(x_i) = ar^k$. Therefore $[\log f(x) - \log f(x_i)] = j \in \mathbb{Z}$ and $[\log f(y) + \log f(x_i) - \log a] = k \in \mathbb{Z}$. Let

$$\mu = \lceil \max\{2n\log a + n,\, 2n\log f(x_i) : i = 1, \ldots, n-1\}\rceil.$$

Define a map $g : V \mapsto \mathbb{Z}$ as follows.

For all $x \in U$, $g(x) = n[2\log f(x) - \log a] + \mu$; if $i \in \{1, 2, \ldots, n-1\}$, then for all $x \in A_i$, $g(x) = 2n[\log f(x) - \log f(x_i)] + i + \mu$ and for all $x \in B_i$, $g(x) = 2n[\log f(x) + \log f(x_i) - \log a] - i + \mu$.

It is easy to verify that if $x \in W$ and $y \in V \setminus W$ where $W \in \{U, A_i, B_i : i = 1, \ldots, n-1\}$, then $g(x) \not\equiv g(y) \pmod{2n}$; using this fact, we find that g is injective. By the choice of μ, clearly for each $v \in V(G)$, $g(v) > 0$. For each $e \in E$, it can be verified that $g^+(e) = 2n\log f^\times(e) + 2\mu - 2n\log a$. Therefore g^+ is injective and the elements of $g^+(E)$ form an arithmetic progression. Thus it follows that g is an arithmetic labeling. ■

Corollary 3. *If a graph is finite and geometric, then it is arithmetic.*

For another proof of Corollary 3, see [6]. It is easy to find the graphs—finite and infinite—which are arithmetic; e.g., it can be verified that every finite path, the infinite path with one leaf and every complete graph whose order is less than 5 are arithmetic and it can be shown that every complete graph with at least 5 vertices—it can be infinite—is not arithmetic. (For more examples of finite arithmetic graphs, see [1, 5].) The next result which yields a class of infinite graphs which are arithmetic, plays a decisive role in settling the main result, viz., Theorem 5.

Theorem 4. *Let $G = (V, E)$ be a graph such that its order is countable and the number of its nontrivial components which are bipartite is infinite. Then G is arithmetic.*

Proof. First consider the possibility that the number of components of order 2 is infinite. Let H be the union of all other components. Let f be an injective map from $V(H)$ to $\{2^{n+3} : n \in \mathbb{N}\}$. It can be verified that f^+ is injective and $J := \{n \in \mathbb{N} : 8n \notin f^+(E(H))\}$ is infinite. We can take

$\{H_j : j \in J\}$ as the set of all components which are copies of K_2. Let f be extended to a map $g : V \mapsto \mathbb{N}$ so that for each $j \in J$, $g(V(H_j)) = \{4j-1, 4j+1\}$. Then g and g^+ are injective and $g^+(E) = \{8n : n \in \mathbb{N}\}$; i.e., g is an arithmetic labeling of G.

So, let us assume that the number of components which are of order 2 is finite. Let $V = \{v_1, v_2, v_3, \ldots\}$. Let Z be the union of the vertex sets of all components each of which is either non-bipartite or K_2 or K_1. Inductively we construct for each $n \in \mathbb{N}$, an induced subgraph G_n and a map $f_n : V(G_n) \mapsto \mathbb{N}$ such that the conditions (a), (b), …, (f) and (g) given below are fulfilled.

(a) For each $x \in V(G_n)$, there exists an integer $k \in \{0, 1, 3, 5, 7\}$ such that $f_n(x) \equiv k$. (8 is the modulus of every congruence occurred in this proof.)

(b) $X_n \cap Z = \emptyset = Y_n \cap Z$ and $Z_n \subset Z$ where $X_n = \{x \in V(G_n) : f_n(x) \equiv 1 \text{ or } 5\}$, $Y_n = \{x \in V(G_n) : f_n(x) \equiv 3 \text{ or } 7\}$ and $Z_n = \{x \in V(G_n) : f_n(x) \equiv 0\}$.

(c) f_n is injective and $f_n^+(E(G_n)) = \{8k : k = 1, \ldots, |E(G_n)|\}$.

(d) If $n > 1$, then $V(G_{n-1}) \subseteq V(G_n)$ and f_n is an extension of f_{n-1}.

(e) There exist a vertex $\alpha_n \in X_n$ and a vertex $\beta_n \in Y_n$, such that they are adjacent and for each $x \in X_n$, $f_n(x) \leqslant f_n(\alpha_n)$, for each $x \in Y_n$, $f_n(x) \leqslant f_n(\beta_n)$ and for each $x \in Z_n$, $f_n(\alpha_n) > f_n(x) < f_n(\beta_n)$.

(f) $\{v_1, v_2, \ldots, v_n\} \subseteq V(G_n)$.

(g) If (A, B) is the bipartition of a component of $G - Z$ and x, y belong to $V(G_m) \cap (A \cup B)$, then the following hold.

$$\begin{aligned} x, y \in A \text{ or } x, y \in B \quad &\Rightarrow \quad f_n(x) \equiv f_n(y) \text{ and} \\ x \in A,\ y \in B \quad &\Rightarrow \quad f_n(x) + f_n(y) \equiv 0. \end{aligned}$$

We can assume that $v_1, v_2, v_3 \notin Z$ and v_1 is adjacent to both v_2 and v_3. Let $G_1 = G[v_1, v_2, v_3]$. Set $f_1(v_1) = 1$, $f_1(v_2) = 7$ and $f_1(v_3) = 15$. Let $\alpha_1 = v_1$ and $\beta_1 = v_3$. Then (a), …, (f) and (g) are satisfied for $n = 1$. Now suppose for some $m \in \mathbb{N}$, there exists an induced subgraph G_m and a map $f_m : V(G_m) \mapsto \mathbb{N}$ satisfying the conditions listed above. Let us construct the subgraph G_{m+1}, the map f_{m+1} and set the vertices $\alpha_{m+1}, \beta_{m+1}$ as described below.

Case 1. $v_{m+1} \in V(G_m)$.
Let $G_{m+1} = G_m$, $f_{m+1} = f_m$, $\alpha_{m+1} = \alpha_m$ and $\beta_{m+1} = \beta_m$.

Case 2. $v_{m+1} \in (Z \setminus V(G_m))$.
Let $J = \{j \in \mathbb{N} : v_j \in V(G_m) \cap N(v_{m+1})\}$. ($J$ may be empty.) Let $L = \{1, 2, \ldots, 2\ell - 1, 2\ell\} \setminus \{\frac{1}{8} f_m(v_j) : j \in J\}$ where $\ell = \frac{1}{8} f_m(\alpha_m \beta_m)$. For each $k \in L$, choose a distinct component H_k of $G - Z$ so that $V(H_k) \cap V(G_m) = \emptyset$ and let x_k, y_k be two adjacent vertices of H_k. Let $V(G_{m+1}) = V(G_m) \cup \{v_{m+1}, x_j, y_j : j \in L\}$. For each $x \in V(G_m)$, let $f_{m+1}(x) = f_m(x)$. Set $f_{m+1}(v_{m+1}) = f_m^+(\alpha_m \beta_m)$. For each $k \in L$, set $f_{m+1}(x_k) = f_m(\alpha_m) + 4k$ and $f_{m+1}(y_k) = f_m(\beta_m) + 4k$. Note that $X_{m+1} = X_m \cup \{x_k : k \in L\}$, $Y_{m+1} = Y_m \cup \{y_k : k \in L\}$ and $Z_{m+1} = Z_m \cup \{v_{m+1}\}$. Let $\alpha_{m+1} = x_{2\ell}$ and $\beta_{m+1} = y_{2\ell}$. It can be verified that $|E(G_{m+1}) - E(G_m)| = |J| + |L| = 2\ell$ and $\{f_{m+1}^+(e) : e \in E(G_{m+1}) - E(G_m)\} = \{f_{m+1}^+(v_k v_{m+1}) : k \in J\} \cup \{f_{m+1}^+(x_j y_j) : j \in L\} = \{f_m^+(\alpha_m \beta_m) + 8k : 1 \leqslant k \leqslant 2\ell\}$.

Case 3. $v_{m+1} \notin (Z \cup V(G_m))$.
Let (S, T) be the bipartition of the component which contains v_{m+1}. By (g), we can assume that $X_m \cap T = Y_m \cap S = \emptyset$. Assume that $v_{m+1} \in S$. Let $J = \{j \in \mathbb{N} : v_j \in V(G_m) \cap N(v_{m+1})\}$. ($J$ may be empty.) Let the number ρ be chosen in $\{0, 4\}$ as follows: If $V(G_m) \cap (S \cup T) = \emptyset$, $\rho = 0$. Otherwise, let $u \in V(G_m) \cap (S \cup T)$; when $u \in S$, $\rho + f_m(\alpha_m) \equiv f(u)$; when $u \in T$, $\rho + f_m(\alpha_m) + f(u) \equiv 0$. (Note that by (g), for all $x \in S \cap V(G_m)$, $\rho + f_m(\alpha_m) \equiv f_m(x)$ and for all $x \in T \cap V(G_m)$, $\rho + f_m(\alpha_m) + f_m(x) \equiv 0$.) Let $\ell = \frac{1}{4}[f_m^+(\alpha_m \beta_m) + \rho] - 1$. Note that for each $j \in J$, $f_m(\alpha_m) + f_m(v_j) + \rho < 8\ell$. Let $L = \{1, 2, \ldots, \ell\} \setminus \{\frac{1}{8}[f_m(\alpha_m) + f_m(v_j) + \rho] : j \in J\}$. For each $k \in L$, choose a distinct component H_k of $G - Z$ so that $V(H_k) \cap V(G_m) = \emptyset$ and let x_k, y_k be two adjacent vertices in H_k. Let y_ℓ be chosen so that $\deg y_\ell \geqslant 2$. Let $x_{\ell+1}$ be a neighbour of y_ℓ, other than x_ℓ. Let $V(G_{m+1}) = V(G_m) \cup \{v_{m+1}, x_{\ell+1}\} \cup \{x_k, y_k : k \in L\}$. For each $x \in V(G_m)$, let $f_{m+1}(x) = f_m(x)$. Set $f_{m+1}(v_{m+1}) = f_m^+(\alpha_m \beta_m) + f_m(\alpha_m) + \rho$. For each $k \in L$, set $f_{m+1}(x_k) = f_m(\alpha_m) + 4k$ and $f_{m+1}(y_k) = f_m(\beta_m) + 4k$. Set $f_{m+1}(x_{\ell+1}) = f_{m+1}(x_\ell) + 8$. Note that $X_{m+1} = X_m \cup \{x_k : k \in L\} \cup \{x_{\ell+1}, v_{m+1}\}$, $Y_{m+1} = Y_m \cup \{y_k : k \in L\}$ and $Z_{m+1} = Z_m$. Let $\alpha_{m+1} = x_{\ell+1}$ and $\beta_{m+1} = y_\ell$. We find that $|E(G_{m+1}) - E(G_m)| = |J| + |L| + 1 = \ell + 1$ and $\{f_{m+1}^+(e) : e \in E(G_{m+1}) - E(G_m)\} = \{f_{m+1}^+(v_j v_{m+1}) : j \in J\} \cup \{f_{m+1}^+(x_k y_k) : k \in L\} \cup \{f_{m+1}^+(y_\ell x_{\ell+1})\} = \{f_m^+(\alpha_m \beta_m) + 8k : 1 \leqslant k \leqslant \ell + 1\}$.

In each of the three cases, it can be routinely verified that the conditions (a), …, (f) and (g) hold when $n = m + 1$. Since $\cup_{n=1}^{\infty} V(G_n) = V(G)$ and for each $n \in \mathbb{N}$, f_{n+1} is the extension of f_n, we have an injective map $f : V \mapsto \mathbb{N}$ such that for each $n \in \mathbb{N}$, f_n is the restriction of f to $V(G_n)$. Since $f^+(E) = \{8k : k \in \mathbb{N}\}$, f is an arithmetic labeling of G. ■

Combining Proposition 2 and Theorem 4, we find that if a graph is geometric, then it is arithmetic. We have already noted that every arithmetic graph is geometric. Thus we obtain the following.

Theorem 5. *A graph is arithmetic if and only if it is geometric.*

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