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FACTORING DIRECTED GRAPHS WITH RESPECT TO THE CARDINAL PRODUCT IN POLYNOMIAL TIME II

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Abstract

By a result of McKenzie [7] all finite directed graphs that satisfy certain connectivity conditions have unique prime factorizations with respect to the cardinal product. McKenzie does not provide an algorithm, and even up to now no polynomial algorithm that factors all graphs satisfying McKenzie's conditions is known. Only partial results [1, 3, 5] have been published, all of which depend on certain thinness conditions of the graphs to be factored.

In this paper we weaken the thinness conditions and thus significantly extend the class of graphs for which the prime factorization can be found in polynomial time.

Keywords: directed graphs, cardinal product, graph algorithms.

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1. Introduction

McKenzie [7] proved the so-called common refinement property for factorizations of finite or infinite relational structures with respect to the cardinal product. In the context of finite directed graphs his result implies the uniqueness of the prime factor decomposition (PFD) of directed graphs with respect to the cardinal product under certain conditions.

Here we are concerned with the problem of finding an efficient algorithm for the computation of the PFD. This question has evolved from a purely

academic problem to one of importance for the recognition of "characters" (traits or Merkmale) in theoretical biology; see [2].

The problem of finding an algorithm for the factorization was first addressed by Feigenbaum and Schäffer in [1], where they present a polynomial algorithm for the case of reflexive and symmetric structures. In the language of undirected graphs this means that the prime factorization of connected graphs with respect to the strong product can be found in polynomial time.

Later this was extended to symmetric structures that are not necessarily reflexive by Imrich [3], that is, it was shown that the PFD of nonbipartite connected graphs with respect to the cardinal product can also be found in polynomial time.

The first algorithm for asymmetric relations is due to Imrich and Klöckl [5]. It presents a class of graphs with unique PFD, which can be computed in polynomial time. This class, say K, has nonempty intersection with the class M of graphs for which McKenzie's result implies uniqueness of the PFD. Since K contains elements not in M, [5] also extends the uniqueness result of McKenzie. Unfortunately the graphs in K must satisfy a rather strong thinness condition.

In this paper we weaken the thinness condition. In other words, we further extend the class of graphs whose PFD can be computed in polynomial time. The approach of the algorithm is to simplify the given graph G until one obtains a graph S that fulfills the assumptions of our algorithm in [5]. We then factor S and derive the factorization of G from that of S.

2. Definitions and Preliminary Results

By a *directed graph* $G = (V, A)$ we mean a set of vertices V together with a set of arcs A, that are ordered pairs $\langle x, y\rangle$ of vertices. We allow that both $\langle x, y\rangle$ and $\langle y, x\rangle$ are in A and do not require x, y to be distinct. Thus, A is a subset of the Cartesian product $V \times V$.

The vertex x is the *origin* and y the *terminus* of $\langle x, y\rangle$. In the case when $x = y$ we speak of a *loop*. In analogy to the undirected case we call a graph G with $A(G) = V(G) \times V(G)$ complete. If it has n vertices it will be denoted by K_n^d to distinguish it from the ordinary complete graph K_n (where any two distinct vertices are connected by an undirected edge.)

We say $A(G)$ is *reflexive* if A contains all loops $\langle x, x\rangle$, where $x \in V(G)$. It is *symmetric* if $\langle x, y\rangle \in A(G) \Leftrightarrow \langle y, x\rangle \in A(G)$. By abuse of language we simply say that G is reflexive, respectively symmetric. Symmetric directed

graphs correspond to undirected graphs by identification of pairs of arcs $\langle x, y\rangle$, $\langle y, x\rangle$ with undirected edges $[x, y]$.

The *out-neighborhood* $N^+(x)$ of a vertex x is defined as the set

$$\{y \in V \mid \langle x, y\rangle \in A\}.$$

Analogously one defines the *in-neighborhood* $N^-(x)$. A directed graph is uniquely defined by its vertex set and the out-neighborhoods of the vertices.

The *cardinal product* $G_1 \times G_2$ of two directed graphs G_1, G_2 is defined on the Cartesian product $V(G_1) \times V(G_2)$ of the vertex sets of the factors such that the out-neighborhood of a vertex $(x_1, x_2) \in V(G_1 \times G_2)$ is the Cartesian product of the out-neighborhoods of x_1 in G_1 and x_2 in G_2:

$$N^+_{G_1\times G_2}(x_1, x_2) = N^+_{G_1}(x_1) \times N^+_{G_2}(x_2).$$

The cardinal product is commutative, associative, and the single vertex with a loop, that is K_1^d, is a unit. The cardinal product of reflexive symmetric graphs corresponds to the strong product of undirected graphs.

A graph G is *prime* with respect to the cardinal product if $G = G_1 \times G_2$ implies that G_1 or G_2 is equal to K_1^d.

A graph G is $R^+|R^-$*-connected* if to all pairs x, $y \in V(G)$ and $n \in \mathbb{N}$ a sequence $x = x_0, x_1, \ldots, x_n = y$ can be found where

$$N^+(x_i) \cap N^+(x_{i+1}) \neq \emptyset$$

for $0 \leq i < n$; compare the factors in Figure 1. We also say that x and y are $R^+|R^-$-connected. $R^-|R^+$*-connectedness* is defined analogously.

Lemma 2.1. *Let $G = G_1 \times G_2 \times \cdots \times G_k$ be the cardinal product of directed graphs. Then the following conditions are equivalent:*

(i) *G is $R^+|R^-$-connected.*

(ii) *G_i, $i \in \{1, 2, \ldots, k\}$, is $R^+|R^-$-connected.*

The statement remains true if one replaces $R^+|R^-$ by $R^-|R^+$.

Proof. It suffices to prove the lemma for $k = 2$.

(i)$\Longrightarrow$(ii) Given two vertices v_1, $w_1 \in G_1$, arbitrarily choose $v_2 \in G_2$. By condition (i) the vertices $x_0 = (v_1, v_2)$ and $x_n = (w_1, v_2)$ are $R^+|R^-$-connected. Thus there exists a sequence $(v_1, v_2) = x_0, x_1, \ldots, x_n = (w_1, v_2)$ such that $N^+(x_i) \cap N^+(x_{i+1}) \neq \emptyset$ for $0 \leq i < n$. Setting $x_i = (x_{i,1}, x_{i,2})$

for $0 \leq i \leq n$, it is clear that v_1 and w_1 are $R^+|R^-$-connected in G_1 by the sequence $v_1 = x_{0,1}, x_{1,1}, \ldots, x_{n,1} = w_1$.

Reversing the roles of G_1 and G_2 it is easily seen that G_2 is $R^+|R^-$-connected too.

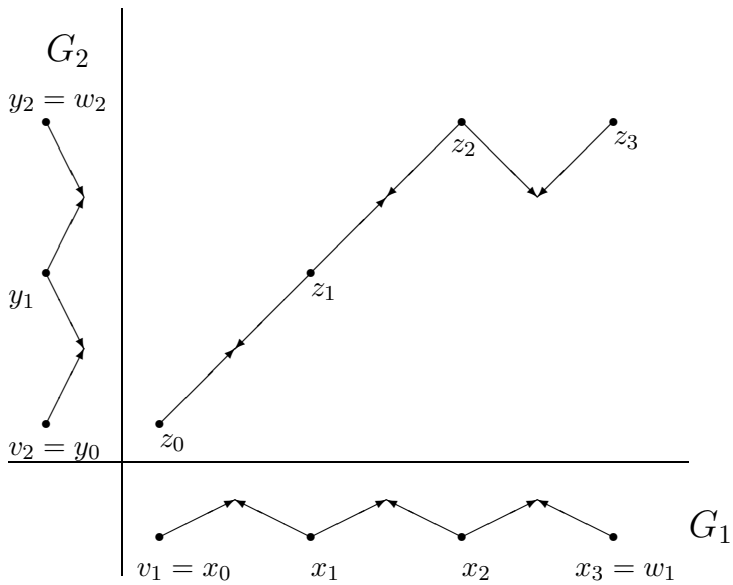

Figure 1. Connectedness.

(ii)$\Longrightarrow$(i) Given two vertices $v = (v_1, v_2)$, $w = (w_1, w_2) \in G$. Then there exist sequences $v_1 = x_0, x_1, \ldots, x_n = w_1$ and $v_2 = y_0, y_1, \ldots, y_m = w_2$, where $N_1^+(x_i) \cap N_1^+(x_{i+1}) \neq \emptyset$ for $0 \leq i < n$, and $N_2^+(y_i) \cap N_2^+(y_{i+1}) \neq \emptyset$ for $0 \leq i < m$. Let $m \leq n$ and set $z_i = (x_i, y_i)$ for $0 \leq i \leq m$ and (x_i, y_m) for $m < i \leq n$.

Then (v_1, v_2) and (w_1, w_2) are $R^+|R^-$-connected in G by $z_0, z_1, \ldots, z_n$. ■

McKenzie showed that these connectivity conditions ensure the uniqueness of the PFD with respect to cardinal multiplication:

Theorem 2.2 [7]. *Let G be an $R^+|R^-$- and $R^-|R^+$-connected finite graph. Then G is uniquely representable as a cardinal product of prime graphs, up to isomorphisms and the order of the factors.*

The first algorithm for the actual computation of such a factorization is due to Feigenbaum and Schäffer. They treat the case of reflexive, symmetric relations. In the language of graphs it reads as follows:

Theorem 2.3 [1]. *Let $G = (V, A)$ be an undirected, finite, connected graph. Then the prime factor decomposition of G with respect to the cardinal product can be found in polynomial time.*

This was extended by Imrich to the case of symmetric relations that are not necessarily reflexive, that is, to finite undirected graphs. (In this setting nonbipartiteness ensures $R^+|R^-$- and $R^-|R^+$-connectedness.)

Theorem 2.4 [3]. *Let $G = (V, A)$ be a connected, nonbipartite finite undirected graph. Then the prime factor decomposition of G with respect to the cardinal product can be found in polynomial time.*

The first step in these algorithms is the reduction of the graphs under consideration to so-called thin graphs. Subsequently the thin graphs are factored, and then this factorization is extended to the given graphs.

A graph is *thin* if no two vertices have the same out- and the same in-neighborhood. More formally we say two vertices are in the relation R if their out-neighborhoods as well as their in-neighborhoods coincide. Clearly R is an equivalence relation. A graph is thin if every equivalence class of R consists of only one vertex.

For the reduction to thin graphs we introduce the concept of quotient graphs. As usual the *quotient graph* G/R is defined as follows: the vertex set of G/R is the set of all equivalence classes $\{\overline{x} \mid x \in V(G)\}$ of $V(G)$ with respect to R, and $\langle \overline{x}, \overline{y} \rangle \in A(G/R)$ if there are vertices $a \in \overline{x}$, $b \in \overline{y}$ with $\langle a, b \rangle \in A(G)$.

It is clear that G/R is thin. In a thin graph it is possible that two different vertices with different in-neighborhoods have the same out-neighborhood and vice versa. Since the algorithm of the present paper essentially assumes that different vertices have different out-neighborhoods, we introduce the relation R^+: Two vertices of G are in the relation R^+ if their N^+-neighborhoods are the same.

R^- is defined analogously. Clearly R^+ and R^- are equivalence relations. A graph is then called R^+-*thin*, respectively R^--*thin*, if all equivalence classes of the relation R^+, respectively R^-, consist of just one element. For R^+- or R^--thin graphs we have the following result of Imrich and Klöckl:

Theorem 2.5 [5]. *Let $G = (V, A)$ be an $R^+|R^-$- connected R^+-thin graph, or an $R^-|R^+$-connected R^--thin graph. Then the prime factor decomposition of G with respect to the cardinal product is unique and can be found in polynomial time.*

In order to describe the subgraphs induced by the R^+-classes we introduce new notation. Let r and s be integers, where $r > 0,\ s \geq 0$. Then $R^+_{s,r}$ is the graph consisting of a complete subgraph K^d_s and $r - s$ other vertices that have empty in-neighborhoods and whose out-neighborhoods are exactly the vertices of the complete subgraph K^d_s.

Lemma 2.6. *The graphs induced by the R^+-classes are $R^+_{s,r}$-graphs.*

Proof. Let v be a vertex of $R^+_{s,r}$ with nonempty in-neighborhood. Then its out-neighborhood must be the entire R^+-class. Thus the vertices with non-empty out-neighborhood induce a complete subgraph K^d_s.

On the other hand, if v has empty in-neighborhood, then its out-neighborhood consists exactly of the vertices in K^d_s. ■

3. Computation of the G-Factorization from that of G/R^+

The goal of this paper is to generalize Theorem 2.5 in a way that we can apply it to graphs with a relaxed thinness condition. In order to do this we will simplify a given graph first, apply Theorem 2.5 to compute the PFD of the simplified graph and then extend this decomposition to the one of the given graph.

In this paper the simplified graphs are R^+-quotients, whereas R-quotients were used in [7, 1, 3]. Since the subgraphs induced by the R-classes have a much simpler structure than those induced by the R^+-classes the present setting is considerably harder.

Interestingly, G/R^+ need neither be R^+-thin nor thin as Figure 2 shows. The out-neighborhoods of d' and e' coincide and the same holds for their in-neighborhoods. Unfortunately, also the relation R^- can fail to be thin in such situations.

Compare Figures 2 and 3. They show a graph G whose quotient graphs G/R^+ and G/R^- are not thin. Nonetheless, after a finite number of R^+-contractions we must arrive at an R^+-thin graph; similarly for R^-.

The main aim of this section is the construction of the G-factorization from that of G/R^+. To do this we first consider the special case, where the PFD of G/R^+ consists of only two factors, say A_1 and B_1.

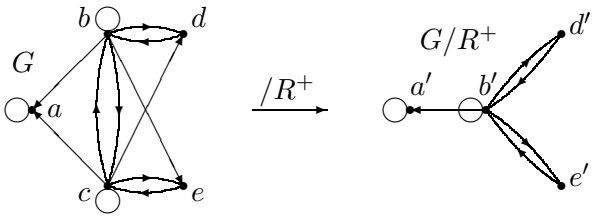

Figure 2. $N^+(b) = N^+(c)$.

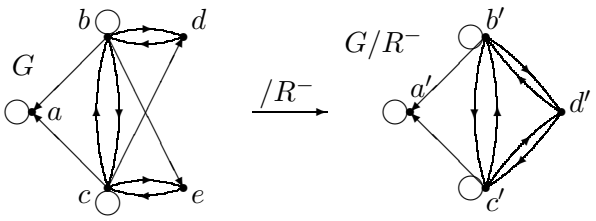

Figure 3. $N^-(d) = N^-(e)$.

Lemma 3.1. *Let G be a finite, thin $R^+|R^-$- and $R^-|R^+$-connected graph and $A_1 \times B_1$ the prime factor decomposition of G/R^+.*

Then G has at most two nontrivial prime divisors. This means that G is prime or that it has a prime factorization $A \times B$, where $A/R^+ = A_1$ and $B/R^+ = B_1$.

Proof. Let us assume that $A \times B \times C$ is a decomposition of G into three nontrivial factors. Then, by Lemma 4.1 in [5], $A/R^+ \times B/R^+ \times C/R^+$ is a decomposition of G/R^+. Since G is thin, no nontrivial factor of G is a K_s^d. Furthermore, Lemma 2.1 implies that all factors are $R^-|R^+$-connected. Thus, all in-neighborhoods in the factors are nonempty and none of the factors A, B, C is an $R^+_{s,r}$-graph with $s < r$. We conclude that each of the three factors of G/R^+ is nontrivial, contrary to the assumption that G/R^+ has exactly two prime factors. ■

In the sequel we try to obtain the decomposition of the graph G from the lemma by expanding the vertices of A_1 and B_1 to R^+-classes of possible G-factors. The question for an estimation of the number of different ways to do this will be answered by Lemma 3.2; it bounds this number polynomially.

We thus consider the situation where $A_1 \times B_1$ is the PFD of G/R^+ and label the vertex sets as follows: $V(A_1) = \{a_1, a_2, \ldots, a_l\}$, $V(B_1) = \{b_1, b_2, \ldots, b_m\}$ and $V(G/R^+) = V(A_1) \times V(B_1)$. $V(G/R^+) = \{v_1, \ldots, v_n\}$. In the proofs of the next lemmas we will use a matrix M that consists of all subgraphs of G that are induced by the vertices in R^+-classes in G. $M(a_i, b_j)$, the element in row i and column j of M, is the graph induced by the vertices in the R^+-class (a_i, b_j). Note that all $M(a_i, b_j)$ are $R^+_{s,r}$-graphs. The idea is that a decomposition $G = A \times B$ induces a decomposition $M = \mathbf{a} \times \mathbf{b}$, where $\mathbf{a}$ is a column-vector containing the graphs induced by the R^+-classes of A and $\mathbf{b}$ is a row-vector defined analogously. If G/R^+ is thin, then the matrix M is unique up to the order of the rows and the columns.

Lemma 3.2. *Let G, A_1, B_1, $V(A_1)$, $V(B_1)$ and M be defined as above. If G is not prime, then the following statements hold:*

(i) *M is the product of a column-vector $\mathbf{a}$ and a row-vector $\mathbf{b}$, therefore $rk(M) = 1$.*

(ii) *The number of M-decompositions $\mathbf{a} \times \mathbf{b}$ is bounded by n^2, where n is the number of vertices of G.*

Proof. (i) We define the column-vector $\mathbf{a}$ as $(D(a_1), D(a_2), \ldots, D(a_l))^t$, where $D(a_i)$ is the subgraph of A that is induced by the R^+-class (a_i) of A, and the vector $\mathbf{b}$ analogously. By Lemma 4.1 in [5] the following equations hold:

$$a \times b = \begin{pmatrix} D(a_1) \\ D(a_2) \\ \vdots \\ D(a_m) \end{pmatrix} \times \begin{pmatrix} D(b_1) & D(b_2) & \cdots & D(b_l) \end{pmatrix} =$$

$$\begin{pmatrix} M(a_1, b_1) & M(a_1, b_2) & \cdots & M(a_1, b_l) \\ M(a_2, b_1) & M(a_2, b_2) & \cdots & M(a_2, b_l) \\ \vdots & \vdots & \cdots & \vdots \\ M(a_m, b_1) & M(a_m, b_2) & \cdots & M(a_m, b_l) \end{pmatrix} = M$$

This proves (i).

(ii) Let $\mathbf{a}$ and $\mathbf{b}$ be as in (i). Clearly, for all i and j we have $D(a_i) \times D(b_j) = M(a_i, b_j)$. Note that all graphs in this equation are $R^+_{s,r}$-graphs and that $D(a_i)$ is uniquely defined if $D(b_j)$ and $M(a_i, b_j)$ are fixed.

Hence, the entry $D(b_1)$ in $\mathbf{b}$ determines all entries of $\mathbf{a}$ uniquely, and thus also all entries of $\mathbf{b}$. Since $D(b_1)$ is uniquely determined by its order and its number of loops, the number of different ways to define $D(b_1)$ is at most n^2. Clearly this also bounds the number of M-decompositions. ■

Note that the upper bound for the number of M-decompositions gives us also an upper bound for the number of ways to extend the vertex-sets of the G/R^+-factors to those of the perhaps existing G-factors denoted by A and B. For every one of these extension possibilities we have to check whether it leads to a G-decomposition. This means we have to test whether it is possible to define $E(A)$ and $E(B)$ such that A and B are nontrivial divisors of G. In Lemma 3.3 we describe this test.

Lemma 3.3. *Let G be a finite, thin, $R^+|R^-$-connected graph where all in-neighborhoods are nonempty. We assume G/R^+ to be thin, that $A_1 \times B_1$ is its prime factorization and that M is the matrix of the R^+-classes of G as above.*

Furthermore, let $\mathbf{a} \times \mathbf{b}$ be a (not necessarily unique) decomposition of M. Then we can check in polynomial time whether there are two graphs A and B such that the R^+-classes of A are the entries in $\mathbf{a}$, the R^+-classes of B the entries in $\mathbf{b}$, and $A \times B = G$.

Proof. Since the R^+-classes are known, $V(A)$ and $V(B)$ are determined as well as all edges between vertices in one and the same R^+-class. Therefore we only have to check if we can define the edges between vertices in different R^+-classes such that $A \times B = G$.

In order to do this we first compute the coordinates of the vertices in G. This is the main and most difficult part of the proof. But then we can define the edge sets of A and B by projecting $E(G)$ into A and B. Testing whether $A \times B \cong G$ can then be effected by counting the edges in $A \times B$ and in G.

Note that the thinness of G implies unique coordinatization of G. This means uniqueness of the A-, resp. B-layers, that is, the maximal subsets of G, where all vertices have the same A-, resp. B-coordinate.

Suppose that $V(A_1) = \{a_i \mid 1 \leq i \leq k\}$ and that $V(B_1) = \{b_j \mid 1 \leq j \leq l\}$. Then the vertex set of G/R^+ is $\{(a_i, b_j) \mid 1 \leq i \leq k,\ 1 \leq j \leq l\}$.

By the thinness of G/R^+ we know that this coordinatization is also unique. Without loss of generality we assume that $A_1 = A/R^+$ and $B_1 = B/R^+$. By a_i we also mean the set $\{x \in V(A) \mid x \in a_i\}$, analogously for b_j and (a_i, b_j) also denotes the set $\{x \in V(G) \mid x \in (a_i, b_j)\}$. In this sense the equation $(a_i, b_j) = a_i \times b_j$ holds, because R^+-classes in G are products of R^+-classes in the G divisors.

Let $l(a_i)$ be the subset of $V(G)$ that consists of all vertices in (a_i, b_j) with $j \in \{1, 2, \ldots, k\}$. Since $l(a_1) = V(B) \times a_1$, we call $l(a_1)$ a *generalized layer*. The $l(b_j)$ are defined analogously. Before layers are computed we list three important basic facts:

(i) Layers (as vertex sets) are subsets of generalized layers.

(ii) From the definition of R^+-classes we know that the in-neighborhoods are unions of R^+-classes.

(iii) The in-neighborhoods of every factor of G are nonempty.

Set

$$I_A(x) = \{j \mid \exists i \in \{1, 2, \ldots, l\} \ s.t. \ (a_i, b_j) \subset N^-(x)\}.$$

From (i)–(iii) we conclude, as shown below in detail, that:

Two vertices $x, y \in l(b_j)$ are in the same A-layer if and only if $I_A(x) = I_A(y)$.

Suppose first that $x, y \in l(b_j)$ are in the same A-layer. Then by definition $x_B = y_B$. From (ii) we infer that

$$N^-(x_B) = N^-(y_B) = \bigcup_{j \in I} b_j$$

for some subset I of $\{1, 2, \ldots, k\}$. One can obtain this information about the in-neighborhoods also from the generalized layers, which allows to prove: $I_A(x) = I = I_A(y)$. We will show the first equation in detail, the second can be proved analogously.

Given some $j_0 \in I$. By (iii) we know that $N^-(x_A) \neq \emptyset$. Hence, we have by (ii) that there is some $i_0 \in \{1, 2, \ldots, l\}$, such that $a_{i_0} \subset N^-(x_A)$. This implies $(b_{j_0}, a_{i_0}) \subset N^-(x)$. By the definition of $I_A(x)$ we conclude that $j_0 \in I_A(x)$.

For any given $j_0 \in I_A(x)$ there exists an i_0, such that $(b_{j_0}, a_{i_0}) \subset N^-(x)$. This implies that the Cartesian product of R^+-classes $b_{j_0} \times a_{i_0}$ is a subset of $N^-(x_B) \times N^-(x_A)$. Since $a_{i_0} \neq \emptyset$, $b_{j_0} \subset N^-(x_B)$. By the definition of I we know now that $j_0 \in I$, which completes the proof of the first equation.

For the proof of the second assertion we suppose that $I_A(x) = I_A(y)$. This implies $N^-(x_B) = N^-(y_B)$. Otherwise there would exist without loss of generality a $b_{j_0} \subset N^-(x_B) \setminus N^-(y_B)$. Furthermore there would also exist a $(b_{j_0}, a_{i_0}) \subset N^-(x) \setminus N^-(y)$, and hence a $j_0 \in I_A(x) \setminus I_A(y)$, which is not possible. Since x and y are in the same generalized layer $l(b_j)$, we have $N^+(x_B) = N^+(y_B)$. As G is thin, all divisors of G are thin. Thus x_B is y_B and the vertices x and y are in the same A-layer.

By statement (i) A-layers are maximal subsets L of generalized layers that satisfy the condition $I_A(x) = I_A(y)$ for all $x,\ y \in L$. B-layers are characterized analogously.

In the sequel we present an algorithm to compute the layers in $O(n^3)$ time, where $n = |V(G)|$. It consists of three parts:

1.) Compute the index set $I_A(x)$ for all vertices $x \in V(G)$.

For every x
  For every R^+-class (b_j, a_i) of G
    Take one $y \in (b_j, a_i)$ and check
    If $y \in N^-(x)$
      $I_A(x) = I_A(x) \cup \{j\}$

2.) Put the $I_A(x)$ in ascending order. The cardinality of the index sets is of course not larger than $|B_1|$. Hence the cost for this part is bounded by $n \cdot |B_1| \cdot \log(|B_1|)$, which is $O(n^2 \cdot \log(n))$.

3.) Compute the layers. To find the A-layer through $x \in V(G)$, we have to check for all y in the same generalized A-layer as x whether

$$I_A(x) = I_A(y).$$

The effort for checking the last equation is bounded by n, because the index sets are ordered. Totally we have a bound n^3 for the last part of the algorithm.

Since all coordinates of G with respect to A and B are known now, we can derive the edge sets of A and B by projection of $E(G)$. By this definition of the edge sets we clearly have

$$E(G) \subset E(A \times B)$$

which allows to prove $G = A \times B$ by checking the equation $|E(G)| = |E(A \times B)|$ $(= |E(A)| * |E(B)|)$. ■

Lemma 3.4. *Let G be a finite, thin, $R^+|R^-$-connected graph where all in-neighborhoods are nonempty and $A_1 \times \cdots \times A_l$ a PFD of the thin graph G/R^+. Then the PFD of G can be found in polynomial time $O(n^6)$, where n is the number of vertices of G.*

Proof. For $l = 2$ the statement of the corollary follows immediately from the two last lemmas. In general we take all prime factors A_i ($i \in I = \{1, 2, \ldots, l\}$) times the cofactor $G : A_i$ and try to blow up this decomposition of G/R^+ to a decomposition of G.

For any way of blowing A_{i_0} up (complexity $O(n^2)$ by Lemma 3.2) we have to determine all layers. From Lemma 3.3 we know that the complexity of this computation for one way of blowing up A_{i_0} to a graph A'_{i_0} ($A'_{i_0}/R^+ = A_{i_0}$) and for checking if A'_{i_0} is a divisor of G is bounded by $O(n^3)$.

In the case where the blown up graph A'_{i_0} over A_{i_0} is really a divisor of G it must be prime by Lemma 3.3 of [5]. We extract it from G and cancel i_0 from I.

Then we try to blow up decompositions $A_i \times A_j$ ($i, j \in I$) times the cofactor with respect to G/R^+ to decompositions of G. Note that the factor over $A_i \times A_j$, if existing, must be prime, because otherwise we had found factors over A_i and A_j. Analogously we proceed with factors consisting of three prime factors and so on.

Since G/R^+ has at most $\log_2(n)$ prime factors, it has at most n different divisors. For this reason Lemma 3.3 must be applied at most n^3 times. Hence the complexity for finding the PFD of G is bounded by $O(n^6)$. ■

Lemma 3.5. *Let G be a finite, $R^+|R^-$-connected graph where all in-neighborhoods are nonempty. If $G/R = \prod_{i \in I} A_i$ is a PFD, then the PFD of G can be found in $O(n^2)$ time, where n is the number of vertices of G.*

Proof. We can proceed as in the case of undirected graphs, compare [3]: Determine all minimal subsets S of $I = \{1, 2, \ldots, r\}$ so that there are graphs A and B with $G = A \times B$, $A/R = \prod_{j \in S} A_j$ and $B/R = \prod_{j \in J \setminus S} A_j$. From minimality of S we can conclude that A is prime. Now we can extract A and compute the other divisors of G analogously.

Those graphs A and B can be found by the blowing up procedure described in Lemma 13 of [3]. Since $r \leq \log_2(n)$, the lemma must be applied at most n times.

To get one R-class of A respectively B, we just have to do one gcd-computation of less than $\sqrt{n}$ natural numbers, because it is not possible

that both A and B consist of more than or of exactly $\sqrt{n}$ R-classes if G is not thin. We can get every other R-class of A and B by one division. Clearly the number of divisions is bounded by n.

The effort of the gcd-computation of less than $\sqrt{n}$ natural numbers smaller than n is $O(n)$. Hence, the total complexity of this procedure is bounded by $O(n^2)$. ■

Theorem 3.6. *Let G be a finite, thin, $R^+|R^-$–connected graph where all in-neighborhoods are nonempty. Furthermore we assume that all quotient graphs $T_{i+1} = T_i/R^+$ ($k \geq i \geq 0$ and $T_0 = G$), which we compute until we arrive at an R^+-thin graph T_k, are thin. Then the PFD of G can be computed in $O(n^7)$ time.*

Proof. Clearly the quotient-graphs T_k inherit the connectivity-properties of G. Hence we can use Theorem 2.5 to determine the PFD of the R^+-thin graph T_k. Now we have to compute the PFDs of the graphs T_i ($i \in \{0, 1, 2, \ldots, k-1\}$). The idea at this point is of course to win the T_i-factorization from the T_{i+1}-factorization ($i \in \{0, 1, 2, \ldots, k-1\}$). Since

$$T_{i+1} = T_i/R^+ \tag{1}$$

we apply Lemma 3.4. It tells us that the complexity for finding the PFD of T_i is bounded by $O(n^6)$.

After at most k ($k \leq n$) steps we know the PFD of G. Thus the complexity of our algorithm is bounded by $O(n^7)$. ■

Corollary 3.7. *If the first thinness condition of Theorem 3.6 is not satisfied, that is, if G is not thin, then the PFD of G can still be computed in $O(n^7)$ time.*

Proof. If G is not thin we simply apply the relation R before we start, if necessary, with the R^+-applications. To get the G-decomposition from the G/R-decomposition we just have to apply Lemma 3.5. ■

Corollary 3.7 is our most general result concerning PFDs with respect to directed cardinal products. It gives us a polynomial algorithm to compute the PFD of graphs that are $R^+|R^-$- and $R^-|R^+$-connected — those are shown to have a unique PFD by McKenzie (Theorem 2.2) — and additionally fulfill the assumption that all quotient graphs in Theorem 3.6 are thin.

Clearly all results in the third section, especially Theorem 3.6, also hold if we replace R^+ by R^-, N^+ by N^- and vice versa. In view of this theorem we assume that it should be possible to prove Theorem 3.6 without using the thinness conditions concerning the quotient graphs of G.

The following conjecture was proposed by one of the referees. We are sure that it is true, but could not find an easy proof.

Conjecture 3.8. For any digraph H there is a digraph G such that G/R^+ is isomorphic to H.

References

[1] J. Feigenbaum and A.A. Schäffer, *Finding the prime factors of strong direct product graphs in polynomial time*, Discrete Math. **109** (1992) 77–102.

[2] M. Hellmuth, W. Imrich, W. Klöckl and P. Stadler, *Approximate graph products*, Europ. J. Combinatorics **30** (2009) 1119–1133.

[3] W. Imrich, *Factoring cardinal product graphs in polynomial time*, Discrete Math. **192** (1998) 119–144.

[4] W. Imrich and S. Klavžar, Product graphs, Wiley-Interscience Series in Discrete Mathematics and Optimization (Wiley-Interscience, New York, 2000), Structure and recognition, With a foreword by Peter Winkler.

[5] W. Imrich and W. Klöckl, *Factoring directed graphs with respect to the cardinal product in polynomial time*, Discuss. Math. Graph Theory **27** (2007) 593–601.

[6] W. Klöckl, On the cardinal product, Ph.D. thesis (Montanuniversität Leoben, Austria, 2007).

[7] R. McKenzie, *Cardinal multiplication of structures with a reflexive relation*, Fund. Math. **70** (1971) 59–101.

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