Discussiones Mathematicae
Graph Theory 30 (2010) 449–459

THE COMPETITION NUMBERS OF JOHNSON GRAPHS

Suh-Ryung Kim*, Boram Park*†‡

Department of Mathematics Education
Seoul National University, Seoul 151–742, Korea

Yoshio Sano§

Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606–8502, Japan

Abstract

The competition graph of a digraph D is a graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of a graph G is defined to be the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and to characterize all graphs with given competition number k has been one of the important research problems in the study of competition graphs.

The Johnson graph $J(n, d)$ has the vertex set $\{v_X \mid X \in \binom{[n]}{d}\}$, where $\binom{[n]}{d}$ denotes the set of all d-subsets of an n-set $[n] = \{1, \ldots, n\}$, and two vertices v_{X_1} and v_{X_2} are adjacent if and only if $|X_1 \cap X_2| = d - 1$. In this paper, we study the edge clique number and the competition number of $J(n, d)$. Especially we give the exact competition numbers of $J(n, 2)$ and $J(n, 3)$.

*This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2008-531-C00004).

†The author was supported by Seoul Fellowship.

‡Corresponding author. *E-mail address:* kawa22@snu.ac.kr

§The author was supported by JSPS Research Fellowships for Young Scientists. The author was also supported partly by Global COE program "Fostering Top Leaders in Mathematics".

Keywords: competition graph, competition number, edge clique cover, Johnson graph.

2010 Mathematics Subject Classification: 05C69, 05C75.

1. Introduction

The *competition graph* $C(D)$ of a digraph D is a simple undirected graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there is a vertex v in D such that (x, v) and (y, v) are arcs of D. The notion of a competition graph was introduced by Cohen [3] as a means of determining the smallest dimension of ecological phase space (see also [4]). Since then, various variations have been defined and studied by many authors (see [11, 15] for surveys and [1, 2, 7, 8, 9, 10, 12, 14, 19, 20] for some recent results). Besides an application to ecology, the concept of competition graph can be applied to a variety of fields, as summarized in [17].

Roberts [18] observed that, for a graph G, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices is the competition graph of an acyclic digraph.

A subset S of the vertex set of a graph G is called a *clique* of G if the subgraph of G induced by S is a complete graph. For a clique S of a graph G and an edge e of G, we say *e is covered by S* if both of the endpoints of e are contained in S. An *edge clique cover* of a graph G is a family of cliques such that each edge of G is covered by some clique in the family. The *edge clique cover number* $\theta_E(G)$ of a graph G is the minimum size of an edge clique cover of G. We call an edge clique cover of G with the minimum size $\theta_E(G)$ a *minimum edge clique cover* of G. A *vertex clique cover* of a graph G is a family of cliques such that each vertex of G is contained in some clique in the family. The *vertex clique cover number* $\theta_V(G)$ of a graph G is the minimum size of a vertex clique cover of G. Dutton and Brigham [5] characterized the competition graphs of acyclic digraphs using edge clique covers of graphs.

Roberts [18] observed that the characterization of competition graphs is equivalent to the computation of competition numbers. It does not seem to be easy in general to compute $k(G)$ for a graph G, as Opsut [16] showed

that the computation of the competition number of a graph is an NP-hard problem (see [11, 13] for graphs whose competition numbers are known). For some special graph families, we have explicit formulae for computing competition numbers. For example, if G is a chordal graph without isolated vertices then $k(G) = 1$, and if G is a nontrivial triangle-free connected graph then $k(G) = |E(G)| - |V(G)| + 2$ (see [18]).

In this paper, we study the competition numbers of Johnson graphs. We denote an n-set $\{1, \ldots, n\}$ by $[n]$ and the set of all d-subsets of an n-set by $\binom{[n]}{d}$. The *Johnson graph* $J(n, d)$ has the vertex set $\{v_X \mid X \in \binom{[n]}{d}\}$, and two vertices v_{X_1} and v_{X_2} are adjacent if and only if $|X_1 \cap X_2| = d - 1$ (for reference, see [6]). For example, the Johnson graph $J(5, 2)$ is given in Figure 1.

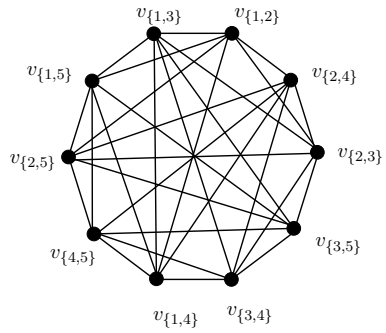

Figure 1. The Johnson graph $J(5, 2)$.

As it is known that $J(n, d) \cong J(n, n-d)$, we assume that $n \geq 2d$. Our main results are the following.

Theorem 1. *For $n \geq 4$, we have $k(J(n, 2)) = 2$.*

Theorem 2. *For $n \geq 6$, we have $k(J(n, 3)) = 4$.*

We use the following notation and terminology in this paper. For a digraph D, an ordering $v_1, v_2, \ldots, v_n$ of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D. For a digraph D and a vertex v of D, the *out-neighborhood* of v in D is the set $\{w \in V(D) \mid (v, w) \in A(D)\}$. A vertex in the out-neighborhood of a vertex v in a digraph D is called a *prey* of v in D. For simplicity, we denote the

out-neighborhood of a vertex v in a digraph D by $P_D(v)$ instead of usual notation $N_D^+(v)$. For a graph G and a vertex v of G, we define the (*open*) *neighborhood* $N_G(v)$ of v in G to be the set $\{u \in V(G) \mid uv \in E(G)\}$. We sometimes also use $N_G(v)$ to stand for the subgraph induced by its vertices.

2. A Lower Bound for the Competition Number of $J(n, d)$

In this section, we give lower bounds for the competition number of the Johnson graph $J(n, d)$.

Lemma 3. *Let n and d be positive integers with $n \geq 2d$. For any vertex x of the Johnson graph $J(n, d)$, we have $\theta_V(N_{J(n,d)}(x)) = d$.*

Proof. If $d = 1$, then $J(n, d)$ is a complete graph and the lemma is trivially true. Assume that $d \geq 2$. Take any vertex x in $J(n, d)$. Then $x = v_A$ for some $A \in \binom{[n]}{d}$. For any vertex v_A in $J(n, d)$, the set

$$S_i(v_A) := \{v_B \mid B = (A \setminus \{i\}) \cup \{j\} \text{ for some } j \in [n] \setminus A\}$$

forms a clique of $J(n, d)$ for each $i \in A$. To see why, take two distinct vertices v_B and v_C in $S_i(v_A)$. Then $B = (A \setminus \{i\}) \cup \{j\}$ and $C = (A \setminus \{i\}) \cup \{k\}$ for some distinct $j, k \in [n] \setminus A$. Clearly $|B \cap C| = d - 1$, and so v_B and v_C are adjacent in $J(n, d)$.

Take a vertex v_B in $N_{J(n,d)}(v_A)$. Then $B = (A \setminus \{i\}) \cup \{j\}$ for some $i \in A$ and $j \in [n] \setminus A$ and so $v_B \in S_i(v_A)$. Thus $\{S_i(v_A) \mid i \in A\}$ is a vertex clique cover of $N_{J(n,d)}(v_A)$. Thus $\theta_V(N_{J(n,d)}(v_A)) \leq d$. On the other hand,

$$|((A \setminus \{i\}) \cup \{j\}) \cap ((A \setminus \{i'\}) \cup \{j'\})| = d - 2$$

if $i, i' \in A$ and $j, j' \in [n] \setminus A$ satisfy $i \neq i'$ and $j \neq j'$ (such i, i', j, j' exist since $n \geq 2d \geq 4$). This implies that $\theta_V(N_{J(n,d)}(v_A)) \geq d$. Hence $\theta_V(N_{J(n,d)}(v_A)) = d$. ■

Opsut [16] gave a lower bound for the competition number of a graph G as follows:

$$k(G) \geq \min\{\theta_V(N_G(v)) \mid v \in V(G)\}.$$

Together with Lemma 3, we have $k(J(n, d)) \geq d$ for positive integers n and d satisfying $n \geq 2d$. The following theorem gives a better lower bound for $k(J(n, d))$ if $d \geq 2$.

Theorem 4. *For $n \geq 2d \geq 4$, we have $k(J(n,d)) \geq 2d-2$.*

Proof. Put $k := k(J(n,d))$. Then there exists an acyclic digraph D such that $C(D) = J(n,d) \cup I_k$, where $I_k = \{z_1, z_2, \ldots, z_k\}$ is a set of isolated vertices. Let $x_1, x_2, \ldots, x_{\binom{n}{d}}, z_1, z_2, \ldots, z_k$ be an acyclic ordering of D. Let $v_1 := x_{\binom{n}{d}}$ and $v_2 := x_{\binom{n}{d}-1}$. By Lemma 3, we have $\theta_V(N_{J(n,d)}(x_i)) = d$ for $i = 1, \ldots, \binom{n}{d}$. Thus v_i has at least d distinct prey in D, that is,

$$|P_D(v_i)| \geq d. \tag{2.1}$$

Since $x_1, x_2, \ldots, x_{\binom{n}{d}}, z_1, z_2, \ldots, z_k$ is an acyclic ordering of D, we have

$$P_D(v_1) \cup P_D(v_2) \subset I_k \cup \{v_1\}. \tag{2.2}$$

Moreover, we may claim the following:

Claim. For any two adjacent vertices v_{X_1} and v_{X_2} of $J(n,d)$, we have $|P_D(v_{X_1}) \setminus P_D(v_{X_2})| \geq d-1$.

Proof of Claim. Suppose that v_{X_1} and v_{X_2} are adjacent in $J(n,d)$. Then $|X_1 \cap X_2| = d-1$ and

$$|[n] \setminus (X_1 \cup X_2)| \geq 2d - |X_1| - |X_2| + |X_1 \cap X_2| = d-1.$$

We take $d-1$ elements from $[n] \setminus (X_1 \cup X_2)$, say $z_1, z_2, \ldots, z_{d-1}$, and put $X_1 \cap X_2 := \{y_1, y_2, \ldots, y_{d-1}\}$.

For each $1 \leq j \leq d-1$, we put $Z_j := X_1 \cup \{z_j\} \setminus \{y_j\}$. Then $|Z_j| = d$ and so v_{Z_j} is a vertex in $J(n,d)$. Note that $|Z_j \cap X_1| = d-1$ and $|Z_j \cap X_2| = d-2$. Thus v_{Z_j} is adjacent to v_{X_1} while it is not adjacent to v_{X_2}. Therefore

$$P_D(v_{X_1}) \cap P_D(v_{Z_j}) \neq \emptyset \quad \text{and} \quad P_D(v_{X_2}) \cap P_D(v_{Z_j}) = \emptyset.$$

This implies

$$P_D(v_{X_1}) \setminus P_D(v_{X_2}) \supseteq \bigcup_{j=1}^{d-1} \left(P_D(v_{X_1}) \cap P_D(v_{Z_j})\right), \tag{2.3}$$

and, trivially, for each $j \in \{1, \ldots, d-1\}$,

$$|P_D(v_{X_1}) \cap P_D(v_{Z_j})| \geq 1. \tag{2.4}$$

Note that $|Z_j \cap Z_i| = d-2$ for $i \neq j$. Therefore v_{Z_i} and v_{Z_j} are not adjacent and so $P_D(v_{Z_i}) \cap P_D(v_{Z_j}) = \emptyset$. Thus, for $i \neq j$,

$$(P_D(v_{X_1}) \cap P_D(v_{Z_i})) \cap \big(P_D(v_{X_1}) \cap P_D(v_{Z_j})\big) = \emptyset. \tag{2.5}$$

From (2.3), (2.4), and (2.5), it follows that

$$|P_D(v_{X_1}) \setminus P_D(v_{X_2})| \geq \sum_{j=1}^{d-1} |P_D(v_{X_1}) \cap P_D(v_{Z_j})| \geq d-1.$$

This completes the proof of the claim. □

Now suppose that v_1 and v_2 are not adjacent in $J(n,d)$. Then v_1 and v_2 do not have a common prey in D, that is,

$$P_D(v_1) \cap P_D(v_2) = \emptyset. \tag{2.6}$$

By (2.1), (2.2) and (2.6), we have

$$k+1 \geq |P_D(v_1) \cup P_D(v_2)| = |P_D(v_1)| + |P_D(v_2)| \geq 2d.$$

Hence $k \geq 2d-1 > 2d-2$.

Next suppose that v_1 and v_2 are adjacent in $J(n,d)$. Then v_1 and v_2 have at least one common prey in D, that is,

$$|P_D(v_1) \cap P_D(v_2)| \geq 1. \tag{2.7}$$

By the above claim,

$$|P_D(v_1) \setminus P_D(v_2)| \geq d-1 \quad \text{and} \quad |P_D(v_2) \setminus P_D(v_1)| \geq d-1. \tag{2.8}$$

Then

$$\begin{aligned} k+1 &\geq |P_D(v_1) \cup P_D(v_2)| \qquad \text{(by (2.2))} \\ &= |P_D(v_1) \setminus P_D(v_2)| + |P_D(v_2) \setminus P_D(v_1)| + |P_D(v_1) \cap P_D(v_2)| \\ &\geq (d-1) + (d-1) + 1 \qquad \text{(by (2.7) and (2.8))} \\ &= 2d-1. \end{aligned}$$

Hence it holds that $k \geq 2d-2$. ■

3. A MINIMUM EDGE CLIQUE COVER OF $J(n,d)$

In this section, we build a minimum edge clique cover of $J(n,d)$.

Given a Johnson graph $J(n,d)$, we define a family $\mathcal{F}_d^n$ of cliques of $J(n,d)$ as follows. For each $Y \in \binom{[n]}{d-1}$, we put

$$S_Y := \{v_X \mid X = Y \cup \{j\} \text{ for } j \in [n] - Y\}.$$

Note that S_Y is a clique of $J(n,d)$ with size $n-d+1$. We let

$$\mathcal{F}_d^n := \{S_Y \mid Y \in \binom{[n]}{d-1}\}. \tag{3.1}$$

Then it is not difficult to show that $\mathcal{F}_d^n$ is the collection of cliques of maximum size. Moreover the family $\mathcal{F}_d^n$ is an edge clique cover of $J(n,d)$. To see why, take any edge $v_{X_1}v_{X_2}$ of $J(n,d)$. Then $|X_1 \cap X_2| = d-1$ and both of v_{X_1} and v_{X_2} belong to the clique $S_{X_1 \cap X_2} \in \mathcal{F}_d^n$. Thus $\mathcal{F}_d^n$ is an edge clique cover of $J(n,d)$.

We will show that $\mathcal{F}_d^n$ is a minimum edge clique cover of $J(n,d)$. Prior to that, we present the following theorem. For two distinct cliques S and S' of a graph G, we say S and S' are *edge disjoint* if $|S \cap S'| \leq 1$.

Theorem 5. *$\theta_E(J(n,d)) = \binom{n}{d-1}$ and any minimum edge clique cover of $J(n,d)$ consists of edge disjoint maximum cliques.*

Proof. Let $\mathcal{E}$ be a minimum edge clique cover for $J(n,d)$, that is, $\theta_E(J(n,d)) = |\mathcal{E}|$. Since $\mathcal{F}_d^n$ is an edge clique cover with $|\mathcal{F}_d^n| = \binom{n}{d-1}$, we have $\theta_E(J(n,d)) \leq \binom{n}{d-1}$.

Now we show that $|\mathcal{E}| \geq \binom{n}{d-1}$. Since the size of a maximum clique is $n-d+1$, we have $|E(S)| \leq \binom{n-d+1}{2}$ for each $S \in \mathcal{E}$ where $E(S) = \binom{S}{2}$. Therefore,

$$|E(J(n,d))| \leq \sum_{S \in \mathcal{E}} |E(S)| \leq \binom{n-d+1}{2} \times |\mathcal{E}|, \tag{3.2}$$

and the first equality holds if and only if none of two distinct cliques in $\mathcal{E}$ have a common edge, and the second equality holds if and only if any element of $\mathcal{E}$ is a maximum clique in $J(n,d)$.

Since the Johnson graph $J(n,d)$ is a $d(n-d)$-regular graph and the number of vertices of $J(n,d)$ is $\binom{n}{d}$,

$$|E(J(n,d))| = \frac{1}{2}d(n-d) \times \binom{n}{d} = \binom{n-d+1}{2} \times \binom{n}{d-1}. \tag{3.3}$$

From (3.2) and (3.3), it follows that $\binom{n-d+1}{2} \times \binom{n}{d-1} \leq \binom{n-d+1}{2} \times |\mathcal{E}|$. Thus we have $\binom{n}{d-1} \leq |\mathcal{E}|$. Hence we can conclude that $\theta_E(J(n,d)) = \binom{n}{d-1}$.

Furthermore, two equalities in (3.2) must hold, and therefore any minimum edge clique cover of $J(n,d)$ consists of edge disjoint maximum cliques. ■

Since $|\mathcal{F}_d^n| = \binom{n}{d-1}$, the following corollary is an immediate consequence of Theorem 5:

Corollary 6. *The edge clique cover $\mathcal{F}_d^n$ of $J(n,d)$ defined in* (3.1) *is a minimum edge clique cover of $J(n,d)$.*

4. Proofs of Theorems 1 and 2

First, we define an order $\prec$ on the set $\binom{[n]}{d}$ as follows. Take two distinct elements X_1 and X_2 in $\binom{[n]}{d}$. Let $X_1 = \{i_1, i_2, \ldots, i_d\}$ and $X_2 = \{j_1, j_2, \ldots, j_d\}$ where $i_1 < \cdots < i_d$ and $j_1 < \cdots < j_d$. Then we define $X_1 \prec X_2$ if there exists $t \in \{1, \ldots, d\}$ such that $i_s = j_s$ for $1 \leq s \leq t-1$ and $i_t < j_t$. It is easy to see that $\prec$ is a total order.

Now we prove Theorem 1.

***Proof of Theorem* 1.** As $k(J(n,2)) \geq 2$ by Theorem 4, it remains to show $k(J(n,2)) \leq 2$. We define a digraph D as follows:

$$V(D) = V(J(n,2)) \cup I_2$$

where $I_2 = \{z_1, z_2\}$, and

$$\begin{aligned} A(D) = &\bigcup_{i=1}^{n-2} \left\{(x, v_{\{i+1,i+2\}}) \mid x \in S_{\{i\}} \in \mathcal{F}_2^n\right\} \\ &\cup \bigcup_{i=1}^{2} \left\{(x, z_i) \mid x \in S_{\{n-2+i\}} \in \mathcal{F}_2^n\right\}. \end{aligned}$$

Since the vertices of each clique in the edge clique cover $\mathcal{F}_2^n$ has a common prey in D, it holds that $C(D) = J(n,2) \cup I_2$. Each vertex in $S_{\{i\}}$ is denoted by v_X for some $X \in \binom{[n]}{2}$ which contains i. Then by the definition of $\prec$, $v_X \prec v_{\{i+1,i+2\}}$ for $i = 1, \ldots, n-2$. Thus, there exists an arc from a vertex x to a vertex y in D if and only if either $x = v_X$ and $y = v_Y$ with $X \prec Y$, or $x = v_X$ and $y = z_i$ with $X \in S_{\{n-1\}} \cup S_{\{n\}}$ and $i \in \{1,2\}$. Therefore D is acyclic. Thus we have $k(J(n,2)) \leq 2$ and this completes the proof. ■

***Proof of Theorem* 2.** By Theorem 4, we have $k(J(n,3)) \geq 4$. It remains to show $k(J(n,3)) \leq 4$. We define a digraph D as follows:

$$V(D) = V(J(n,3)) \cup I_4$$

where $I_4 = \{z_1, z_2, z_3, z_4\}$, and

$$\begin{aligned} A(D) = & \bigcup_{i=1}^{n-3} \bigcup_{j=i+1}^{n-2} \left\{(x, v_{\{i,j+1,j+2\}}) \mid x \in S_{\{i,j\}} \in \mathcal{F}_3^n\right\} \\ & \cup \bigcup_{i=1}^{n-3} \left\{(x, v_{\{i+1,i+2,i+3\}}) \mid x \in S_{\{i,n-1\}} \in \mathcal{F}_3^n\right\} \\ & \cup \bigcup_{i=1}^{n-4} \left\{(x, v_{\{i+1,i+2,i+4\}}) \mid x \in S_{\{i,n\}} \in \mathcal{F}_3^n\right\} \\ & \cup \bigcup_{i=1}^{3} \{(x, z_i) \mid x \in S_{\{n-4+i,n\}} \in \mathcal{F}_3^n\} \\ & \cup \{(x, z_4) \mid x \in S_{\{n-2,n-1\}} \in \mathcal{F}_3^n\}. \end{aligned}$$

It is easy to check that

$$\begin{aligned} \mathcal{F}_3^n = & \{S_{\{i,j\}} \mid i = 1, \ldots, n-3; j = i+1, \ldots, n-2\} \\ & \cup \{S_{\{i,n-1\}} \mid i = 1, \ldots, n-3\} \cup \{S_{\{i,n\}} \mid i = 1, \ldots, n-4\} \\ & \cup \{S_{\{n-3,n\}}, S_{\{n-2,n\}}, S_{\{n-1,n\}}\} \cup \{S_{\{n-2,n-1\}}\}. \end{aligned}$$

Thus $C(D) = J(n,3) \cup I_4$. Moreover, any vertex $x \in S_{\{i,j\}}$ is denoted by v_X for some $X \in \binom{[n]}{3}$ which contains i and j. By the definition of $\prec$,

$X \prec \{i, j+1, j+2\}$. In a similar manner, for x in other cliques in $\mathcal{F}_3^n$, we may show that $(x, y) \in A(D)$ if and only if either $x = v_X$ and $y = v_Y$ with $X \prec Y$, or $x = v_X$ and $y = z_i$ with $X \in S_{\{n-3,n\}} \cup S_{\{n-2,n\}} \cup S_{\{n-1,n\}} \cup S_{\{n-2,n-1\}}$ and $i \in \{1, 2, 3, 4\}$. Thus D is acyclic. Hence $k(J(n, 3)) \leq 4$. ■

5. Concluding Remarks

In this paper, we gave some lower bounds for the competition numbers of Johnson graphs, and computed the competition numbers of Johnson graphs $J(n, 2)$ and $J(n, 3)$. It would be natural to ask: What is the exact value of the competition number of a Johnson graph $J(n, 4)$ for $n \geq 8$? Eventually, what are the exact values of the competition numbers of the Johnson graphs $J(n, q)$ for $q \geq 5$?

References

[1] H.H. Cho and S.-R. Kim, *The competition number of a graph having exactly one hole*, Discrete Math. **303** (2005) 32–41.

[2] H.H. Cho, S.-R. Kim and Y. Nam, *On the trees whose 2-step competition numbers are two*, Ars Combin. **77** (2005) 129–142.

[3] J.E. Cohen, *Interval graphs and food webs: a finding and a problem*, Document 17696-PR, RAND Corporation (Santa Monica, CA, 1968).

[4] J.E. Cohen, Food webs and Niche space (Princeton University Press, Princeton, NJ, 1978).

[5] R.D. Dutton and R.C. Brigham, *A characterization of competition graphs*, Discrete Appl. Math. **6** (1983) 315–317.

[6] C. Godsil and G. Royle, Algebraic Graph Theory, Graduate Texts in Mathematics **207** (Springer-Verlag, 2001).

[7] S.G. Hartke, *The elimination procedure for the phylogeny number*, Ars Combin. **75** (2005) 297–311.

[8] S.G. Hartke, *The elimination procedure for the competition number is not optimal*, Discrete Appl. Math. **154** (2006) 1633–1639.

[9] G.T. Helleloid, *Connected triangle-free m-step competition graphs*, Discrete Appl. Math. **145** (2005) 376–383.

[10] W. Ho, *The m-step, same-step, and any-step competition graphs*, Discrete Appl. Math. **152** (2005) 159–175.

[11] S.-R. Kim, *The competition number and its variants*, in: Quo Vadis, Graph Theory, (J. Gimbel, J.W. Kennedy, and L.V. Quintas, eds.), Annals of Discrete Mathematics **55** (North-Holland, Amsterdam, 1993) 313–326.

[12] S.-R. Kim, *Graphs with one hole and competition number one*, J. Korean Math. Soc. **42** (2005) 1251–1264.

[13] S.-R. Kim and F.S. Roberts, *Competition numbers of graphs with a small number of triangles*, Discrete Appl. Math. **78** (1997) 153–162.

[14] S.-R. Kim and Y. Sano, *The competition numbers of complete tripartite graphs*, Discrete Appl. Math. **156** (2008) 3522–3524.

[15] J.R. Lundgren, *Food Webs, Competition Graphs, Competition-Common Enemy Graphs, and Niche Graphs*, in: Applications of Combinatorics and Graph Theory to the Biological and Social Sciences, IMH Volumes in Mathematics and Its Application **17** (Springer-Verlag, New York, 1989) 221–243.

[16] R.J. Opsut, *On the computation of the competition number of a graph*, SIAM J. Algebraic Discrete Methods **3** (1982) 420–428.

[17] A. Raychaudhuri and F.S. Roberts, *Generalized competition graphs and their applications*, Methods of Operations Research, **49** (Anton Hain, Königstein, West Germany, 1985) 295–311.

[18] F.S. Roberts, *Food webs, competition graphs, and the boxicity of ecological phase space*, in: Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976) (1978) 477–490.

[19] F.S. Roberts and L. Sheng, *Phylogeny numbers for graphs with two triangles*, Discrete Appl. Math. **103** (2000) 191–207.

[20] M. Sonntag and H.-M. Teichert, *Competition hypergraphs*, Discrete Appl. Math. **143** (2004) 324–329.

Received 14 April 2009
Revised 9 October 2009
Accepted 10 October 2009