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# THE COMPETITION NUMBERS OF JOHNSON GRAPHS

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### Abstract

The competition graph of a digraph D is a graph which has the same vertex set as D and has an edge between two distinct vertices xand y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number k(G) of a graph G is defined to be the smallest number of such isolated vertices. In general, it is hard to compute the competition number k(G) for a graph G and to characterize all graphs with given competition number k has been one of the important research problems in the study of competition graphs.

The Johnson graph J(n, d) has the vertex set  $\{v_X \mid X \in {[n] \choose d}\}$ , where  ${[n] \choose d}$  denotes the set of all *d*-subsets of an *n*-set  $[n] = \{1, \ldots, n\}$ , and two vertices  $v_{X_1}$  and  $v_{X_2}$  are adjacent if and only if  $|X_1 \cap X_2| = d - 1$ . In this paper, we study the edge clique number and the competition number of J(n, d). Especially we give the exact competition numbers of J(n, 2) and J(n, 3).

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## 1. INTRODUCTION

The competition graph C(D) of a digraph D is a simple undirected graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there is a vertex v in D such that (x, v) and (y, v) are arcs of D. The notion of a competition graph was introduced by Cohen [3] as a means of determining the smallest dimension of ecological phase space (see also [4]). Since then, various variations have been defined and studied by many authors (see [11, 15] for surveys and [1, 2, 7, 8, 9, 10, 12, 14, 19, 20] for some recent results). Besides an application to ecology, the concept of competition graph can be applied to a variety of fields, as summarized in [17].

Roberts [18] observed that, for a graph G, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* k(G) of a graph G to be the smallest number k such that G together with k isolated vertices is the competition graph of an acyclic digraph.

A subset S of the vertex set of a graph G is called a *clique* of G if the subgraph of G induced by S is a complete graph. For a clique S of a graph G and an edge e of G, we say e is covered by S if both of the endpoints of e are contained in S. An edge clique cover of a graph G is a family of cliques such that each edge of G is covered by some clique in the family. The edge clique cover number  $\theta_E(G)$  of a graph G is the minimum size of an edge clique cover of G. We call an edge clique cover of G with the minimum size  $\theta_E(G)$  a minimum edge clique cover of G. A vertex clique cover of a graph G is a family of cliques such that each vertex of G is contained in some clique in the family. The vertex clique cover number  $\theta_V(G)$  of a graph G is the minimum size of a vertex clique cover of G. Dutton and Brigham [5] characterized the competition graphs of acyclic digraphs using edge clique covers of graphs.

Roberts [18] observed that the characterization of competition graphs is equivalent to the computation of competition numbers. It does not seem to be easy in general to compute k(G) for a graph G, as Opsut [16] showed that the computation of the competition number of a graph is an NP-hard problem (see [11, 13] for graphs whose competition numbers are known). For some special graph families, we have explicit formulae for computing competition numbers. For example, if G is a chordal graph without isolated vertices then k(G) = 1, and if G is a nontrivial triangle-free connected graph then k(G) = |E(G)| - |V(G)| + 2 (see [18]).

In this paper, we study the competition numbers of Johnson graphs. We denote an *n*-set  $\{1, \ldots, n\}$  by [n] and the set of all *d*-subsets of an *n*-set by  $\binom{[n]}{d}$ . The Johnson graph J(n,d) has the vertex set  $\{v_X \mid X \in \binom{[n]}{d}\}$ , and two vertices  $v_{X_1}$  and  $v_{X_2}$  are adjacent if and only if  $|X_1 \cap X_2| = d - 1$  (for reference, see [6]). For example, the Johnson graph J(5,2) is given in Figure 1.



Figure 1. The Johnson graph J(5, 2).

As it is known that  $J(n, d) \cong J(n, n-d)$ , we assume that  $n \ge 2d$ . Our main results are the following.

**Theorem 1.** For  $n \ge 4$ , we have k(J(n,2)) = 2.

**Theorem 2.** For  $n \ge 6$ , we have k(J(n,3)) = 4.

We use the following notation and terminology in this paper. For a digraph D, an ordering  $v_1, v_2, \ldots, v_n$  of the vertices of D is called an *acyclic ordering* of D if  $(v_i, v_j) \in A(D)$  implies i < j. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D. For a digraph D and a vertex v of D, the *out-neighborhood* of v in D is the set  $\{w \in V(D) \mid (v, w) \in A(D)\}$ . A vertex in the out-neighborhood of a vertex v in a digraph D is called a *prey* of v in D. For simplicity, we denote the

out-neighborhood of a vertex v in a digraph D by  $P_D(v)$  instead of usual notation  $N_D^+(v)$ . For a graph G and a vertex v of G, we define the (open) neighborhood  $N_G(v)$  of v in G to be the set  $\{u \in V(G) \mid uv \in E(G)\}$ . We sometimes also use  $N_G(v)$  to stand for the subgraph induced by its vertices.

2. A Lower Bound for the Competition Number of J(n, d)

In this section, we give lower bounds for the competition number of the Johnson graph J(n, d).

**Lemma 3.** Let n and d be positive integers with  $n \ge 2d$ . For any vertex x of the Johnson graph J(n,d), we have  $\theta_V(N_{J(n,d)}(x)) = d$ .

**Proof.** If d = 1, then J(n, d) is a complete graph and the lemma is trivially true. Assume that  $d \ge 2$ . Take any vertex x in J(n, d). Then  $x = v_A$  for some  $A \in {\binom{[n]}{d}}$ . For any vertex  $v_A$  in J(n, d), the set

$$S_i(v_A) := \{ v_B \mid B = (A \setminus \{i\}) \cup \{j\} \text{ for some } j \in [n] \setminus A \}$$

forms a clique of J(n, d) for each  $i \in A$ . To see why, take two distinct vertices  $v_B$  and  $v_C$  in  $S_i(v_A)$ . Then  $B = (A \setminus \{i\}) \cup \{j\}$  and  $C = (A \setminus \{i\}) \cup \{k\}$  for some distinct  $j, k \in [n] \setminus A$ . Clearly  $|B \cap C| = d - 1$ , and so  $v_B$  and  $v_C$  are adjacent in J(n, d).

Take a vertex  $v_B$  in  $N_{J(n,d)}(v_A)$ . Then  $B = (A \setminus \{i\}) \cup \{j\}$  for some  $i \in A$  and  $j \in [n] \setminus A$  and so  $v_B \in S_i(v_A)$ . Thus  $\{S_i(v_A) \mid i \in A\}$  is a vertex clique cover of  $N_{J(n,d)}(v_A)$ . Thus  $\theta_V(N_{J(n,d)}(v_A)) \leq d$ . On the other hand,

$$|((A \setminus \{i\}) \cup \{j\}) \cap ((A \setminus \{i'\}) \cup \{j'\})| = d - 2$$

if  $i, i' \in A$  and  $j, j' \in [n] \setminus A$  satisfy  $i \neq i'$  and  $j \neq j'$  (such i, i', j, j' exist since  $n \geq 2d \geq 4$ ). This implies that  $\theta_V(N_{J(n,d)}(v_A)) \geq d$ . Hence  $\theta_V(N_{J(n,d)}(v_A)) = d$ .

Opsut [16] gave a lower bound for the competition number of a graph G as follows:

$$k(G) \ge \min\{\theta_V(N_G(v)) \mid v \in V(G)\}.$$

Together with Lemma 3, we have  $k(J(n,d)) \ge d$  for positive integers n and d satisfying  $n \ge 2d$ . The following theorem gives a better lower bound for k(J(n,d)) if  $d \ge 2$ .

**Theorem 4.** For  $n \ge 2d \ge 4$ , we have  $k(J(n,d)) \ge 2d - 2$ .

**Proof.** Put k := k(J(n,d)). Then there exists an acyclic digraph D such that  $C(D) = J(n,d) \cup I_k$ , where  $I_k = \{z_1, z_2, \ldots, z_k\}$  is a set of isolated vertices. Let  $x_1, x_2, \ldots, x_{\binom{n}{d}}, z_1, z_2, \ldots, z_k$  be an acyclic ordering of D. Let  $v_1 := x_{\binom{n}{d}}$  and  $v_2 := x_{\binom{n}{d}-1}$ . By Lemma 3, we have  $\theta_V(N_{J(n,d)}(x_i)) = d$  for  $i = 1, \ldots, \binom{n}{d}$ . Thus  $v_i$  has at least d distinct prey in D, that is,

$$(2.1) |P_D(v_i)| \ge d.$$

Since  $x_1, x_2, \ldots, x_{\binom{n}{d}}, z_1, z_2, \ldots, z_k$  is an acyclic ordering of D, we have

(2.2) 
$$P_D(v_1) \cup P_D(v_2) \subset I_k \cup \{v_1\}.$$

Moreover, we may claim the following:

**Claim.** For any two adjacent vertices  $v_{X_1}$  and  $v_{X_2}$  of J(n,d), we have  $|P_D(v_{X_1}) \setminus P_D(v_{X_2})| \ge d-1$ .

**Proof of Claim.** Suppose that  $v_{X_1}$  and  $v_{X_2}$  are adjacent in J(n,d). Then  $|X_1 \cap X_2| = d - 1$  and

$$|[n] \setminus (X_1 \cup X_2)| \ge 2d - |X_1| - |X_2| + |X_1 \cap X_2| = d - 1.$$

We take d - 1 elements from  $[n] \setminus (X_1 \cup X_2)$ , say  $z_1, z_2, \ldots, z_{d-1}$ , and put  $X_1 \cap X_2 := \{y_1, y_2, \ldots, y_{d-1}\}.$ 

For each  $1 \leq j \leq d-1$ , we put  $Z_j := X_1 \cup \{z_j\} \setminus \{y_j\}$ . Then  $|Z_j| = d$  and so  $v_{Z_j}$  is a vertex in J(n, d). Note that  $|Z_j \cap X_1| = d-1$  and  $|Z_j \cap X_2| = d-2$ . Thus  $v_{Z_j}$  is adjacent to  $v_{X_1}$  while it is not adjacent to  $v_{X_2}$ . Therefore

$$P_D(v_{X_1}) \cap P_D(v_{Z_j}) \neq \emptyset$$
 and  $P_D(v_{X_2}) \cap P_D(v_{Z_j}) = \emptyset$ .

This implies

(2.3) 
$$P_D(v_{X_1}) \setminus P_D(v_{X_2}) \supseteq \bigcup_{j=1}^{d-1} \left( P_D(v_{X_1}) \cap P_D(v_{Z_j}) \right)$$

and, trivially, for each  $j \in \{1, \ldots, d-1\}$ ,

(2.4) 
$$|P_D(v_{X_1}) \cap P_D(v_{Z_j})| \ge 1.$$

Note that  $|Z_j \cap Z_i| = d - 2$  for  $i \neq j$ . Therefore  $v_{Z_i}$  and  $v_{Z_j}$  are not adjacent and so  $P_D(v_{Z_i}) \cap P_D(v_{Z_j}) = \emptyset$ . Thus, for  $i \neq j$ ,

(2.5) 
$$(P_D(v_{X_1}) \cap P_D(v_{Z_i})) \cap (P_D(v_{X_1}) \cap P_D(v_{Z_j})) = \emptyset.$$

From (2.3), (2.4), and (2.5), it follows that

$$|P_D(v_{X_1}) \setminus P_D(v_{X_2})| \ge \sum_{j=1}^{d-1} |P_D(v_{X_1}) \cap P_D(v_{Z_j})| \ge d-1.$$

This completes the proof of the claim.

Now suppose that  $v_1$  and  $v_2$  are not adjacent in J(n, d). Then  $v_1$  and  $v_2$  do not have a common prey in D, that is,

$$(2.6) P_D(v_1) \cap P_D(v_2) = \emptyset.$$

By (2.1), (2.2) and (2.6), we have

$$k+1 \ge |P_D(v_1) \cup P_D(v_2)| = |P_D(v_1)| + |P_D(v_2)| \ge 2d.$$

Hence  $k \ge 2d - 1 > 2d - 2$ .

Next suppose that  $v_1$  and  $v_2$  are adjacent in J(n, d). Then  $v_1$  and  $v_2$  have at least one common prey in D, that is,

$$(2.7) |P_D(v_1) \cap P_D(v_2)| \ge 1.$$

By the above claim,

(2.8) 
$$|P_D(v_1) \setminus P_D(v_2)| \ge d-1$$
 and  $|P_D(v_2) \setminus P_D(v_1)| \ge d-1$ .

Then

$$k+1 \ge |P_D(v_1) \cup P_D(v_2)| \quad \text{(by (2.2))}$$
  
=  $|P_D(v_1) \setminus P_D(v_2)| + |P_D(v_2) \setminus P_D(v_1)| + |P_D(v_1) \cap P_D(v_2)|$   
 $\ge (d-1) + (d-1) + 1 \quad \text{(by (2.7) and (2.8))}$   
=  $2d-1.$ 

Hence it holds that  $k \ge 2d - 2$ .

3. A MINIMUM EDGE CLIQUE COVER OF J(n, d)

In this section, we build a minimum edge clique cover of J(n, d).

Given a Johnson graph J(n,d), we define a family  $\mathcal{F}_d^n$  of cliques of J(n,d) as follows. For each  $Y \in {[n] \choose d-1}$ , we put

$$S_Y := \{ v_X \mid X = Y \cup \{ j \} \text{ for } j \in [n] - Y \}.$$

Note that  $S_Y$  is a clique of J(n, d) with size n - d + 1. We let

(3.1) 
$$\mathcal{F}_d^n := \{ S_Y \mid Y \in \binom{[n]}{d-1} \}.$$

Then it is not difficult to show that  $\mathcal{F}_d^n$  is the collection of cliques of maximum size. Moreover the family  $\mathcal{F}_d^n$  is an edge clique cover of J(n, d). To see why, take any edge  $v_{X_1}v_{X_2}$  of J(n, d). Then  $|X_1 \cap X_2| = d - 1$  and both of  $v_{X_1}$  and  $v_{X_2}$  belong to the clique  $S_{X_1 \cap X_2} \in \mathcal{F}_d^n$ . Thus  $\mathcal{F}_d^n$  is an edge clique cover of J(n, d).

We will show that  $\mathcal{F}_d^n$  is a minimum edge clique cover of J(n, d). Prior to that, we present the following theorem. For two distinct cliques S and S' of a graph G, we say S and S' are *edge disjoint* if  $|S \cap S'| \leq 1$ .

**Theorem 5.**  $\theta_E(J(n,d)) = \binom{n}{d-1}$  and any minimum edge clique cover of J(n,d) consists of edge disjoint maximum cliques.

**Proof.** Let  $\mathcal{E}$  be a minimum edge clique cover for J(n, d), that is,  $\theta_E(J(n, d)) = |\mathcal{E}|$ . Since  $\mathcal{F}_d^n$  is an edge clique cover with  $|\mathcal{F}_d^n| = \binom{n}{d-1}$ , we have  $\theta_E(J(n, d)) \leq \binom{n}{d-1}$ .

Now we show that  $|\mathcal{E}| \ge {n \choose d-1}$ . Since the size of a maximum clique is n-d+1, we have  $|E(S)| \le {n-d+1 \choose 2}$  for each  $S \in \mathcal{E}$  where  $E(S) = {S \choose 2}$ . Therefore,

(3.2) 
$$|E(J(n,d))| \le \sum_{S \in \mathcal{E}} |E(S)| \le \binom{n-d+1}{2} \times |\mathcal{E}|,$$

and the first equality holds if and only if none of two distinct cliques in  $\mathcal{E}$  have a common edge, and the second equality holds if and only if any element of  $\mathcal{E}$  is a maximum clique in J(n, d).

Since the Johnson graph J(n, d) is a d(n-d)-regular graph and the number of vertices of J(n, d) is  $\binom{n}{d}$ ,

$$(3.3) \qquad |E(J(n,d))| = \frac{1}{2}d(n-d) \times \binom{n}{d} = \binom{n-d+1}{2} \times \binom{n}{d-1}.$$

From (3.2) and (3.3), it follows that  $\binom{n-d+1}{2} \times \binom{n}{d-1} \leq \binom{n-d+1}{2} \times |\mathcal{E}|$ . Thus we have  $\binom{n}{d-1} \leq |\mathcal{E}|$ . Hence we can conclude that  $\theta_E(J(n,d)) = \binom{n}{d-1}$ .

Furthermore, two equalities in (3.2) must hold, and therefore any minimum edge clique cover of J(n, d) consists of edge disjoint maximum cliques.

Since  $|\mathcal{F}_d^n| = \binom{n}{d-1}$ , the following corollary is an immediate consequence of Theorem 5:

**Corollary 6.** The edge clique cover  $\mathcal{F}_d^n$  of J(n,d) defined in (3.1) is a minimum edge clique cover of J(n,d).

# 4. Proofs of Theorems 1 and 2

First, we define an order  $\prec$  on the set  $\binom{[n]}{d}$  as follows. Take two distinct elements  $X_1$  and  $X_2$  in  $\binom{[n]}{d}$ . Let  $X_1 = \{i_1, i_2, \ldots, i_d\}$  and  $X_2 = \{j_1, j_2, \ldots, j_d\}$  where  $i_1 < \cdots < i_d$  and  $j_1 < \cdots < j_d$ . Then we define  $X_1 \prec X_2$  if there exists  $t \in \{1, \ldots, d\}$  such that  $i_s = j_s$  for  $1 \le s \le t - 1$  and  $i_t < j_t$ . It is easy to see that  $\prec$  is a total order.

Now we prove Theorem 1.

**Proof of Theorem 1.** As  $k(J(n, 2)) \ge 2$  by Theorem 4, it remains to show  $k(J(n, 2)) \le 2$ . We define a digraph D as follows:

$$V(D) = V(J(n,2)) \cup I_2$$

where  $I_2 = \{z_1, z_2\}$ , and

$$\begin{split} A(D) &= \bigcup_{i=1}^{n-2} \left\{ (x, v_{\{i+1, i+2\}}) \mid x \in S_{\{i\}} \in \mathcal{F}_2^n \right\} \\ &\cup \bigcup_{i=1}^2 \left\{ (x, z_i) \mid x \in S_{\{n-2+i\}} \in \mathcal{F}_2^n \right\} \end{split}$$

Since the vertices of each clique in the edge clique cover  $\mathcal{F}_2^n$  has a common prey in D, it holds that  $C(D) = J(n, 2) \cup I_2$ . Each vertex in  $S_{\{i\}}$  is denoted by  $v_X$  for some  $X \in {\binom{[n]}{2}}$  which contains i. Then by the definition of  $\prec$ ,  $v_X \prec v_{\{i+1,i+2\}}$  for  $i = 1, \ldots, n-2$ . Thus, there exists an arc from a vertex x to a vertex y in D if and only if either  $x = v_X$  and  $y = v_Y$  with  $X \prec Y$ , or  $x = v_X$  and  $y = z_i$  with  $X \in S_{\{n-1\}} \cup S_{\{n\}}$  and  $i \in \{1, 2\}$ . Therefore Dis acyclic. Thus we have  $k(J(n, 2)) \leq 2$  and this completes the proof.

**Proof of Theorem 2.** By Theorem 4, we have  $k(J(n,3)) \ge 4$ . It remains to show  $k(J(n,3)) \le 4$ . We define a digraph D as follows:

$$V(D) = V(J(n,3)) \cup I_4$$

where  $I_4 = \{z_1, z_2, z_3, z_4\}$ , and

$$A(D) = \bigcup_{i=1}^{n-3} \bigcup_{j=i+1}^{n-2} \left\{ (x, v_{\{i,j+1,j+2\}}) \mid x \in S_{\{i,j\}} \in \mathcal{F}_3^n \right\}$$
$$\cup \bigcup_{i=1}^{n-3} \left\{ (x, v_{\{i+1,i+2,i+3\}}) \mid x \in S_{\{i,n-1\}} \in \mathcal{F}_3^n \right\}$$
$$\cup \bigcup_{i=1}^{n-4} \left\{ (x, v_{\{i+1,i+2,i+4\}}) \mid x \in S_{\{i,n\}} \in \mathcal{F}_3^n \right\}$$
$$\cup \bigcup_{i=1}^{3} \left\{ (x, z_i) \mid x \in S_{\{n-4+i,n\}} \in \mathcal{F}_3^n \right\}$$
$$\cup \left\{ (x, z_4) \mid x \in S_{\{n-2,n-1\}} \in \mathcal{F}_3^n \right\}.$$

It is easy to check that

$$\begin{aligned} \mathcal{F}_{3}^{n} &= \{S_{\{i,j\}} \mid i = 1, \dots, n-3; j = i+1, \dots, n-2\} \\ &\cup \{S_{\{i,n-1\}} \mid i = 1, \dots, n-3\} \cup \{S_{\{i,n\}} \mid i = 1, \dots, n-4\} \\ &\cup \{S_{\{n-3,n\}}, S_{\{n-2,n\}}, S_{\{n-1,n\}}\} \cup \{S_{\{n-2,n-1\}}\}. \end{aligned}$$

Thus  $C(D) = J(n,3) \cup I_4$ . Moreover, any vertex  $x \in S_{\{i,j\}}$  is denoted by  $v_X$  for some  $X \in {[n] \choose 3}$  which contains *i* and *j*. By the definition of  $\prec$ ,  $X \prec \{i, j+1, j+2\}$ . In a similar manner, for x in other cliques in  $\mathcal{F}_3^n$ , we may show that  $(x, y) \in A(D)$  if and only if either  $x = v_X$  and  $y = v_Y$  with  $X \prec Y$ , or  $x = v_X$  and  $y = z_i$  with  $X \in S_{\{n-3,n\}} \cup S_{\{n-2,n\}} \cup S_{\{n-1,n\}} \cup S_{\{n-2,n-1\}}$ and  $i \in \{1, 2, 3, 4\}$ . Thus D is acyclic. Hence  $k(J(n, 3)) \leq 4$ .

### 5. Concluding Remarks

In this paper, we gave some lower bounds for the competition numbers of Johnson graphs, and computed the competition numbers of Johnson graphs J(n, 2) and J(n, 3). It would be natural to ask: What is the exact value of the competition number of a Johnson graph J(n, 4) for  $n \ge 8$ ? Eventually, what are the exact values of the competition numbers of the Johnson graphs J(n,q) for  $q \ge 5$ ?

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