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GRAPHS FOR n-CIRCULAR MATROIDS

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Abstract

We give "if and only if" characterization of graphs with the following property: given $n \geq 3$, edges of such graphs form matroids with circuits from the collection of all graphs with n fundamental cycles. In this way we refer to the notion of matroidal family defined by Simões-Pereira [2].

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1. Introduction

The connection between graph theory and matroid theory was first noticed by Whitney [1], who defined a matroid in order to generalize the idea of linear dependence. Simões-Pereira [2] introduced a notion of matroidal family. Among known examples of such families we find the collections of cycles and bicycles which give well-known cycle matroid [4] and bicircular matroid [5]. Cycles and bicycles are graphs with, respectively, one and two fundamental cycles. Thus, it is interesting whether we can define, for given $n \geq 3$, a matroidal family that consists of graphs from $\mathfrak{C}_n$, a family of all graphs with n fundamental cycles. One of the results of this paper is a theorem which explains why it is impossible. However, we may put a question the other way. Given $n \geq 3$, can we describe graphs, such that their edges form matroids with circuits from $\mathfrak{C}_n$? Positive answer to this question is the main result of this paper.

Next section we devote to remind basic facts about graphs and matroids. We recall a precise definition of matroidal family and a few important examples. Some facts concerning fundamental cycles are also given. Third section contains the main results of the paper and some technical lemmas.

2. Preliminaries

2.1. Graphs and matroids

At first we are going to recall some terminology connected with graphs. In this paper, all considered graphs are finite, undirected and simple (without loops or parallel edges). By a *cycle* we mean a closed path, with no repeated vertices except for the starting and ending vertices. We call a path *simple* if it has no repeated vertices. A *subdivision of an edge d* with endpoints $\{w, y\}$ is an operation in a graph such that the edge d is replaced by two new edges with endpoints $\{w, x\}$ and $\{x, y\}$, where x is a new vertex. By a *subdivision of a graph G* we mean a graph obtained from G by subdivisions of some edges in G. A *bridge* (also known as a cut-edge) of a graph G is an edge whose deletion increases the number of connected components of G. By a *pendant edge* of a graph G we mean an edge which is incident to a vertex of degree 1. Such a vertex is also called *pendant*. A *pendant cycle* in a graph G is a cycle where all vertices have degree 2, except one vertex with degree equal to 3. We say that a graph G is *homeomorphic* to a graph H if G and H have isomorphic subdivisions. If F_1 and F_2 are subgraphs of a graph G, their sum $F_1 \cup F_2$ is a graph which is built of all vertices and edges which belong to F_1 or F_2.

We use the following definition of a matroid:

Definition 1. A *matroid M* is an ordered pair $(E, \mathcal{C})$, where E is a finite set and $\mathcal{C}$ is a collection of subsets of E satisfying the following three conditions:

1. $\emptyset \notin \mathcal{C}$;
2. If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$;
3. If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exist $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

The members of the family $\mathcal{C}$ in the above definition are called the *circuits* of the matroid M. The set E is called the *ground* set of M. As we will see in the next subsection, we may use a set of edges of an arbitrary graph G as a ground set of a matroid M. In this case we say that M comes from a given graph G or is defined on G.

2.2. Matroidal families

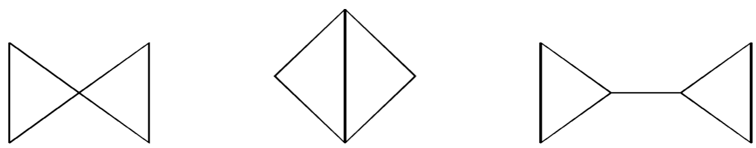
An interesting approach to the relations between graphs and matroids is widely presented in [3], and is based on the following definition:

Definition 2. A *matroidal family* of graphs is a non-empty collection $\mathcal{P}$ of connected graphs with the following property: given an arbitrary graph G, the edge sets of the subgraphs of G, which are isomorphic to some members of $\mathcal{P}$, are the circuits of a matroid on G.

To abridge further statements, we introduce the following: if G is a graph and $\mathcal{P}$ is a collection of connected graphs, we say that G *forms a* $\mathcal{P}$*-matroid* if we can define a matroid on G such that the circuits of this matroid are the sets of edges of subgraphs of G which are isomorphic to some members of $\mathcal{P}$.

Example 3. *There were only four matroidal families known until* 1978*:*

1. $\mathcal{P}_0$ *consists of one graph only, namely the complete graph on two points.*
2. $\mathcal{P}_1$ *consists of all cycles with at least* 3 *edges.* $\mathcal{P}_1$*-matroid formed by an arbitrary graph is a well-known cycle matroid* (*see* [4]).
3. $\mathcal{P}_2$ *consists of subdivisions of the following graphs* (*called bicycles*)*:*



   $\mathcal{P}_2$*-matroid formed by an arbitrary graph is called a bicircular matroid* (*see* [5]).
4. $\mathcal{P}_3$ *consists of the even cycles with at least* 4 *edges and bicycles with no even cycle.*

Andreae [6] showed the existence of a countably infinite series of matroidal families. Schmidt [7] extended this result and proved that there is uncountably many of such families. Proofs of these results are also available in [3].

2.3. Fundamental cycles

This section is devoted to the notion of a fundamental cycle. A general reference for fundamental cycles is [8]. Let us fix a connected graph G with e edges and v vertices. A *spanning tree* of G is its connected subgraph which contains all vertices of G and does not contain any cycle. The edges of a spanning tree are called the *branches* and all remaining edges are called the *chords*. It is obvious that every bridge of G must be a branch. Since the graph may have many different spanning trees, division of the set of edges of G into branches and chords depends on a choice of a spanning tree.

Let us fix a spanning tree T of G and a corresponding set $\{c_1, \ldots, c_k\}$ of chords. Notice that adding any chord c to the spanning tree T produces a unique cycle L in G. This cycle is called a *fundamental cycle* of G with respect to the chord c. The fundamental cycle with respect to the chord c contains only one chord, namely, the chord c. Moreover, this chord is not present in any other fundamental cycle with respect to T. Thanks to this one-to-one correspondence, fundamental cycles are also uniquely determined by a spanning tree. However, number of these cycles does not depend on the chosen spanning tree. Indeed, every spanning tree has $v - 1$ branches, thus the number of chords is equal to $k = e - (v - 1)$. This number is called a cyclomatic number of G and is denoted by $\mu(G)$. We assume that $\mu(\emptyset) = 0$.

A *spanning forest* is a notion that generalises the concept of a spanning tree to graphs which are not connected, i.e., it is a subgraph that consists of all spanning trees of connected components of a graph. A cyclomatic number for such a graph can also be counted. If G is a graph with e edges, v vertices and a connected components, then $\mu(G) = e - (v - a)$.

Notation 4. Let $\mathfrak{C}_n$ denote the family of all connected graphs without pendant edges and $\mu(G) = n$.

It is evident that for $i \neq j$ we have $\mathfrak{C}_i \cap \mathfrak{C}_j = \emptyset$.

3. Main Results

As we mentioned in Example 3, sets of edges of cycles as well as bicycles of an arbitrary graph G may be considered as circuits of a matroid whose ground set is the set of edges of G. Notice that cycles and bicycles are exactly those graphs which have one and two fundamental cycles, thus form

$\mathfrak{C}_1$ and $\mathfrak{C}_2$, respectively. Obviously $\mathfrak{C}_i = \mathcal{P}_i$, $i = 1, 2$. Can we go further and ask whether the collection $\mathfrak{C}_n$, for a certain $n \geq 3$, can be a matroidal family? Next theorem explains why the answer for this question is negative. Before we state it, we introduce the following definition.

Definition 5. Let $r, s \in \mathbb{N}$, $r, s \geq 2$, and suppose that H and F are disjoint graphs such that $H \in \mathfrak{C}_r$, $F \in \mathfrak{C}_s$. A graph which arises from H and F joined by a bridge we call an (r, s)-bridge-graph.

Notice that an (r, s)-bridge-graph belongs to $\mathfrak{C}_{r+s}$. Before we continue, we will need a definition of a certain operation on graphs.

Let G be a connected graph. We define a graph $\varphi(G)$ in the following way: delete, one by one, all pendant edges from G together with incident vertices of degree 1. In this way we receive a graph with no pendant edge. It is easy to see that $\varphi(G)$ is also connected. Notice that $\mu(G) = \mu(\varphi(G))$. Indeed, all pendant edges are branches of any spanning tree of G. If we remove them, we do not change the number of chords of G. Hence the number of fundamental cycles also remains the same.

Lemma 6. *If $G \in \mathfrak{C}_n$ and an edge d belongs to some cycle of G, then $G \setminus \{d\}$ has $n - 1$ fundamental cycles and $\varphi(G \setminus \{d\}) \in \mathfrak{C}_{n-1}$.*

Proof. If we remove d from G, the number of vertices in $G \setminus \{d\}$ remains unchanged, say v. On the other hand, if e is the number of edges of G, then graph $G \setminus \{d\}$ has $e - 1$ of them. Hence

$$\mu(\varphi(G \setminus \{d\})) = \mu(G \setminus \{d\}) = (e - 1) - v + 1 = e - v = \mu(G) - 1. \qquad \blacksquare$$

Lemma 7. *Let $F \in \mathfrak{C}_s$, $s \geq 2$, and w be an arbitrary vertex of F. There exist connected subgraphs F_1, F_2 of F, $F_1 \neq F_2$, with $s - 1$ fundamental cycles, such that w is a common vertex of F_1 and F_2, and w is their only possible pendant vertex.*

Proof. We choose two different chords c_1 and c_2 of F. Next we remove c_1 from F and, by Lemma 6, we obtain a subgraph $\hat{F}_1$ with $s - 1$ fundamental cycles. If $\hat{F}_1$ has pendant vertices, different from w, we delete them and edges incident to them, one by one, not changing the number of fundamental cycles. A graph obtained in this way we denote by F_1. We define F_2 in an analogous manner, removing c_2 from F. Thus F_1 and F_2 have different sets of fundamental cycles, and as a result, $F_1 \neq F_2$. Moreover, in this way, w is a common vertex of F_1 and F_2 and their only possible pendant vertex. ■

Theorem 8. *Given any $n \in \mathbb{N}$, $n \geq 3$, suppose that G is an (r,s)-bridge-graph such that $r, s \in \{2, \dots, n-1\}$ and $r + s = n + 1$. Then graph G does not form a $\mathfrak{C}_n$-matroid.*

Proof. To prove this, we will show a pair of subgraphs C_1 and C_2 of G, isomorphic to some members of $\mathfrak{C}_n$, which do not fulfil the third condition of Definition 1. Since G is an (r,s)-bridge-graph, it consists of disjoint subgraphs $F \in \mathfrak{C}_s$ and $H \in \mathfrak{C}_r$ connected by a bridge b, and we can suppose that $r \leq s$. Let us denote by w a common vertex of b and F. From the Lemma 7 we obtain connected subgraphs F_1 and F_2 of F with the properties: each of F_1 and F_2 has $s-1$ fundamental cycles, $F_1 \neq F_2$, w is their common vertex and their only possible pendant vertex. For $i = 1, 2$, we build graphs C_i from graphs F_i, H and the bridge b. It is easily seen that $C_1, C_2 \in \mathfrak{C}_n$ and $C_1 \neq C_2$. Since bridge b is in the intersection of C_1 and C_2, we look for a graph $C_3 \subseteq (C_1 \cup C_2) \setminus \{b\}$ such that $C_3 \in \mathfrak{C}_n$. However $(C_1 \cup C_2) \setminus \{b\}$ consists of two connected components and each of them has at most $n - 1$ fundamental cycles. This observation ends the proof. ■

The following is obvious.

Corollary 9. *Given any $n \in \mathbb{N}$, $n \geq 3$, suppose that G is an (r,s)-bridge-graph such that $r, s \in \{2, \dots, n-1\}$ and $r + s = n + 1$. If a graph E has a subgraph homeomorphic to G, then E does not form a $\mathfrak{C}_n$-matroid.*

It follows from the above theorem that for $n \geq 3$ we cannot construct a matroidal family which consists of all graphs with n fundamental cycles. However, for each such n, we are able to describe graphs forming $\mathfrak{C}_n$-matroids. Before we do that, we need a few lemmas. The proof of the following is straightforward.

Lemma 10. *An arbitrary edge of a connected graph G either belongs to a cycle of G or is a bridge.*

Lemma 11. *If $C_1, C_2 \in \mathfrak{C}_n$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*

Proof. Suppose that $C_1 \subsetneq C_2$. Hence there is an edge $d \in C_2$ such that $d \notin C_1$. Let us consider two cases. If d is a bridge, then C_1 is contained in one of two connected components of $C_2 \setminus \{d\}$. Each of them has at least 1 fundamental cycle (otherwise C_2 would have a pendant edge) and at most $n-1$ fundamental cycles. Hence $C_1 \in \mathfrak{C}_k$ for some $k \leq n-1$, which contradicts our assumptions. If d is not a bridge, then according to Lemma 10, it belongs to some cycle of C_2. It follows from Lemma 6 that $\varphi(C_2 \setminus \{d\}) \in \mathfrak{C}_{n-1}$. Since C_1 is contained in $C_2 \setminus \{d\}$ and has no pendant edge, it is also contained in $\varphi(C_2 \setminus \{d\})$. Thus $C_1 \in \mathfrak{C}_k$ for some $k \leq n-1$, which is impossible. This finishes the proof. ∎

Lemma 12. *Let $F \in \mathfrak{C}_n$ and $H \in \mathfrak{C}_n$ be subgraphs of a connected graph G such that they have at least one common vertex and $F \neq H$. Then $F \cup H \in \mathfrak{C}_k$ for a certain $k > n$.*

Proof. Let v denote the number of vertices that belong to both F and H, v_F denote the number of vertices that belong to F and do not belong to H, and v_H denote the number of vertices that belong to H and do not belong to F. Let e, e_F and e_H denote numbers of edges defined in an analogous manner. We will prove that $e_H > v_H$.

Let us notice that case $v_H = e_H = 0$ implies $H \subseteq F$, thus from Lemma 11 we obtain $H = F$, against our assumption. Thus we may assume $v_H > 0$. Since H is connected, every vertex of H that does not belong to F, must be incident to an edge, which also belongs to H and does not belong to F, hence $e_H \geq v_H$.

Let us suppose that $e_H = v_H$, and let R denote a set of those vertices which belong to F and are also incident to the edges that belong to H and do not belong to F. Next, let a denote a number of connected components of a graph $H \setminus F$, which is obtained from H after removing all edges belonging to F and the vertices belonging to F, except for those belonging to R. Every such a component contains at least one vertex from R, hence $a < |R|$ and we obtain

$$\begin{aligned} \mu(H \setminus F) &= e_H - (v_H + |R|) + a \\ &\leq e_H - (v_H + a) + a \\ &= e_H - v_H = 0. \end{aligned}$$

Therefore H has a pendant edges, as a graph with no fundamental cycles. This shows that $e_H > v_H$, and since $n = \mu(F) = e + e_F - v - v_F + 1$ we have

$$\begin{aligned}\mu(F \cup H) &= e + e_F + e_H - v - v_F - v_H + 1 \\ &= (e + e_F - v - v_F + 1) + e_H - v_H \\ &= n + e_H - v_H > n.\end{aligned}$$

Since it is evident that $F \cup H$ is connected and has no pendant edge, this ends the proof. ■

Theorem 13. *Given any $n, k \in \mathbb{N}$, $k > n \geq 3$, let $G \in \mathfrak{C}_k$ be a graph which does not have a subgraph homeomorphic to an (r, s)-bridge-graph for any $r, s \in \{2, \ldots, n-1\}$ such that $r + s = n + 1$. Then G forms a $\mathfrak{C}_n$-matroid.*

Proof. We have to check if all conditions of Definition 1 are fulfilled by arbitrary subgraphs C_1, C_2 of G such that $C_1, C_2 \in \mathfrak{C}_n$. Condition 1 is fulfilled because we assumed that $\mu(\emptyset) = 0$. Condition 2 is exactly the assertion of Lemma 11. Now, assume that C_1 and C_2 have common edges $\{d_1, \ldots, d_j\}$, $j \in \mathbb{N}$, and $C_1 \neq C_2$. It suffices to show that for all $i \in \{1, \ldots, j\}$ a graph $(C_1 \cup C_2) \setminus \{d_i\}$ contains a graph $C_3 \in \mathfrak{C}_n$. Let us fix $i \in \{1, \ldots, j\}$ and consider two cases:

1. If d_i is not a bridge of the graph $C_1 \cup C_2$, then according to Lemma 10, it belongs to some cycle of $C_1 \cup C_2$. From Lemma 12, it follows that $C_1 \cup C_2 \in \mathfrak{C}_m$, where $m > n$. By this and Lemma 6 we obtain that $\varphi(C_1 \cup C_2 \setminus \{d_i\}) \in \mathfrak{C}_l$, for some $l \geq n$. Hence there is a graph $C_3 \in \mathfrak{C}_n$ such that $C_3 \subset \varphi(C_1 \cup C_2 \setminus \{d_i\}) \subset (C_1 \cup C_2) \setminus \{d_i\}$.
2. Suppose d_i is a bridge in $C_1 \cup C_2$. It follows from Lemma 12 that $C_1 \cup C_2$ has at least $n+1$ fundamental cycles. Because of the assumption that G does not contain a subgraph homeomorphic to (r, s)-bridge-graph with $n+1$ fundamental cycles, d_i can only be a bridge between a pendant cycle connected with a certain simple path and the remaining part of $C_1 \cup C_2$. It means that d_i divides $C_1 \cup C_2$ into two connected components: a pendant cycle connected with a simple path, which has 1 fundamental cycle, and a subgraph F with at least n fundamental cycles. Hence there exist $C_3 \in \mathfrak{C}_n$ such that $C_3 \subseteq F \subseteq (C_1 \cup C_2) \setminus \{d_i\}$.

This completes the proof. ■

By an analogy to cases $n = 1, 2$, matroids described in the above theorem we will call an *n-circular matroids*.

Corollary 14. *Let $G \in \mathfrak{C}_k$, $k > 3$, be a graph having a vertex w, with the following properties:*

1. *any two cycles of G are disjoint or w is their only common vertex,*
2. *for any cycle L of G, if w does not belong to L, then there exist a vertex $u \in L$ such that $\{u, w\}$ are endpoints of a bridge.*

Then G forms a $\mathfrak{C}_n$-matroid for $3 \leq n < k$.

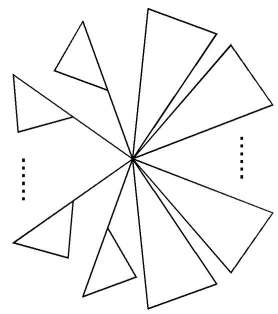


Figure 1. An example of a graph that meets conditions of Corollary 14.

Proof. To prove this corollary, we will check that all assumptions of Theorem 13 are fulfilled. For any $n \in \{3, \ldots, k-1\}$ let us fix $r, s \in \{2, \ldots, n-1\}$ such that $r+s = n+1$ and suppose, on the contrary, that G has a subgraph E homeomorphic to an (r, s)-bridge-graph. Let us denote a subgraph of E which belongs to $\mathfrak{C}_r$ by H and, similarly, a subgraph of E which belongs to $\mathfrak{C}_s$ by F. It is immediate that every connected subgraph of G without pendant edges and with at least 2 fundamental cycles must have w as its vertex. Hence H and F have a common vertex w, what contradicts the fact that they are disjoint. ∎

Corollary 15. *Let $W \in \mathfrak{C}_k$, $k \geq 3$, be a wheel, $L \in \mathfrak{C}_1$, and assume that W and L are disjoint graphs. A graph G which arises from W and L joined by a bridge b forms $\mathfrak{C}_n$-matroid for $3 \leq n < k+1$.*

Proof. To prove this, for any $n \in \{3, \ldots, k\}$ let us fix $r, s \in \{2, \ldots, n-1\}$ such that $r+s = n+1$ and suppose that G has a subgraph E homeomorphic to an (r, s)-bridge-graph. Denote by H and F subgraphs of E which belong to $\mathfrak{C}_r$ and $\mathfrak{C}_s$, respectively. Notice that any two cycles of W have at least one common vertex. Since H and F are disjoint, L must be a unique cycle of H or F. This is a contradiction because both H and F should have at least two fundamental cycles. ∎

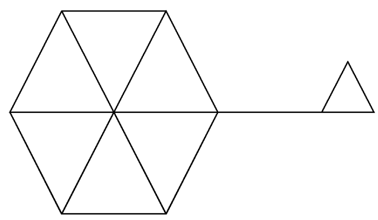


Figure 2. An example of a graph that meets conditions of Corollary 15.

Corollary 16. *Let $W \in \mathfrak{C}_k$, $k \geq 6$, be a wheel, $L_1, L_2 \in \mathfrak{C}_1$, and assume that W, L_1 and L_2 are disjoint graphs. Let G be a graph which arises from W, L_1 and L_2 in the following way:*

1. *L_1 and L_2 are joined with W by bridges b_1 and b_2, respectively,*
2. *b_i is incident to a vertex w_i of W for $i = 1, 2$,*
3. *w_1 and w_2 are adjacent and neither of them is a hub.*

Then G forms $\mathfrak{C}_n$-matroid for $k - 1 \leq n < k + 2$, however does not form $\mathfrak{C}_n$-matroid for $3 \leq n \leq k - 2$.

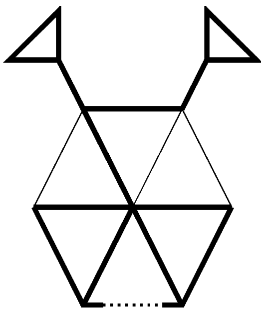


Figure 3. A graph that meets conditions of Corollary 16.

Proof. Our procedure will be to find for which r and s graph G has a subgraph E homeomorphic to an (r, s)-bridge-graph. Denote by H and F subgraphs of E which belong to $\mathfrak{C}_r$ and $\mathfrak{C}_s$, respectively. Any two cycles of W have at least one common vertex, and since subgraphs H and F are disjoint, then one of them, say F, does not have common cycles with W. By the fact that F has at least two fundamental cycles, then L_1 and L_2 are subgraphs of F. Since H also has at least two fundamental cycles, a hub of W belongs to H. By this and from connectivity we infer that F is built out (exclusively) of the following components: L_1, b_1, L_2, b_2, an edge with endpoints w_1 and w_2. As a result we obtain $r = 2$. Now it is easily seen that we can find subgraph $H \in \mathfrak{C}_s$ of W, disjoint with F, if and only if $s \in \{2, \dots, k-3\}$. Moreover, for those values of s it is easy to find a bridge joining H and F described above. On account of Theorems 8 and 13 we obtain both assertions. ∎

References

[1] H. Whitney, *On the abstract properties of linear dependence*, Amer. J. Math. **57** (1935) 509–533.

[2] J.M.S. Simões-Pereira, *On matroids on edge sets of graphs with connected subgraphs as circuits* II, Discrete Math. **12** (1975) 55–78.

[3] J.M.S. Simões-Pereira, *Matroidal Families of Graphs*, in: N. White (ed.) Matroid Applications (Cambridge University Press, 1992).

[4] J.G. Oxley, Matroid Theory (Oxford University Press, 1992).

[5] L.R. Matthews, *Bicircular matroids*, Quart. J. Math. Oxford **28** (1977) 213–228.

[6] T. Andreae, *Matroidal families of finite connected nonhomeomorphic graphs exist*, J. Graph Theory **2** (1978) 149–153.

[7] R. Schmidt, *On the existence of uncountably many matroidal families*, Discrete Math. **27** (1979) 93–97.

[8] J.L. Gross and J. Yellen, Handbook of Graph Theory (CRC Press, 2004).

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