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# A CHARACTERIZATION OF $(\gamma_t, \gamma_2)$ -TREES \*

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#### Abstract

Let  $\gamma_t(G)$  and  $\gamma_2(G)$  be the total domination number and the 2domination number of a graph G, respectively. It has been shown that:  $\gamma_t(T) \leq \gamma_2(T)$  for any tree T. In this paper, we provide a constructive characterization of those trees with equal total domination number and 2-domination number.

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### 1. INTRODUCTION

Let G = (V(G), E(G)) be a simple graph with vertex set V(G) and edge set E(G). The open neighborhood, the closed neighborhood and the degree of a vertex  $v \in V(G)$  are denoted by  $N_G(v) = \{u \in V(G) \mid uv \in E(G)\},$  $N_G[v] = N_G(v) \cup \{v\}$  and  $deg_G(v) = |N_G(v)|$ , respectively. For  $u \in V(G)$ , u is a leaf of G if  $deg_G(u) = 1$  and a support vertex of G if u has a leaf as its neighbor in G. For a pair of vertices  $u, v \in V(G)$ , the distance  $d_G(u, v)$ of u and v is the length of a shortest uv-path in G. The diameter of G is  $d(G) = \max\{d_G(u, v) : u, v \in V(G)\}.$ 

For any set  $S \subseteq V(G)$ , the subgraph induced by S is denoted by G[S]and we write G - S for G[V(G) - S]. For convenience, we write G - v for

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 $G - \{v\}$  for  $v \in V(G)$ . For any edge  $xy \in E(G)$ , we use G - xy to denote the subgraph induced by  $E(G) - \{xy\}$ .

Total domination in graphs was introduced by Cockayne *et al.* [3]. A subset  $S \subseteq V(G)$  is a total dominating set (denoted by TDS) if every vertex of V(G) has at least one neighbor in S. The total domination number (denoted by  $\gamma_t(G)$ ) is the minimum cardinality among the total dominating sets of G. The total dominating set of G with cardinality  $\gamma_t(G)$  will be called a  $\gamma_t$ -set of G. For a survey on total domination in graphs one can refer to Henning [12].

Let p be a positive integer. In [6], Fink and Jacobson introduced the concept of p-domination. A p-dominating set of G is a subset S of V(G) such that every vertex not in S has at least p neighbors in S. The p-domination number  $\gamma_p(G)$  is the minimum cardinality of a p-dominating set of G. The p-dominating set of G with cardinality  $\gamma_p(G)$  will be called a  $\gamma_p$ -set of G. Note that p-domination is the classic domination when p = 1. For any  $S, T \subseteq V(G)$ , S p-dominates T in G if every vertex of T not in S has at least p neighbors in S.

An area of research in domination of graphs that has received considerable attention is the characterization of classes of graphs with equal domination parameters. For any two graph parameters  $\lambda$  and  $\mu$ , G is called a  $(\lambda, \mu)$ -graph if  $\lambda(G) = \mu(G)$ . Characterizing the  $(\lambda, \mu)$ -graphs has been investigated in many papers (for example [1, 4, 7, 11, 13]).

In [8], Haynes *et al.* showed that for all trees the total domination number is equal or less than the 2-domination number, and they also gave a necessary condition for all trees with equal total domination number and 2-domination number. In this paper, we give a constructive characterization of trees with equal total domination number and 2-domination number.

### 2. A CHARACTERIZATION

Let  $P_n = u_1 \cdots u_n$   $(n \ge 1)$  be a path with vertex set  $\{u_1, \ldots, u_n\}$  and K(t) $(t \ge 2)$  be the tree obtained from a star  $K_{1,t}$  with support vertex u by adding a path  $P_2$  to every leaf of  $K_{1,t}$ . Denote u by cent(K(t)). For convenience, we denote a path  $P_4$  by K(1) and let cent(K(1)) represent one leaf of  $P_4$ .

To state the characterization of  $(\gamma_t, \gamma_2)$ -trees, we introduce the six types of operations.

**Type-1 operation:** Attach a path  $P_1$  to each of the two vertices u, w of a tree T, respectively, where u, w locate at a component  $P_l$  of T - xy for some edge xy such that either x is in a  $\gamma_2$ -set of T and  $P_l = P_4 = uvwx$  or y is in a  $\gamma_2$ -set of T and  $P_l = P_5 = uvwxx'$ .

**Type-2 operation:** Attach a path  $P_2$  to a vertex v of a tree T by joining one leaf of  $P_2$  to v, where v is a vertex such that T - v has a component  $P_2$ .

**Type-3 operation:** Attach  $t (\geq 1)$  paths  $P_3$  to a vertex v of a tree T by joining one leaf of each  $P_3$  to v, where v is a vertex such that either T - v has a component  $P_2$  or T - v has two components  $P_1$  and  $P_3$  that a leaf of  $P_3$  is adjacent to v in T.

**Type-4 operation:** Attach a path  $P_3$  to a vertex v of a tree T by joining its support vertex to v, where v is a vertex such that v is not contained in any  $\gamma_t$ -set of T and T - v has a component  $P_3$  that one of its leaves is adjacent to v in T.

**Type-5 operation:** Attach a tree K(t)  $(t \ge 1)$  to a vertex v of a tree T by joining cent(K(t)) to v, where v is in a  $\gamma_2$ -set of T if t = 1.

**Type-6 operation:** Attach a path  $P_5$  to a vertex v of a tree T by joining one of its support vertices to v, where v is a vertex such that T - v has a component  $H \in \{P_2, P_3, P_5\}$  and v is adjacent to a support vertex of H if  $H = P_5$ .

From the survey on total domination in graphs [12], it is hard to recognize whether a vertex v is in no  $\gamma_t$ -set or no  $\gamma_2$ -set.

Let  $\mathscr{A}$  be the family of trees with equal total domination number and 2-domination number, that is

 $\mathscr{A} = \{T : T \text{ is a tree satisfying } \gamma_t(T) = \gamma_2(T)\}.$ 

We also define the family  $\mathcal{B}$  as:

 $\mathscr{B} = \{T: T \text{ is obtained from } P_3 \text{ by a finite sequence of operations of Type-}i, where <math>1 \leq i \leq 6\}.$ 

We shall show that

Theorem 1.  $\mathscr{A} = \mathscr{B} \cup \{P_2\}.$ 

#### 3. The Proof of Theorem 1

We need some known results.

**Lemma 2** ([8]). Let T be a tree without isolated vertices, then  $\gamma_t(T) \leq \gamma_2(T)$ .

**Lemma 3** ([2]). Every 2-dominating set of a graph G contains all leaves of G.

**Lemma 4** ([8]). If T is a tree satisfying  $\gamma_t(T) = \gamma_2(T)$ , then every support vertex of T is adjacent to at most two leaves.

Let T be a rooted tree. For every  $v \in V(T)$ , let C(v) and D(v) denote the set of children and descendants of v, respectively, and  $D[v] = D(v) \cup \{v\}$ . Define

 $C'(v) = \{u \in C(v) : \text{every vertex of } D[u] \text{ has distance at most two}$ from  $v \text{ in } T\}.$ 

By Lemma 4, each vertex of C'(v) has degree at most three. Hence we can partition C'(v) into  $C'_1(v), C'_2(v), C'_3(v)$  such that every vertex of  $C'_i(v)$  has degree i in T, i = 1, 2 or 3.

**Lemma 5.** Let T be a rooted tree satisfying  $\gamma_t(T) = \gamma_2(T)$  and  $w \in V(T)$ . We have

(1) If  $C'_2(w) \neq \emptyset$ , then  $C'_1(w) = C'_3(w) = \emptyset$ .

(2) If 
$$C'_{3}(w) \neq \emptyset$$
, then  $C'_{1}(w) = C'_{2}(w) = \emptyset$  and  $|C'_{3}(w)| = 1$ .

(3) If  $C(w) = C'(w) \neq C'_1(w)$ , then  $C'_1(w) = C'_3(w) = \emptyset$ .

**Proof.** Let  $C'_1(w) = \{x_1, \ldots, x_r\}, C'_2(w) = \{y_1, \ldots, y_s\}$  and  $C'_3(w) = \{z_1, \ldots, z_t\}$ . Then  $|C'_1(w)| = r$ ,  $|C'_2(w)| = s$  and  $|C'_3(w)| = t$ . For each  $i = 1, \ldots, t$ , let  $u_i$  be a leaf adjacent with  $z_i$  in T. Let  $T' = T - \{x_1, \ldots, x_r, u_1, \ldots, u_t\}$ .

(1). We prove that if  $s \ge 1$  then r + t = 0. Assume  $r + t \ge 1$ . Since  $s \ge 1$ , we can choose a  $\gamma_2$ -set D of T such that  $w \in D$ , and a  $\gamma_t$ -set S' of T' such that  $w \in S'$ . It is not difficult to check that  $D - \{x_1, \ldots, x_r, u_1, \ldots, u_t\}$  is a 2-dominating set of T' and S' is a TDS of T. Hence,

$$\begin{aligned} \gamma_t(T') &= |S'| \ge \gamma_t(T) = \gamma_2(T) \\ &= |D| > |D - \{x_1, \dots, x_r, u_1, \dots, u_t\}| \ge \gamma_2(T'), \end{aligned}$$

a contradiction with Lemma 2.

(2) and (3). Suppose either  $C'_3(w) \neq \emptyset$  or  $C(w) = C'(w) \neq C'_1(w)$ . Then  $s + t \ge 1$ . Choose a  $\gamma_t$ -set S' of T' such that  $w \in S'$ . Then S' is also a TDS of T. Hence  $\gamma_t(T') = |S'| \ge \gamma_t(T)$ . By the definition of  $\gamma_2$ -set and Lemma 3, there is a  $\gamma_2$ -set, denoted by D, of T satisfying  $D \cap \{y_1, \ldots, y_s, z_1, \ldots, z_t\} = \emptyset$ . Then  $(D \cap V(T')) \cup \{w\}$  is a 2-dominating set of T'. Hence

$$\begin{aligned} \gamma_2(T') &\leq |(D \cap V(T')) \cup \{w\}| \\ &\leq |D| - (r+t) + 1 \\ &= \gamma_2(T) - (r+t) + 1 \\ &= \gamma_t(T) - (r+t) + 1. \end{aligned}$$

If  $t \ge 1$ , then  $\gamma_2(T') \le \gamma_t(T) \le \gamma_t(T') \le \gamma_2(T')$ , the last inequality is by Lemma 2, which implies that r + t = 1 and  $w \notin D$ . So r = 0 and t = 1. By (1), we have s = 0. Hence (2) is valid.

If  $C(w) = C'(w) \neq C'_1(w)$ , then  $s + t \geq 1$ . By (1) and (2), r = 0. We show that t = 0. If not, similar to the proof of (2), we have  $w \notin D$ , t = 1 and s = 0. Since C(w) = C'(w), we know that  $deg_T(w) = 2$ . To 2-dominate  $w, z_1 \in D$ , which contradicts with the choice of D.

**Lemma 6.** If  $T' \in \mathscr{A}$  with order at least three and T is obtained from T' by an operation of Type-i,  $1 \leq i \leq 6$ , then  $T \in \mathscr{A}$ .

**Proof.** Since  $T' \in \mathscr{A}$ , we have  $\gamma_t(T') = \gamma_2(T')$ . By Lemma 2, we only need to prove that  $\gamma_t(T) \ge \gamma_2(T)$ .

Case 1. i = 1. Assume that T is obtained from T' by attaching u' and w' to u and w, respectively, where u and w satisfy the conditions of Type-1 operation. Then there is an edge xy in T' such that either x is in a  $\gamma_2$ -set of T' and T' - xy has a component  $P_4 = uvwx$ , or y is in a  $\gamma_2$ -set of T' and T' - xy has a component  $P_5 = uvwxx'$ . Clearly,  $\gamma_t(T') = \gamma_t(T) - 1$ .

If T' - xy contains a path  $P_4 = uvwx$ , then let D' be a  $\gamma_2$ -set of T' containing x. From Lemma 3 and the definition of  $\gamma_2$ -set, we have  $D' \cap$ 

 $\{u, v, w, x\} = \{u, w\}$  or  $\{u, v\}$ . Thus  $D = (D' - \{u, v, w\}) \cup \{u', v, w'\}$  is a 2-dominating set of T with  $|D| = |D'| + 1 = \gamma_2(T') + 1$ . So,  $\gamma_t(T) = \gamma_t(T') + 1 = \gamma_2(T') + 1 = |D| \ge \gamma_2(T)$ .

If T' - xy contains a path  $P_5 = uvwxx'$ , then let D' be a  $\gamma_2$ -set of T' containing y. By Lemma 3 and the definition of  $\gamma_2$ -set, we have  $D' \cap \{u, v, w, x, x'\} = \{u, w, x'\}$ . Thus  $D = (D' \setminus \{u, w\}) \cup \{u', v, w'\}$  is a 2-dominating set of T with  $|D| = |D'| + 1 = \gamma_2(T') + 1$ . So,  $\gamma_t(T) = \gamma_t(T') + 1 = \gamma_2(T') + 1 = |D| \ge \gamma_2(T)$ .

Case 2. i = 2. Assume that T is obtained from T' by attaching a path  $P_2 = uu'$  to a vertex v of T' such that  $uv \in E(T)$ , where T' - v has a component  $P_2 = wx$  satisfying  $vw \in E(T')$ . It is easy to show that  $\gamma_t(T) = \gamma_t(T') + 1$ . By the definition of  $\gamma_2$ -set, there exists a  $\gamma_2$ -set D' of T' containing the vertex v. Then  $D' \cup \{u'\}$  is a 2-dominating set of T. Hence,  $\gamma_t(T) = \gamma_t(T') + 1 = \gamma_2(T') + 1 = |D' \cup \{u'\}| \ge \gamma_2(T)$ .

Case 3. i = 3. Assume that T is obtained from T' by attaching  $t (\geq 1)$  paths  $P_3$ , denoted by  $\{x_i y_i z_i : 1 \leq i \leq t\}$ , to a vertex v of T' such that  $x_i v \in E(T)$  for  $1 \leq i \leq t$ , where either T' - v has a component  $P_2 = uu'$  satisfying  $uv \in E(T')$ , or T' - v has a component  $P_1 = u_0$  and a component  $P_3 = uu'u''$  satisfying  $uv \in E(T')$ . By the definitions of  $\gamma_t$ -set and  $\gamma_2$ -set, we can easily prove that  $\gamma_t(T) \geq \gamma_t(T') + 2t$  and  $\gamma_2(T') + 2t \geq \gamma_2(T)$ . Since  $\gamma_t(T') = \gamma_2(T')$ , we have  $\gamma_t(T) \geq \gamma_t(T') + 2t = \gamma_2(T') + 2t \geq \gamma_2(T)$ .

Case 4. i = 4. Assume that T is obtained from T' by attaching a path  $P_2 = xyz$  to a vertex v of T' such that  $yv \in E(T)$ , where v is not in any  $\gamma_t$ -set of T' and T' - v has a component  $P_3 = uu'u''$  satisfying  $uv \in E(T')$ . For any  $\gamma_2$ -set D' of T',  $D' \cup \{x, z\}$  is a 2-dominating set of T. So  $\gamma_2(T') + 2 \ge \gamma_2(T)$ . Let S be a  $\gamma_t$ -set of T containing the vertex u, then  $y \in S$  and  $|S \cap \{v, x, z\}| = 1$ .

If  $v \notin S$ , then  $|S \cap V(T')| = |S| - 2 = \gamma_t(T) - 2 \ge \gamma_t(T')$  since  $S \cap V(T')$ is a TDS of T'. By  $\gamma_t(T') = \gamma_2(T'), \gamma_t(T) \ge \gamma_t(T') + 2 = \gamma_2(T') + 2 \ge \gamma_2(T)$ .

If  $v \in S$ , then  $S \cap \{v, x, z\} = \{v\}$  and  $|S \cap V(T')| = |S| - 1 = \gamma_t(T) - 1 \ge \gamma_t(T')$  since  $S \cap V(T')$  is a TDS of T'. Suppose that  $\gamma_t(T) \le \gamma_2(T) - 1$ , then, by  $\gamma_t(T') = \gamma_2(T')$ ,  $\gamma_2(T) \ge \gamma_t(T) + 1 \ge \gamma_t(T') + 2 = \gamma_2(T') + 2 \ge \gamma_2(T)$ . So  $|S \cap V(T')| = \gamma_t(T) - 1 = \gamma_t(T')$ , and  $S \cap V(T')$  is a  $\gamma_t$ -set of T' containing v, which contradicts with v is not in any  $\gamma_t$ -set of T'. Hence  $\gamma_t(T) \ge \gamma_2(T)$ .

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Case 5. i = 5. Assume that T is obtained from T' by attaching a K(t)  $(t \ge 1)$  to a vertex v of T' by joining u = cent(K(t)) to v, where v satisfies the condition of Type-5 operation. Clearly,  $\gamma_t(T) \ge \gamma_t(T') + 2t$ .

If  $t \ge 2$ , then, by  $\gamma_t(T') = \gamma_2(T')$ , it is obvious that  $\gamma_t(T) \ge \gamma_t(T') + 2t = \gamma_2(T') + 2t \ge \gamma_2(T)$ .

If t = 1, then let K(1) = uxyz and D' be a  $\gamma_2$ -set of T' containing v. Thus  $D' \cup \{z, x\}$  is a 2-dominating set of T. Hence  $\gamma_t(T) \ge \gamma_t(T') + 2 = \gamma_2(T') + 2 = |D' \cup \{z, x\}| \ge \gamma_2(T)$ .

Case 6. i = 6. Assume that T is obtained from T' by attaching a path  $P_5 = x_1 x_2 x_3 x_4 x_5$  to a vertex v of a tree T such that  $x_2 v \in E(T)$ , where T' and v satisfy the condition of Type-6 operation. Then we can choose a subset S of V(T) as a  $\gamma_t$ -set of T such that  $S \cap N_{T'}(v) \neq \emptyset$ . Thus  $S \cap V(T')$  is a TDS of T' and then  $|S \cap V(T')| \geq \gamma_t(T')$ . By the definition of  $\gamma_2$ -set, we have  $\gamma_2(T') + 3 \geq \gamma_2(T)$ . Hence  $\gamma_t(T) = |S| = |S \cap V(P_5)| + |S \cap V(T')| \geq 3 + \gamma_t(T') \geq \gamma_2(T)$ .

# **Lemma 7.** If $T \in \mathscr{A}$ with order at least three, then $T \in \mathscr{B}$ .

**Proof.** Let n = |V(T)|. Since  $T \in \mathscr{A}$ , we have  $\gamma_t(T) = \gamma_2(T)$ . If d(T) = 2, then T is a star  $K_{1,n-1}$ . Since  $2 = \gamma_t(T) = \gamma_2(T) = n - 1$ , n = 3. So  $T = P_3 \in \mathscr{B}$ . If d(T) = 3, then T contains exactly n - 2 leaves. Since  $2 = \gamma_t(T) = \gamma_2(T) \ge n - 2$ , n = 4. So  $T = P_4$ . However,  $\gamma_2(P_4) = 3 \ne \gamma_t(P_4)$ , a contradiction. If d(T) = 4, then there is a vertex w of T with distance at most two from the other vertices in T. Hence  $C(w) = C'(w) \ne C'_1(w)$  if we root T at w. By (3) of Lemma 5, T is a tree obtained from a star  $K_{1,t}$  by attaching a vertex to every leaf of  $K_{1,t}$ , where 2t + 1 = n. Clearly, T can be obtained from  $P_3$  by t - 1 operations of Type-2. By Lemma 6,  $T \in \mathscr{B}$ . In the following, we will assume that  $d(T) \ge 5$  and prove  $T \in \mathscr{B}$  by induction on the order of n = |V(T)|.

If n < 6, then  $d(G) \le 4$ . The result is true from the above proof. If n = 6, then  $T = P_6 \in \mathscr{B}$ . This establishes the base cases. Assume that n > 6 and the result is true for all the trees T' with order |V(T')| < n, that is, if  $T' \in \mathscr{A}$  with order |V(T')| < n then  $T' \in \mathscr{B}$ .

**Claim 1.** If there is a vertex  $a \in V(T)$  such that T - a contains at least two components  $P_2$ , then  $T \in \mathscr{B}$ .

**Proof.** Assume that  $P_2 = bb'$  and  $P_2 = cc'$  are two components of T - a such that  $ab, ac \in E(T)$ . Let  $T' = T - \{b, b'\}$ , then we use S' and D to

denote a  $\gamma_t$ -set of T' containing a and a  $\gamma_2$ -set of T, respectively. Since  $a \in S', S' \cup \{b\}$  is a TDS of T, and so  $\gamma_t(T') \ge \gamma_t(T) - 1$ . Since D is a  $\gamma_2$ -set of  $T, D \cap \{a, b, b'\} = \{a, b'\}$  by the definition of  $\gamma_2$ -set. So  $D \cap V(T')$  is a 2-dominating set of T'. Hence  $\gamma_t(T') \ge \gamma_t(T) - 1 = \gamma_2(T) - 1 = |D \cap V(T')| \ge \gamma_2(T')$ . By Lemma 2,  $\gamma_t(T') = \gamma_2(T')$ , and so  $T' \in \mathscr{A}$ . By the induction on  $T', T' \in \mathscr{B}$ . Since T can be obtained from T' by Type-2 operation. So  $T \in \mathscr{B}$ . The claim holds.

By Claim 1, we only need consider the case that, for every vertex a, T - a has at most one component  $P_2$ . Let  $P = uvwxyz \cdots r$  be a longest path in T and we root T at r.

Clearly,  $C(w) = C'(w) \neq C'_1(w)$ . By (3) of Lemma 5,  $C'_1(w) = C'_3(w) = \emptyset$ . Hence  $P_3 = uvw$  is a component of T - x. Let t be the number of components  $P_3$  of T[D(x)] such that a leaf of every  $P_3$  is adjacent to x. Note that T[D(x)] possible has other components. We suppose T[D(x)] has s components  $P_3$  with its support vertex is adjacent to x, k components  $P_2$  and h components  $P_1$ . By Lemmas 4 and 5,  $s, k \in \{0, 1\}$  and  $h \in \{0, 1, 2\}$ . Denote the t components  $P_3$  of T[D(x)] with one of its leaves is adjacent to x in T by  $P_3 = u_i v_i w_i$   $(1 \le i \le t)$ , where  $xw_i \in E(T)$  for  $1 \le i \le t$ . We prove the result according to the values of  $\{s, k, h\}$ .

Case 1. s = k = h = 0.

Then  $T[D[x]] = K(t), t \ge 1$ . Let T' = T - D[x]. Then  $3 \le |V(T')| < n$ . Clearly,  $\gamma_t(T') \ge \gamma_t(T) - 2t$ . Let D be a  $\gamma_2$ -set of T such that D contains as few vertices of T[D[x]] as possible. Then,  $x \notin D$  and  $|D \cap D[x]| = 2t$ by the definition of  $\gamma_2$ -set. So  $D \cap V(T')$  is a 2-dominating set of T'. Thus  $\gamma_t(T') \ge \gamma_t(T) - 2t = \gamma_t(T) - 2t = |D \cap V(T')| \ge \gamma_2(T)$ . By Lemma 2,  $\gamma_t(T') = \gamma_2(T')$  and  $D \cap V(T')$  is a  $\gamma_2$ -set of T'. So  $T' \in \mathscr{A}$ . Applying the inductive hypothesis on  $T', T' \in \mathscr{B}$ .

If  $t \geq 2$ , then it is obvious that T is obtained from T' by Type-5 operation, and so  $T \in \mathscr{B}$ .

If t = 1, then  $T[D[x]] = K(1) = P_4 = uvwx$ , and so  $D \cap \{u, v, w, x\} = \{u, w\}$ . To 2-dominate  $x, y \in D$ , and so  $y \in D \cap V(T')$ , which implies that y is in some  $\gamma_2$ -set of T'. Hence T can be obtained from T' by Type-5 operation, and  $T \in \mathcal{B}$ , too.

Case 2.  $s \neq 0$ . By the proof procedure of Lemma 5, s = 1 and k = h = 0. Denote the component  $P_3$  of T[D[x]] whose support vertex is adjacent to x in T by  $P_3 = abc$  and let  $T' = T - \{a, b, c\}$ . Clearly,  $3 \leq |V(T')| < n$ . Let D be a  $\gamma_2$ -set of T which does not contain b.

## A CHARACTERIZATION OF $(\gamma_t, \gamma_2)$ -TREES

We claim that x is not in any  $\gamma_t$ -set of T'. Suppose that T' has a  $\gamma_t$ -set containing x, denoted by S', then  $S' \cup \{b\}$  is a TDS of T. So  $\gamma_t(T') \geq \gamma_t(T) - 1$ . Since  $b \notin D$ , then  $D \cap V(T')$  is a 2-dominating set of T'. Hence  $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2(T) - 1 = |D \cap V(T')| + 1 \geq \gamma_2(T') + 1$ , which contradicts  $\gamma_t(T') \leq \gamma_2(T')$ . The claim holds. Therefore, T can be obtained from T' by Type-4 operation.

Now we prove that  $T' \in \mathscr{B}$ . Let S' be a  $\gamma_t$ -set of T'. By the above claim,  $x \notin S'$ . Since  $S' \cup \{x, b\}$  is a TDS of T,  $\gamma_t(T') \ge \gamma_t(T) - 2$ . Since  $b \notin D$ ,  $D \cap V(T')$  is a 2-dominating set of T'. Hence  $\gamma_t(T') \ge \gamma_t(T) - 2 = \gamma_2(T) - 2 = |D \cap V(T')| \ge \gamma_2(T')$ . By Lemma 2,  $\gamma_t(T') = \gamma_2(T')$ , which implies  $T' \in \mathscr{A}$ . Applying the inductive hypothesis on T',  $T' \in \mathscr{B}$ , and so  $T \in \mathscr{B}$ .

Case 3.  $k \neq 0$ . By the proof procedure of Lemma 5, s = h = 0. Let  $T' = T - \bigcup_{i=1}^{t} \{u_i, v_i, w_i\}$ . It is clearly that  $3 \leq |V(T')| < n$  and T is obtained from T' by Type-3 operation.

We only need to prove that  $T' \in \mathscr{B}$ . Let  $S' \subseteq V(T')$  be a  $\gamma_t$ -set of T', then  $S' \cup (\cup_{i=1}^t \{v_i, w_i\})$  is a TDS of T. So  $\gamma_t(T') \geq \gamma_t(T) - 2t$ . Since T - x has a component  $P_2 = ab$ , we can choose  $D \subseteq V(T)$  as a  $\gamma_2$ -set of T containing x. Then  $D \cap V(T')$  is a 2-dominating set of T', and so  $\gamma_2(T) = |D| = 2t + |D \cap V(T')| \geq 2t + \gamma_2(T')$ . By  $\gamma_t(T) = \gamma_2(T)$ , we have  $\gamma_t(T') = \gamma_2(T')$ , and so  $T' \in \mathscr{A}$ . Applying the inductive hypothesis on T',  $T' \in \mathscr{B}$ .

Case 4.  $h \neq 0$ . By Lemmas 4 and 5,  $h \in \{1, 2\}$  and s = k = 0. We claim that h = 1. If not, then h = 2. We denote the two components  $P_1$  of T[D(x)] by x' and x''. Let T' = T - x''. Clearly,  $\gamma_t(T') = \gamma_t(T)$ . Let D be a  $\gamma_2$ -set of T containing  $\{w_1, \ldots, w_t\}$ . By Lemma 3,  $\{x', x''\} \subseteq D$ . Since  $D \cap V(T')$  is 2-dominating set of T' with  $|D \cap V(T')| = \gamma_2(T) - 1$ , we have  $\gamma_t(T') = \gamma_t(T) = \gamma_2(T) > \gamma_2(T) - 1 \ge \gamma_2(T')$ , which contradicts  $\gamma_t(T') \le \gamma_2(T')$ .

## Case 4.1. $t \geq 2$ .

Let  $T' = T - \bigcup_{i=2}^{t} \{u_i, v_i, w_i\}$ , then T is obtained from T' by Type-3 operation. By the definitions of  $\gamma_t$ -set and  $\gamma_2$ -set, it is easy to see that  $\gamma_t(T')+2(t-1) = \gamma_t(T)$  and  $\gamma_2(T')+2(t-1) = \gamma_2(T)$ . Hence  $\gamma_t(T') = \gamma_2(T')$  and  $T' \in \mathscr{A}$ . Applying the inductive hypothesis on  $T', T' \in \mathscr{B}$ , and so  $T \in \mathscr{B}$ .

Case 4.2. t = 1. Denote the component  $P_1$  of T[D(x)] by  $P_1 = x'$ .

Case 4.2.1. If  $T[D(y) \setminus D[x]]$  has a component  $H \in \{P_2, P_3, P_5\}$ , then let T' = T - D[x]. We can easily check that T is obtained from T' by Type-6 operation. By the definition of  $\gamma_2$ -set,  $\gamma_2(T') + 3 = \gamma_2(T)$ . For any  $\gamma_t$ -set S' of  $T', S' \cup \{v, w, x\}$  is a TDS of T. So  $\gamma_t(T') \ge \gamma_t(T) - 3 = \gamma_2(T) - 3 =$  $\gamma_2(T')$ . By Lemma 2,  $\gamma_t(T') = \gamma_2(T')$  and  $T' \in \mathscr{A}$ . Applying the inductive hypothesis on  $T', T' \in \mathscr{B}$ , and so  $T \in \mathscr{B}$ .

Case 4.2.2. If  $T[D(y) \setminus D[x]]$  has no component  $P_2$ ,  $P_3$  or  $P_5$ , we consider the structure of T[D(y)]. By the above discussion, T[D(y)] consists of a component  $P_5 = uvwxx'$  and  $\ell$  components  $P_1$ , denoted by  $\{y_1, \ldots, y_\ell\}$ . By Lemma 4,  $\ell \leq 2$ . However, if  $\ell = 2$ , then let T' = T - D[y]. It can be easily checked that  $\gamma_t(T') + 4 \geq \gamma_t(T) = \gamma_2(T) = \gamma_2(T') + 5$ , which contradicts  $\gamma_t(T') \leq \gamma_2(T')$ . Hence  $\ell \leq 1$ .

Let  $T' = T - \{u, x'\}$ . Then we can easily check that  $\gamma_t(T') + 1 = \gamma_t(T)$ . Let D be a  $\gamma_2$ -set of T such that D contains as few vertices of D[y] as possible and  $D \cap D[x] = \{u, w, x'\}$ . Then  $D' = (D \setminus \{u, w, x'\}) \cup \{v, x\}$  is a 2-dominating set of T'. So  $\gamma_t(T') = \gamma_t(T) - 1 = \gamma_2(T) - 1 = |D'| \ge \gamma_2(T')$ , which implies that  $\gamma_t(T') = \gamma_2(T')$  and D' is a  $\gamma_2$ -set of T'. By  $\gamma_t(T') = \gamma_2(T')$ ,  $T' \in \mathscr{A}$ . Applying the inductive hypothesis to  $T', T' \in \mathscr{B}$ .

If  $\ell = 0$ , then  $deg_T(y) = 2$ . Since  $x \notin D$ , to 2-dominate  $y, y \in D$ . Thus y is in the  $\gamma_2$ -set D' of T'. Hence T is obtained from T' by Type-1 operation. Thus  $T \in \mathscr{B}$ .

If  $\ell = 1$ , then  $deg_T(y) = 3$ . Since  $x \notin D$ , to 2-dominate y, we have  $y \notin D$  and  $z \in D$  by the choice of D. Thus z is in the  $\gamma_2$ -set D' of T'. Hence T is obtained from T' by Type-1 operation. Thus  $T \in \mathscr{B}$ .

This completes the proof of Lemma 7.

Note that  $\{P_2, P_3\} \subseteq \mathscr{A}$ . Lemma 6 implies that  $\mathscr{B} \subseteq \mathscr{A}$  and Lemma 7 implies that  $\mathscr{A} \subseteq \mathscr{B} \cup \{P_2\}$ . Hence Theorem 1 is true.

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