# A CHARACTERIZATION OF $\left(\gamma_{t}, \gamma_{2}\right)$-TREES * 

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#### Abstract

Let $\gamma_{t}(G)$ and $\gamma_{2}(G)$ be the total domination number and the 2domination number of a graph $G$, respectively. It has been shown that: $\gamma_{t}(T) \leq \gamma_{2}(T)$ for any tree $T$. In this paper, we provide a constructive characterization of those trees with equal total domination number and 2-domination number.


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## 1. Introduction

Let $G=(V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood, the closed neighborhood and the degree of a vertex $v \in V(G)$ are denoted by $N_{G}(v)=\{u \in V(G) \mid u v \in E(G)\}$, $N_{G}[v]=N_{G}(v) \cup\{v\}$ and $\operatorname{deg}_{G}(v)=\left|N_{G}(v)\right|$, respectively. For $u \in V(G)$, $u$ is a leaf of $G$ if $\operatorname{deg}_{G}(u)=1$ and a support vertex of $G$ if $u$ has a leaf as its neighbor in $G$. For a pair of vertices $u, v \in V(G)$, the distance $d_{G}(u, v)$ of $u$ and $v$ is the length of a shortest $u v$-path in $G$. The diameter of $G$ is $d(G)=\max \left\{d_{G}(u, v): u, v \in V(G)\right\}$.

For any set $S \subseteq V(G)$, the subgraph induced by $S$ is denoted by $G[S]$ and we write $G-S$ for $G[V(G)-S]$. For convenience, we write $G-v$ for

[^0]$G-\{v\}$ for $v \in V(G)$. For any edge $x y \in E(G)$, we use $G-x y$ to denote the subgraph induced by $E(G)-\{x y\}$.

Total domination in graphs was introduced by Cockayne et al. [3]. A subset $S \subseteq V(G)$ is a total dominating set (denoted by TDS) if every vertex of $V(G)$ has at least one neighbor in $S$. The total domination number (denoted by $\gamma_{t}(G)$ ) is the minimum cardinality among the total dominating sets of $G$. The total dominating set of $G$ with cardinality $\gamma_{t}(G)$ will be called a $\gamma_{t}$-set of $G$. For a survey on total domination in graphs one can refer to Henning [12].

Let $p$ be a positive integer. In [6], Fink and Jacobson introduced the concept of $p$-domination. A $p$-dominating set of $G$ is a subset $S$ of $V(G)$ such that every vertex not in $S$ has at least $p$ neighbors in $S$. The $p$-domination number $\gamma_{p}(G)$ is the minimum cardinality of a $p$-dominating set of $G$. The $p$-dominating set of $G$ with cardinality $\gamma_{p}(G)$ will be called a $\gamma_{p}$-set of $G$. Note that $p$-domination is the classic domination when $p=1$. For any $S, T \subseteq V(G), S p$-dominates $T$ in $G$ if every vertex of $T$ not in $S$ has at least $p$ neighbors in $S$.

An area of research in domination of graphs that has received considerable attention is the characterization of classes of graphs with equal domination parameters. For any two graph parameters $\lambda$ and $\mu, G$ is called a $(\lambda, \mu)$-graph if $\lambda(G)=\mu(G)$. Characterizing the $(\lambda, \mu)$-graphs has been investigated in many papers (for example $[1,4,7,11,13]$ ).

In [8], Haynes et al. showed that for all trees the total domination number is equal or less than the 2-domination number, and they also gave a necessary condition for all trees with equal total domination number and 2 -domination number. In this paper, we give a constructive characterization of trees with equal total domination number and 2-domination number.

## 2. A Characterization

Let $P_{n}=u_{1} \cdots u_{n}(n \geq 1)$ be a path with vertex set $\left\{u_{1}, \ldots, u_{n}\right\}$ and $K(t)$ $(t \geq 2)$ be the tree obtained from a star $K_{1, t}$ with support vertex $u$ by adding a path $P_{2}$ to every leaf of $K_{1, t}$. Denote $u$ by $\operatorname{cent}(K(t))$. For convenience, we denote a path $P_{4}$ by $K(1)$ and let cent $(K(1))$ represent one leaf of $P_{4}$.

To state the characterization of $\left(\gamma_{t}, \gamma_{2}\right)$-trees, we introduce the six types of operations.

Type-1 operation: Attach a path $P_{1}$ to each of the two vertices $u, w$ of a tree $T$, respectively, where $u, w$ locate at a component $P_{l}$ of $T-x y$ for some edge $x y$ such that either $x$ is in a $\gamma_{2}$-set of $T$ and $P_{l}=P_{4}=u v w x$ or $y$ is in a $\gamma_{2}$-set of $T$ and $P_{l}=P_{5}=u v w x x^{\prime}$.

Type-2 operation: Attach a path $P_{2}$ to a vertex $v$ of a tree $T$ by joining one leaf of $P_{2}$ to $v$, where $v$ is a vertex such that $T-v$ has a component $P_{2}$.

Type-3 operation: Attach $t(\geq 1)$ paths $P_{3}$ to a vertex $v$ of a tree $T$ by joining one leaf of each $P_{3}$ to $v$, where $v$ is a vertex such that either $T-v$ has a component $P_{2}$ or $T-v$ has two components $P_{1}$ and $P_{3}$ that a leaf of $P_{3}$ is adjacent to $v$ in $T$.

Type-4 operation: Attach a path $P_{3}$ to a vertex $v$ of a tree $T$ by joining its support vertex to $v$, where $v$ is a vertex such that $v$ is not contained in any $\gamma_{t}$-set of $T$ and $T-v$ has a component $P_{3}$ that one of its leaves is adjacent to $v$ in $T$.

Type- 5 operation: Attach a tree $K(t)(t \geq 1)$ to a vertex $v$ of a tree $T$ by joining $\operatorname{cent}(K(t))$ to $v$, where $v$ is in a $\gamma_{2}$-set of $T$ if $t=1$.

Type-6 operation: Attach a path $P_{5}$ to a vertex $v$ of a tree $T$ by joining one of its support vertices to $v$, where $v$ is a vertex such that $T-v$ has a component $H \in\left\{P_{2}, P_{3}, P_{5}\right\}$ and $v$ is adjacent to a support vertex of $H$ if $H=P_{5}$.

From the survey on total domination in graphs [12], it is hard to recognize whether a vertex $v$ is in no $\gamma_{t}$-set or no $\gamma_{2}$-set.

Let $\mathscr{A}$ be the family of trees with equal total domination number and 2 -domination number, that is

$$
\mathscr{A}=\left\{T: T \text { is a tree satisfying } \gamma_{t}(T)=\gamma_{2}(T)\right\} .
$$

We also define the family $\mathscr{B}$ as:
$\mathscr{B}=\left\{T: T\right.$ is obtained from $P_{3}$ by a finite sequence of operations of Type- $i$, where $1 \leq i \leq 6\}$.

We shall show that
Theorem 1. $\mathscr{A}=\mathscr{B} \cup\left\{P_{2}\right\}$.

## 3. The Proof of Theorem 1

We need some known results.

Lemma 2 ([8]). Let $T$ be a tree without isolated vertices, then $\gamma_{t}(T) \leq$ $\gamma_{2}(T)$.

Lemma 3 ([2]). Every 2-dominating set of a graph $G$ contains all leaves of $G$.

Lemma 4 ([8]). If $T$ is a tree satisfying $\gamma_{t}(T)=\gamma_{2}(T)$, then every support vertex of $T$ is adjacent to at most two leaves.

Let $T$ be a rooted tree. For every $v \in V(T)$, let $C(v)$ and $D(v)$ denote the set of children and descendants of $v$, respectively, and $D[v]=D(v) \cup\{v\}$. Define

$$
\begin{aligned}
C^{\prime}(v)= & \{u \in C(v): \text { every vertex of } D[u] \text { has distance at most two } \\
& \text { from } v \text { in } T\} .
\end{aligned}
$$

By Lemma 4, each vertex of $C^{\prime}(v)$ has degree at most three. Hence we can partition $C^{\prime}(v)$ into $C_{1}^{\prime}(v), C_{2}^{\prime}(v), C_{3}^{\prime}(v)$ such that every vertex of $C_{i}^{\prime}(v)$ has degree $i$ in $T, i=1,2$ or 3 .

Lemma 5. Let $T$ be a rooted tree satisfying $\gamma_{t}(T)=\gamma_{2}(T)$ and $w \in V(T)$. We have
(1) If $C_{2}^{\prime}(w) \neq \emptyset$, then $C_{1}^{\prime}(w)=C_{3}^{\prime}(w)=\emptyset$.
(2) If $C_{3}^{\prime}(w) \neq \emptyset$, then $C_{1}^{\prime}(w)=C_{2}^{\prime}(w)=\emptyset$ and $\left|C_{3}^{\prime}(w)\right|=1$.
(3) If $C(w)=C^{\prime}(w) \neq C_{1}^{\prime}(w)$, then $C_{1}^{\prime}(w)=C_{3}^{\prime}(w)=\emptyset$.

Proof. Let $C_{1}^{\prime}(w)=\left\{x_{1}, \ldots, x_{r}\right\}, C_{2}^{\prime}(w)=\left\{y_{1}, \ldots, y_{s}\right\}$ and $C_{3}^{\prime}(w)=$ $\left\{z_{1}, \ldots, z_{t}\right\}$. Then $\left|C_{1}^{\prime}(w)\right|=r,\left|C_{2}^{\prime}(w)\right|=s$ and $\left|C_{3}^{\prime}(w)\right|=t$. For each $i=1, \ldots, t$, let $u_{i}$ be a leaf adjacent with $z_{i}$ in $T$. Let $T^{\prime}=T-\left\{x_{1}, \ldots, x_{r}\right.$, $\left.u_{1}, \ldots, u_{t}\right\}$.
(1). We prove that if $s \geq 1$ then $r+t=0$. Assume $r+t \geq 1$. Since $s \geq 1$, we can choose a $\gamma_{2}$-set $D$ of $T$ such that $w \in D$, and a $\gamma_{t}$-set $S^{\prime}$ of $T^{\prime}$ such that $w \in S^{\prime}$. It is not difficult to check that $D-\left\{x_{1}, \ldots, x_{r}, u_{1}, \ldots, u_{t}\right\}$ is a 2 -dominating set of $T^{\prime}$ and $S^{\prime}$ is a TDS of $T$. Hence,

$$
\begin{aligned}
& \gamma_{t}\left(T^{\prime}\right)=\left|S^{\prime}\right| \geq \gamma_{t}(T)=\gamma_{2}(T) \\
& =|D|>\left|D-\left\{x_{1}, \ldots, x_{r}, u_{1}, \ldots, u_{t}\right\}\right| \geq \gamma_{2}\left(T^{\prime}\right)
\end{aligned}
$$

a contradiction with Lemma 2.
(2) and (3). Suppose either $C_{3}^{\prime}(w) \neq \emptyset$ or $C(w)=C^{\prime}(w) \neq C_{1}^{\prime}(w)$. Then $s+t \geq 1$. Choose a $\gamma_{t}$-set $S^{\prime}$ of $T^{\prime}$ such that $w \in S^{\prime}$. Then $S^{\prime}$ is also a TDS of $T$. Hence $\gamma_{t}\left(T^{\prime}\right)=\left|S^{\prime}\right| \geq \gamma_{t}(T)$. By the definition of $\gamma_{2}$-set and Lemma 3, there is a $\gamma_{2}$-set, denoted by $D$, of $T$ satisfying $D \cap\left\{y_{1}, \ldots, y_{s}, z_{1}, \ldots, z_{t}\right\}=\emptyset$. Then $\left(D \cap V\left(T^{\prime}\right)\right) \cup\{w\}$ is a 2-dominating set of $T^{\prime}$. Hence

$$
\begin{aligned}
\gamma_{2}\left(T^{\prime}\right) & \leq\left|\left(D \cap V\left(T^{\prime}\right)\right) \cup\{w\}\right| \\
& \leq|D|-(r+t)+1 \\
& =\gamma_{2}(T)-(r+t)+1 \\
& =\gamma_{t}(T)-(r+t)+1 .
\end{aligned}
$$

If $t \geq 1$, then $\gamma_{2}\left(T^{\prime}\right) \leq \gamma_{t}(T) \leq \gamma_{t}\left(T^{\prime}\right) \leq \gamma_{2}\left(T^{\prime}\right)$, the last inequality is by Lemma 2, which implies that $r+t=1$ and $w \notin D$. So $r=0$ and $t=1$. By (1), we have $s=0$. Hence (2) is valid.

If $C(w)=C^{\prime}(w) \neq C_{1}^{\prime}(w)$, then $s+t \geq 1$. By (1) and (2), $r=0$. We show that $t=0$. If not, similar to the proof of (2), we have $w \notin D, t=1$ and $s=0$. Since $C(w)=C^{\prime}(w)$, we know that $\operatorname{deg}_{T}(w)=2$. To 2-dominate $w, z_{1} \in D$, which contradicts with the choice of $D$.

Lemma 6. If $T^{\prime} \in \mathscr{A}$ with order at least three and $T$ is obtained from $T^{\prime}$ by an operation of Type-i, $1 \leq i \leq 6$, then $T \in \mathscr{A}$.

Proof. Since $T^{\prime} \in \mathscr{A}$, we have $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$. By Lemma 2, we only need to prove that $\gamma_{t}(T) \geq \gamma_{2}(T)$.

Case 1. $i=1$. Assume that $T$ is obtained from $T^{\prime}$ by attaching $u^{\prime}$ and $w^{\prime}$ to $u$ and $w$, respectively, where $u$ and $w$ satisfy the conditions of Type-1 operation. Then there is an edge $x y$ in $T^{\prime}$ such that either $x$ is in a $\gamma_{2}$-set of $T^{\prime}$ and $T^{\prime}-x y$ has a component $P_{4}=u v w x$, or $y$ is in a $\gamma_{2}$-set of $T^{\prime}$ and $T^{\prime}-x y$ has a component $P_{5}=u v w x x^{\prime}$. Clearly, $\gamma_{t}\left(T^{\prime}\right)=\gamma_{t}(T)-1$.

If $T^{\prime}-x y$ contains a path $P_{4}=u v w x$, then let $D^{\prime}$ be a $\gamma_{2}$-set of $T^{\prime}$ containing $x$. From Lemma 3 and the definition of $\gamma_{2}$-set, we have $D^{\prime} \cap$
$\{u, v, w, x\}=\{u, w\}$ or $\{u, v\}$. Thus $D=\left(D^{\prime}-\{u, v, w\}\right) \cup\left\{u^{\prime}, v, w^{\prime}\right\}$ is a 2-dominating set of $T$ with $|D|=\left|D^{\prime}\right|+1=\gamma_{2}\left(T^{\prime}\right)+1$. So, $\gamma_{t}(T)=$ $\gamma_{t}\left(T^{\prime}\right)+1=\gamma_{2}\left(T^{\prime}\right)+1=|D| \geq \gamma_{2}(T)$.

If $T^{\prime}-x y$ contains a path $P_{5}=u v w x x^{\prime}$, then let $D^{\prime}$ be a $\gamma_{2}$-set of $T^{\prime}$ containing $y$. By Lemma 3 and the definition of $\gamma_{2}$-set, we have $D^{\prime} \cap$ $\left\{u, v, w, x, x^{\prime}\right\}=\left\{u, w, x^{\prime}\right\}$. Thus $D=\left(D^{\prime} \backslash\{u, w\}\right) \cup\left\{u^{\prime}, v, w^{\prime}\right\}$ is a $2-$ dominating set of $T$ with $|D|=\left|D^{\prime}\right|+1=\gamma_{2}\left(T^{\prime}\right)+1$. So, $\gamma_{t}(T)=\gamma_{t}\left(T^{\prime}\right)+1=$ $\gamma_{2}\left(T^{\prime}\right)+1=|D| \geq \gamma_{2}(T)$.

Case 2. $\quad i=2$. Assume that $T$ is obtained from $T^{\prime}$ by attaching a path $P_{2}=u u^{\prime}$ to a vertex $v$ of $T^{\prime}$ such that $u v \in E(T)$, where $T^{\prime}-v$ has a component $P_{2}=w x$ satisfying $v w \in E\left(T^{\prime}\right)$. It is easy to show that $\gamma_{t}(T)=\gamma_{t}\left(T^{\prime}\right)+1$. By the definition of $\gamma_{2}$-set, there exists a $\gamma_{2}$-set $D^{\prime}$ of $T^{\prime}$ containing the vertex $v$. Then $D^{\prime} \cup\left\{u^{\prime}\right\}$ is a 2-dominating set of $T$. Hence, $\gamma_{t}(T)=\gamma_{t}\left(T^{\prime}\right)+1=\gamma_{2}\left(T^{\prime}\right)+1=\left|D^{\prime} \cup\left\{u^{\prime}\right\}\right| \geq \gamma_{2}(T)$.

Case 3. $i=3$. Assume that $T$ is obtained from $T^{\prime}$ by attaching $t(\geq 1)$ paths $P_{3}$, denoted by $\left\{x_{i} y_{i} z_{i}: 1 \leq i \leq t\right\}$, to a vertex $v$ of $T^{\prime}$ such that $x_{i} v \in E(T)$ for $1 \leq i \leq t$, where either $T^{\prime}-v$ has a component $P_{2}=u u^{\prime}$ satisfying $u v \in E\left(T^{\prime}\right)$, or $T^{\prime}-v$ has a component $P_{1}=u_{0}$ and a component $P_{3}=u u^{\prime} u^{\prime \prime}$ satisfying $u v \in E\left(T^{\prime}\right)$. By the definitions of $\gamma_{t}$-set and $\gamma_{2}$-set, we can easily prove that $\gamma_{t}(T) \geq \gamma_{t}\left(T^{\prime}\right)+2 t$ and $\gamma_{2}\left(T^{\prime}\right)+2 t \geq \gamma_{2}(T)$. Since $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$, we have $\gamma_{t}(T) \geq \gamma_{t}\left(T^{\prime}\right)+2 t=\gamma_{2}\left(T^{\prime}\right)+2 t \geq \gamma_{2}(T)$.

Case 4. $i=4$. Assume that $T$ is obtained from $T^{\prime}$ by attaching a path $P_{2}=x y z$ to a vertex $v$ of $T^{\prime}$ such that $y v \in E(T)$, where $v$ is not in any $\gamma_{t}$-set of $T^{\prime}$ and $T^{\prime}-v$ has a component $P_{3}=u u^{\prime} u^{\prime \prime}$ satisfying $u v \in E\left(T^{\prime}\right)$. For any $\gamma_{2}$-set $D^{\prime}$ of $T^{\prime}, D^{\prime} \cup\{x, z\}$ is a 2-dominating set of $T$. So $\gamma_{2}\left(T^{\prime}\right)+2 \geq \gamma_{2}(T)$. Let $S$ be a $\gamma_{t}$-set of $T$ containing the vertex $u$, then $y \in S$ and $|S \cap\{v, x, z\}|=1$.

If $v \notin S$, then $\left|S \cap V\left(T^{\prime}\right)\right|=|S|-2=\gamma_{t}(T)-2 \geq \gamma_{t}\left(T^{\prime}\right)$ since $S \cap V\left(T^{\prime}\right)$ is a TDS of $T^{\prime}$. By $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right), \gamma_{t}(T) \geq \gamma_{t}\left(T^{\prime}\right)+2=\gamma_{2}\left(T^{\prime}\right)+2 \geq \gamma_{2}(T)$.

If $v \in S$, then $S \cap\{v, x, z\}=\{v\}$ and $\left|S \cap V\left(T^{\prime}\right)\right|=|S|-1=\gamma_{t}(T)-1 \geq$ $\gamma_{t}\left(T^{\prime}\right)$ since $S \cap V\left(T^{\prime}\right)$ is a TDS of $T^{\prime}$. Suppose that $\gamma_{t}(T) \leq \gamma_{2}(T)-1$, then, by $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right), \gamma_{2}(T) \geq \gamma_{t}(T)+1 \geq \gamma_{t}\left(T^{\prime}\right)+2=\gamma_{2}\left(T^{\prime}\right)+2 \geq$ $\gamma_{2}(T)$. So $\left|S \cap V\left(T^{\prime}\right)\right|=\gamma_{t}(T)-1=\gamma_{t}\left(T^{\prime}\right)$, and $S \cap V\left(T^{\prime}\right)$ is a $\gamma_{t}$-set of $T^{\prime}$ containing $v$, which contradicts with $v$ is not in any $\gamma_{t}$-set of $T^{\prime}$. Hence $\gamma_{t}(T) \geq \gamma_{2}(T)$.

Case 5. $i=5$. Assume that $T$ is obtained from $T^{\prime}$ by attaching a $K(t)$ $(t \geq 1)$ to a vertex $v$ of $T^{\prime}$ by joining $u=\operatorname{cent}(K(t))$ to $v$, where $v$ satisfies the condition of Type-5 operation. Clearly, $\gamma_{t}(T) \geq \gamma_{t}\left(T^{\prime}\right)+2 t$.

If $t \geq 2$, then, by $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$, it is obvious that $\gamma_{t}(T) \geq \gamma_{t}\left(T^{\prime}\right)+2 t=$ $\gamma_{2}\left(T^{\prime}\right)+2 t \geq \gamma_{2}(T)$.

If $t=1$, then let $K(1)=u x y z$ and $D^{\prime}$ be a $\gamma_{2}$-set of $T^{\prime}$ containing $v$. Thus $D^{\prime} \cup\{z, x\}$ is a 2 -dominating set of $T$. Hence $\gamma_{t}(T) \geq \gamma_{t}\left(T^{\prime}\right)+2=$ $\gamma_{2}\left(T^{\prime}\right)+2=\left|D^{\prime} \cup\{z, x\}\right| \geq \gamma_{2}(T)$.

Case 6. $i=6$. Assume that $T$ is obtained from $T^{\prime}$ by attaching a path $P_{5}=x_{1} x_{2} x_{3} x_{4} x_{5}$ to a vertex $v$ of a tree $T$ such that $x_{2} v \in E(T)$, where $T^{\prime}$ and $v$ satisfy the condition of Type-6 operation. Then we can choose a subset $S$ of $V(T)$ as a $\gamma_{t}$-set of $T$ such that $S \cap N_{T^{\prime}}(v) \neq \emptyset$. Thus $S \cap V\left(T^{\prime}\right)$ is a TDS of $T^{\prime}$ and then $\left|S \cap V\left(T^{\prime}\right)\right| \geq \gamma_{t}\left(T^{\prime}\right)$. By the definition of $\gamma_{2}$-set, we have $\gamma_{2}\left(T^{\prime}\right)+3 \geq \gamma_{2}(T)$. Hence $\gamma_{t}(T)=|S|=\left|S \cap V\left(P_{5}\right)\right|+\left|S \cap V\left(T^{\prime}\right)\right| \geq$ $3+\gamma_{t}\left(T^{\prime}\right)=3+\gamma_{2}\left(T^{\prime}\right) \geq \gamma_{2}(T)$.

Lemma 7. If $T \in \mathscr{A}$ with order at least three, then $T \in \mathscr{B}$.
Proof. Let $n=|V(T)|$. Since $T \in \mathscr{A}$, we have $\gamma_{t}(T)=\gamma_{2}(T)$. If $d(T)=2$, then $T$ is a star $K_{1, n-1}$. Since $2=\gamma_{t}(T)=\gamma_{2}(T)=n-1, n=3$. So $T=P_{3} \in \mathscr{B}$. If $d(T)=3$, then $T$ contains exactly $n-2$ leaves. Since $2=$ $\gamma_{t}(T)=\gamma_{2}(T) \geq n-2, n=4$. So $T=P_{4}$. However, $\gamma_{2}\left(P_{4}\right)=3 \neq \gamma_{t}\left(P_{4}\right)$, a contradiction. If $d(T)=4$, then there is a vertex $w$ of $T$ with distance at most two from the other vertices in $T$. Hence $C(w)=C^{\prime}(w) \neq C_{1}^{\prime}(w)$ if we root $T$ at $w$. By (3) of Lemma $5, T$ is a tree obtained from a star $K_{1, t}$ by attaching a vertex to every leaf of $K_{1, t}$, where $2 t+1=n$. Clearly, $T$ can be obtained from $P_{3}$ by $t-1$ operations of Type-2. By Lemma $6, T \in \mathscr{B}$. In the following, we will assume that $d(T) \geq 5$ and prove $T \in \mathscr{B}$ by induction on the order of $n=|V(T)|$.

If $n<6$, then $d(G) \leq 4$. The result is true from the above proof. If $n=6$, then $T=P_{6} \in \mathscr{B}$. This establishes the base cases. Assume that $n>6$ and the result is true for all the trees $T^{\prime}$ with order $\left|V\left(T^{\prime}\right)\right|<n$, that is, if $T^{\prime} \in \mathscr{A}$ with order $\left|V\left(T^{\prime}\right)\right|<n$ then $T^{\prime} \in \mathscr{B}$.

Claim 1. If there is a vertex $a \in V(T)$ such that $T-a$ contains at least two components $P_{2}$, then $T \in \mathscr{B}$.

Proof. Assume that $P_{2}=b b^{\prime}$ and $P_{2}=c c^{\prime}$ are two components of $T-a$ such that $a b, a c \in E(T)$. Let $T^{\prime}=T-\left\{b, b^{\prime}\right\}$, then we use $S^{\prime}$ and $D$ to
denote a $\gamma_{t}$-set of $T^{\prime}$ containing $a$ and a $\gamma_{2}$-set of $T$, respectively. Since $a \in S^{\prime}, S^{\prime} \cup\{b\}$ is a TDS of $T$, and so $\gamma_{t}\left(T^{\prime}\right) \geq \gamma_{t}(T)-1$. Since $D$ is a $\gamma_{2}$-set of $T, D \cap\left\{a, b, b^{\prime}\right\}=\left\{a, b^{\prime}\right\}$ by the definition of $\gamma_{2}$-set. So $D \cap V\left(T^{\prime}\right)$ is a 2 dominating set of $T^{\prime}$. Hence $\gamma_{t}\left(T^{\prime}\right) \geq \gamma_{t}(T)-1=\gamma_{2}(T)-1=\left|D \cap V\left(T^{\prime}\right)\right| \geq$ $\gamma_{2}\left(T^{\prime}\right)$. By Lemma 2, $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$, and so $T^{\prime} \in \mathscr{A}$. By the induction on $T^{\prime}, T^{\prime} \in \mathscr{B}$. Since $T$ can be obtained from $T^{\prime}$ by Type- 2 operation. So $T \in \mathscr{B}$. The claim holds.

By Claim 1, we only need consider the case that, for every vertex $a$, $T-a$ has at most one component $P_{2}$. Let $P=u v w x y z \cdots r$ be a longest path in $T$ and we root $T$ at $r$.

Clearly, $C(w)=C^{\prime}(w) \neq C_{1}^{\prime}(w)$. By (3) of Lemma 5, $C_{1}^{\prime}(w)=C_{3}^{\prime}(w)$ $=\emptyset$. Hence $P_{3}=u v w$ is a component of $T-x$. Let $t$ be the number of components $P_{3}$ of $T[D(x)]$ such that a leaf of every $P_{3}$ is adjacent to $x$. Note that $T[D(x)]$ possible has other components. We suppose $T[D(x)]$ has $s$ components $P_{3}$ with its support vertex is adjacent to $x, k$ components $P_{2}$ and $h$ components $P_{1}$. By Lemmas 4 and $5, s, k \in\{0,1\}$ and $h \in\{0,1,2\}$. Denote the $t$ components $P_{3}$ of $T[D(x)]$ with one of its leaves is adjacent to $x$ in $T$ by $P_{3}=u_{i} v_{i} w_{i}(1 \leq i \leq t)$, where $x w_{i} \in E(T)$ for $1 \leq i \leq t$. We prove the result according to the values of $\{s, k, h\}$.

Case 1. $s=k=h=0$.
Then $T[D[x]]=K(t), t \geq 1$. Let $T^{\prime}=T-D[x]$. Then $3 \leq\left|V\left(T^{\prime}\right)\right|<n$. Clearly, $\gamma_{t}\left(T^{\prime}\right) \geq \gamma_{t}(T)-2 t$. Let $D$ be a $\gamma_{2}$-set of $T$ such that $D$ contains as few vertices of $T[D[x]]$ as possible. Then, $x \notin D$ and $|D \cap D[x]|=2 t$ by the definition of $\gamma_{2}$-set. So $D \cap V\left(T^{\prime}\right)$ is a 2 -dominating set of $T^{\prime}$. Thus $\gamma_{t}\left(T^{\prime}\right) \geq \gamma_{t}(T)-2 t=\gamma_{t}(T)-2 t=\left|D \cap V\left(T^{\prime}\right)\right| \geq \gamma_{2}(T)$. By Lemma 2, $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$ and $D \cap V\left(T^{\prime}\right)$ is a $\gamma_{2}$-set of $T^{\prime}$. So $T^{\prime} \in \mathscr{A}$. Applying the inductive hypothesis on $T^{\prime}, T^{\prime} \in \mathscr{B}$.

If $t \geq 2$, then it is obvious that $T$ is obtained from $T^{\prime}$ by Type- 5 operation, and so $T \in \mathscr{B}$.

If $t=1$, then $T[D[x]]=K(1)=P_{4}=u v w x$, and so $D \cap\{u, v, w, x\}=$ $\{u, w\}$. To 2-dominate $x, y \in D$, and so $y \in D \cap V\left(T^{\prime}\right)$, which implies that $y$ is in some $\gamma_{2}$-set of $T^{\prime}$. Hence $T$ can be obtained from $T^{\prime}$ by Type- 5 operation, and $T \in \mathscr{B}$, too.

Case 2. $s \neq 0$. By the proof procedure of Lemma $5, s=1$ and $k=h=0$. Denote the component $P_{3}$ of $T[D[x]]$ whose support vertex is adjacent to $x$ in $T$ by $P_{3}=a b c$ and let $T^{\prime}=T-\{a, b, c\}$. Clearly, $3 \leq\left|V\left(T^{\prime}\right)\right|<n$. Let $D$ be a $\gamma_{2}$-set of $T$ which does not contain $b$.

We claim that $x$ is not in any $\gamma_{t}$-set of $T^{\prime}$. Suppose that $T^{\prime}$ has a $\gamma_{t}$-set containing $x$, denoted by $S^{\prime}$, then $S^{\prime} \cup\{b\}$ is a TDS of $T$. So $\gamma_{t}\left(T^{\prime}\right) \geq$ $\gamma_{t}(T)-1$. Since $b \notin D$, then $D \cap V\left(T^{\prime}\right)$ is a 2 -dominating set of $T^{\prime}$. Hence $\gamma_{t}\left(T^{\prime}\right) \geq \gamma_{t}(T)-1=\gamma_{2}(T)-1=\left|D \cap V\left(T^{\prime}\right)\right|+1 \geq \gamma_{2}\left(T^{\prime}\right)+1$, which contradicts $\gamma_{t}\left(T^{\prime}\right) \leq \gamma_{2}\left(T^{\prime}\right)$. The claim holds. Therefore, $T$ can be obtained from $T^{\prime}$ by Type- 4 operation.

Now we prove that $T^{\prime} \in \mathscr{B}$. Let $S^{\prime}$ be a $\gamma_{t}$-set of $T^{\prime}$. By the above claim, $x \notin S^{\prime}$. Since $S^{\prime} \cup\{x, b\}$ is a TDS of $T, \gamma_{t}\left(T^{\prime}\right) \geq \gamma_{t}(T)-2$. Since $b \notin D, D \cap V\left(T^{\prime}\right)$ is a 2-dominating set of $T^{\prime}$. Hence $\gamma_{t}\left(T^{\prime}\right) \geq \gamma_{t}(T)-2=$ $\gamma_{2}(T)-2=\left|D \cap V\left(T^{\prime}\right)\right| \geq \gamma_{2}\left(T^{\prime}\right)$. By Lemma 2, $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$, which implies $T^{\prime} \in \mathscr{A}$. Applying the inductive hypothesis on $T^{\prime}, T^{\prime} \in \mathscr{B}$, and so $T \in \mathscr{B}$.

Case 3. $k \neq 0$. By the proof procedure of Lemma $5, s=h=0$.
Let $T^{\prime}=T-\cup_{i=1}^{t}\left\{u_{i}, v_{i}, w_{i}\right\}$. It is clearly that $3 \leq\left|V\left(T^{\prime}\right)\right|<n$ and $T$ is obtained from $T^{\prime}$ by Type-3 operation.

We only need to prove that $T^{\prime} \in \mathscr{B}$. Let $S^{\prime} \subseteq V\left(T^{\prime}\right)$ be a $\gamma_{t}$-set of $T^{\prime}$, then $S^{\prime} \cup\left(\cup_{i=1}^{t}\left\{v_{i}, w_{i}\right\}\right)$ is a TDS of $T$. So $\gamma_{t}\left(T^{\prime}\right) \geq \gamma_{t}(T)-2 t$. Since $T-x$ has a component $P_{2}=a b$, we can choose $D \subseteq V(T)$ as a $\gamma_{2}$-set of $T$ containing $x$. Then $D \cap V\left(T^{\prime}\right)$ is a 2-dominating set of $T^{\prime}$, and so $\gamma_{2}(T)=|D|=2 t+\left|D \cap V\left(T^{\prime}\right)\right| \geq 2 t+\gamma_{2}\left(T^{\prime}\right)$. By $\gamma_{t}(T)=\gamma_{2}(T)$, we have $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$, and so $T^{\prime} \in \mathscr{A}$. Applying the inductive hypothesis on $T^{\prime}$, $T^{\prime} \in \mathscr{B}$.

Case 4. $h \neq 0$. By Lemmas 4 and $5, h \in\{1,2\}$ and $s=k=0$.
We claim that $h=1$. If not, then $h=2$. We denote the two components $P_{1}$ of $T[D(x)]$ by $x^{\prime}$ and $x^{\prime \prime}$. Let $T^{\prime}=T-x^{\prime \prime}$. Clearly, $\gamma_{t}\left(T^{\prime}\right)=\gamma_{t}(T)$. Let $D$ be a $\gamma_{2}$-set of $T$ containing $\left\{w_{1}, \ldots, w_{t}\right\}$. By Lemma 3, $\left\{x^{\prime}, x^{\prime \prime}\right\} \subseteq D$. Since $D \cap V\left(T^{\prime}\right)$ is 2-dominating set of $T^{\prime}$ with $\left|D \cap V\left(T^{\prime}\right)\right|=\gamma_{2}(T)-1$, we have $\gamma_{t}\left(T^{\prime}\right)=\gamma_{t}(T)=\gamma_{2}(T)>\gamma_{2}(T)-1 \geq \gamma_{2}\left(T^{\prime}\right)$, which contradicts $\gamma_{t}\left(T^{\prime}\right) \leq \gamma_{2}\left(T^{\prime}\right)$.

Case 4.1. $t \geq 2$.
Let $T^{\prime}=T-\cup_{i=2}^{t}\left\{u_{i}, v_{i}, w_{i}\right\}$, then $T$ is obtained from $T^{\prime}$ by Type- 3 operation. By the definitions of $\gamma_{t}$-set and $\gamma_{2}$-set, it is easy to see that $\gamma_{t}\left(T^{\prime}\right)+2(t-1)=\gamma_{t}(T)$ and $\gamma_{2}\left(T^{\prime}\right)+2(t-1)=\gamma_{2}(T)$. Hence $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$ and $T^{\prime} \in \mathscr{A}$. Applying the inductive hypothesis on $T^{\prime}, T^{\prime} \in \mathscr{B}$, and so $T \in \mathscr{B}$.

Case 4.2. $t=1$. Denote the component $P_{1}$ of $T[D(x)]$ by $P_{1}=x^{\prime}$.
Case 4.2.1. If $T[D(y) \backslash D[x]]$ has a component $H \in\left\{P_{2}, P_{3}, P_{5}\right\}$, then let $T^{\prime}=T-D[x]$. We can easily check that $T$ is obtained from $T^{\prime}$ by Type- 6 operation. By the definition of $\gamma_{2}$-set, $\gamma_{2}\left(T^{\prime}\right)+3=\gamma_{2}(T)$. For any $\gamma_{t}$-set $S^{\prime}$ of $T^{\prime}, S^{\prime} \cup\{v, w, x\}$ is a TDS of $T$. So $\gamma_{t}\left(T^{\prime}\right) \geq \gamma_{t}(T)-3=\gamma_{2}(T)-3=$ $\gamma_{2}\left(T^{\prime}\right)$. By Lemma 2, $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$ and $T^{\prime} \in \mathscr{A}$. Applying the inductive hypothesis on $T^{\prime}, T^{\prime} \in \mathscr{B}$, and so $T \in \mathscr{B}$.

Case 4.2.2. If $T[D(y) \backslash D[x]]$ has no component $P_{2}, P_{3}$ or $P_{5}$, we consider the structure of $T[D(y)]$. By the above discussion, $T[D(y)]$ consists of a component $P_{5}=u v w x x^{\prime}$ and $\ell$ components $P_{1}$, denoted by $\left\{y_{1}, \ldots, y_{\ell}\right\}$. By Lemma $4, \ell \leq 2$. However, if $\ell=2$, then let $T^{\prime}=T-D[y]$. It can be easily checked that $\gamma_{t}\left(T^{\prime}\right)+4 \geq \gamma_{t}(T)=\gamma_{2}(T)=\gamma_{2}\left(T^{\prime}\right)+5$, which contradicts $\gamma_{t}\left(T^{\prime}\right) \leq \gamma_{2}\left(T^{\prime}\right)$. Hence $\ell \leq 1$.

Let $T^{\prime}=T-\left\{u, x^{\prime}\right\}$. Then we can easily check that $\gamma_{t}\left(T^{\prime}\right)+1=\gamma_{t}(T)$. Let $D$ be a $\gamma_{2}$-set of $T$ such that $D$ contains as few vertices of $D[y]$ as possible and $D \cap D[x]=\left\{u, w, x^{\prime}\right\}$. Then $D^{\prime}=\left(D \backslash\left\{u, w, x^{\prime}\right\}\right) \cup\{v, x\}$ is a 2-dominating set of $T^{\prime}$. So $\gamma_{t}\left(T^{\prime}\right)=\gamma_{t}(T)-1=\gamma_{2}(T)-1=\left|D^{\prime}\right| \geq \gamma_{2}\left(T^{\prime}\right)$, which implies that $\gamma_{t}\left(T^{\prime}\right)=\gamma_{2}\left(T^{\prime}\right)$ and $D^{\prime}$ is a $\gamma_{2}$-set of $T^{\prime}$. By $\gamma_{t}\left(T^{\prime}\right)=$ $\gamma_{2}\left(T^{\prime}\right), T^{\prime} \in \mathscr{A}$. Applying the inductive hypothesis to $T^{\prime}, T^{\prime} \in \mathscr{B}$.

If $\ell=0$, then $\operatorname{deg}_{T}(y)=2$. Since $x \notin D$, to 2-dominate $y, y \in D$. Thus $y$ is in the $\gamma_{2}$-set $D^{\prime}$ of $T^{\prime}$. Hence $T$ is obtained from $T^{\prime}$ by Type- 1 operation. Thus $T \in \mathscr{B}$.

If $\ell=1$, then $\operatorname{deg}_{T}(y)=3$. Since $x \notin D$, to 2 -dominate $y$, we have $y \notin D$ and $z \in D$ by the choice of $D$. Thus $z$ is in the $\gamma_{2}$-set $D^{\prime}$ of $T^{\prime}$. Hence $T$ is obtained from $T^{\prime}$ by Type- 1 operation. Thus $T \in \mathscr{B}$.

This completes the proof of Lemma 7.
Note that $\left\{P_{2}, P_{3}\right\} \subseteq \mathscr{A}$. Lemma 6 implies that $\mathscr{B} \subseteq \mathscr{A}$ and Lemma 7 implies that $\mathscr{A} \subseteq \mathscr{B} \cup\left\{P_{2}\right\}$. Hence Theorem 1 is true.

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