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# 3-CONSECUTIVE C-COLORINGS OF GRAPHS*

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**Abstract**

A 3-consecutive C-coloring of a graph $G = (V, E)$ is a mapping $\varphi : V \to \mathbb{N}$ such that every path on three vertices has at most two colors. We prove general estimates on the maximum number $\bar{\chi}_{3CC}(G)$ of colors in a 3-consecutive C-coloring of $G$, and characterize the structure of connected graphs with $\bar{\chi}_{3CC}(G) \geq k$ for $k = 3$ and $k = 4$.

**Keywords:** graph coloring, vertex coloring, consecutive coloring, upper chromatic number.

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## 1. Introduction

Motivated by various new developments in the theory of graph and hypergraph coloring, in this note we introduce the notion of 3-consecutive C-coloring of graphs. For a given graph $G = (V, E)$, a mapping

$$\varphi : V \to \mathbb{N}$$

is called a 3-*consecutive C-coloring* — abbreviated as 3CC-*coloring* for short — if there exists no 3-colored path on three vertices; that is, among every three consecutive vertices there exist two having the same color.

Obviously, the trivial coloring that assigns the same color to all vertices of $G$ is 3-consecutive. Therefore, we are interested in the *maximum* number of colors that can occur in a 3CC-coloring of $G$. This number will be called the 3-*consecutive upper chromatic number* of $G$, denoted by $\bar{\chi}_{3CC}(G)$.

It is immediate by definition that, in any 3CC-coloring with more than one color, replacing two color classes with their union results in a 3CC-coloring and the number of colors decreases by precisely one. Consequently, for each integer $k$ between 1 and $\bar{\chi}_{3CC}(G)$ there exists a 3CC-coloring of $G$ with exactly $k$ colors. Moreover, by the pigeon-hole principle, every coloring with only one or two colors is a 3CC-coloring; that is, $\bar{\chi}_{3CC}(G) \geq 2$ holds for each graph having at least two vertices. Note further that $\bar{\chi}_{3CC}(G) = |V(G)|$ if and only if each connected component of $G$ is an isolated vertex or an isolated edge.

If $H$ is a spanning subgraph of a graph $G$, then any three consecutive vertices of $H$ are also consecutive in $G$. Hence any 3CC-coloring of $G$ is also a 3CC-coloring of $H$, thus

$$\bar{\chi}_{3CC}(G) \leq \bar{\chi}_{3CC}(H).$$

But a maximum 3CC-coloring of $G$ need not be a maximum 3CC-coloring of $H$, therefore strict inequality may hold.

Moreover, $\bar{\chi}_{3CC}$ is additive with respect to vertex-disjoint union; that is, the 3-consecutive upper chromatic number of a disconnected graph $G$ is equal to the sum of $\bar{\chi}_{3CC}$ over the connected components of $G$.

**Previous models.** Our new coloring model is closely related to the following earlier ones, which also served as motivation for the present study.

- A 3-*consecutive coloring* of a graph $G = (V, E)$ is a coloring of vertices of $G$ such that if $uvw$ is a path on 3 vertices, then $v$ receives the color of $u$ or $w$. The 3-*consecutive coloring number* $\chi_{3c}(G)$ is the maximum number of colors which can be used in such a coloring. This invariant was introduced in [7] and studied in some detail in [8]. Clearly, any 3-consecutive coloring is a 3CC-coloring, and hence

$$\chi_{3c}(G) \leq \bar{\chi}_{3CC}(G).$$

- A hypergraph is a pair $\mathcal{H} = (V, \mathcal{E})$, where $\mathcal{E}$ is a set system over $V$, and $\emptyset \notin \mathcal{E}$. The elements of $V$ and $\mathcal{E}$ are called vertices and edges of $\mathcal{H}$, respectively. A C-*coloring* of $\mathcal{H}$ is a mapping $\varphi : V \to \mathbb{N}$ such that every edge $E \in \mathcal{E}$ contains at least two vertices with a common color; that is, $|\varphi(E)| < |E|$. The *upper chromatic number*, denoted by $\bar{\chi}(\mathcal{H})$, is the largest possible number of colors that can be used in a C-coloring of $\mathcal{H}$.

  The roots of this notion date back to the early 1970's in the works of Berge (unpublished, cf. [1, p. 151]) and Sterboul [11]; moreover, C-coloring is a particular case of Voloshin's *mixed hypergraph* model (introduced in [12]) where it exactly means coloring of C-hypergraphs.

  The 3CC-colorings of a graph $G = (V, E)$ can be interpreted as C-colorings of the hypergraph $\mathcal{H} = (V, \mathcal{E})$ where $\mathcal{E}$ consists of all 3-element sets $\{u, v, w\} \subseteq V$ inducing a connected subgraph of $G$.

- A *color-bounded hypergraph* $\mathcal{H} = (V, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ is a hypergraph with $\mathcal{E} = \{E_1, \ldots, E_m\}$ where each edge $E_i$ is associated with two integers $s_i$ and $t_i$ such that $1 \leq s_i \leq t_i \leq |E_i|$. A vertex coloring $\varphi : X \to \mathbb{N}$ is considered to be proper if for each edge $E_i$ there occur at least $s_i$ and at most $t_i$ different colors on it; that is, the inequalities $s_i \leq |\varphi(E_i)| \leq t_i$ are satisfied, for every $E_i \in \mathcal{E}$, [2, 3].

  If $G = (V, E)$ is a graph with $V = \{v_1, \ldots, v_n\}$, then its 3CC-colorings are precisely the colorings of the color-bounded hypergraph $\mathcal{H} = (V, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ with $\mathcal{E} = \{E_1, \ldots, E_n\}$, where each hyperedge $E_i$ ($1 \leq i \leq n$) is the closed neighborhood of vertex $v_i$ in $G$, and the color bounds are $\boldsymbol{s}(E_i) = 1$ and $\boldsymbol{t}(E_i) = 2$.

**Our results.** In this paper we study 3CC-colorings of connected graphs. In Section 2 we prove upper bounds on $\bar{\chi}_{3CC}(G)$ in terms of several parameters of $G$ and, particularly, we obtain tight bounds for trees and unicyclic graphs. In Section 3 we give characterizations for graphs admitting proper 3CC-colorings with exactly 3 and exactly 4 colors. These theorems also yield necessary conditions for graphs having 3-consecutive colorings with exactly 3 and 4 colors, respectively, because of the inequality $\chi_{3c}(G) \le \bar{\chi}_{3CC}(G)$.

**Standard notation.** As usual, we write $N[v]$ for the *closed neighborhood* of vertex $v$, and $d(x,y)$ for the *distance* of vertices $x$ and $y$. In the latter, we sometimes put subscript as $d_G(x,y)$, if the graph under consideration has to be emphasized.

## 2. BOUNDS

**Theorem 1.** *For any graph $G=(V,E)$ of order $p$ and minimum degree $\delta$, we have $\bar{\chi}_{3CC}(G) \le \lfloor \frac{2p}{\delta+1} \rfloor$.*

***Proof.*** Consider a 3CC-coloring of $G$ with exactly $k$ colors. Let us call a color class or its color "small" if it contains fewer than $\frac{\delta+1}{2}$ vertices, otherwise call it "big". If all colors are big, then we immediately obtain that the number of colors is at most $2p/(\delta+1)$.

Hence, we can assume that there are $\ell \ge 1$ small color classes. Choose one vertex from each. In this way we have vertices $v_1,\dots,v_\ell$ with small colors $c_1,\dots,c_\ell$, respectively. The closed neighborhood $N[v_i]$ of each $v_i$ contains at least $\delta+1$ vertices, from exactly two colors, namely $c_i$ and another one, say $\alpha_i$. Since $c_i$ is a small color, $\alpha_i$ is a big one. Moreover, the sets $N[v_1],\dots,N[v_\ell]$ are mutually disjoint. Indeed, a common vertex with a small color would imply the identity $\alpha_i = c_j$ for a big and a small color, whilst a common vertex with a big color would yield a polychromatic $P_3$.

Now, the set $A = \bigcup_{1\le i\le \ell} N[v_i]$ contains at least $\ell(\delta+1)$ vertices and at most $2\ell$ different colors. Thus, the average size of color classes intersecting $A$ is at least $\frac{\delta+1}{2}$, and all the remaining classes are big, of size at least $\frac{\delta+1}{2}$ each. This implies $\bar{\chi}_{3CC}(G) \le \lfloor \frac{2p}{\delta+1} \rfloor$. ∎

In a graph $G=(V,E)$ a set $S \subseteq V$ is a *neighborhood set* if $\bigcup_{v\in S}(\langle N[v]\rangle) = G$, where $\langle N(v)\rangle$ is the subgraph induced by $N[v]$, the closed neighborhood of $v$. The *neighborhood number* of a graph $G$, denoted by $n_0(G)$, is the minimum cardinality of a neighborhood set in $G$ (see [9]). For short, we shall write *$N$-set* for neighborhood set in general, and *$N_0$-set* for neighborhood set of minimum cardinality.

**Theorem 2.** *Let $G$ be a connected graph. Then, $\bar{\chi}_{3CC}(G) \le n_0(G)+1$. Further, for a tree $T$, $\bar{\chi}_{3CC}(T) = n_0(T)+1$.*

***Proof.*** Let $k = n_0(G)$ and $S = \{v_1, \ldots, v_k\}$ be an $N_0$-set. Suppose that the vertices of $S$ are labeled in such a way that $(N[v_i] \cap \bigcup_{1 \leq j < i} N[v_j]) \neq \emptyset$ for all $2 \leq i \leq k$. Such an order on $S$ exists because $G$ is connected.

Since each $N[v_i]$ can have at most two colors, and at least one of them occurs in $\bigcup_{1 \leq j < i} N[v_j]$ if $i \neq 1$, $G$ cannot be colored with more than $|S|+1 = n_0(G)+1$ colors. This completes the proof of the first statement.

To prove the second part, we first fix a root in the tree $T$ and choose a smallest $N$-set $S^*$ with $|S^*| = n_0(T)$. If $S^*$ contains some vertex $v$ all of whose children also belong to $S^*$ (or, in particular, if $v$ is a leaf) then $v$ can be replaced by its parent in the $N$-set. Repeatedly applying this replacement, an $N$-set $S$ is obtained, in which every vertex has at least one child not contained in $S$.

Next, we show a procedure which yields a proper 3CC-coloring of $T$ with exactly $n_0(T)+1$ colors. First, assign color 1 to the root, and then in every step choose a vertex $v$ which has already got a color but its children have not yet. To color its children, we apply the following rules:

(i) If $v \notin S$ then every child of $v$ receives the color of $v$.

(ii) If $v \in S$ then we choose precisely one child $u$ not contained in $S$. In the coloring, $u$ will receive its dedicated color, whilst all the remaining children will get the color of $v$.

The number of used colors remains the same when we color the children of a vertex $v \notin S$, whilst it increases by precisely one when $v \in S$. Moreover, there is no leaf belonging to $S$. Taking also into account the color of the root, this means exactly $n_0(T)+1$ colors.

The obtained vertex coloring of $T$ is a proper 3CC-coloring. Indeed, for every vertex $v$, in the neighborhood $N[v]$ there occur at most two different colors because if $u$ gets its dedicated color then $u \notin S$ is assumed, hence its parent $v \in S$ has monochromatic $N[v] \setminus \{u\}$.

This coloring algorithm proves that for a tree $T$ the inequality $\bar{\chi}_{3CC}(T) \geq n_0(T)+1$ holds. We have already proved that also $\bar{\chi}_{3CC}(T) \leq n_0(T)+1$ is valid, hence the second statement follows. ∎

We remark that the algorithm described in the proof actually yields a 3-consecutive coloring. Moreover, it is known that $n_0(G)$ does not exceed the vertex covering number $\alpha_0(G)$, moreover $n_0(G) = \alpha_0(G)$ for every triangle-free graph (see [9]; characterizations for other graph classes and complexity results on $n_0(G)$ can be found in [6] and [4]). As a consequence, we have

**Corollary 1.**

(i) *For a connected graph $G$, $\bar{\chi}_{3CC}(G) \leq \alpha_0(G) + 1$.*

(ii) *For a tree $T$, $\chi_{3c}(T) = \bar{\chi}_{3CC}(T) = \alpha_0(T) + 1 = \beta_1(T) + 1$ where $\beta_1(T)$ is the edge independence number.*

(iii) *For trees, both $\chi_{3c}$ and $\bar{\chi}_{3CC}$ can be determined and an optimal coloring can be found in linear time.*

A set $S \subset V$ of vertices in a connected graph $G = (V, E)$ is called a *connected dominating set* if (i) every vertex $v \in V \setminus S$ is adjacent to at least one vertex in $S$, and (ii) the subgraph $G[S]$ induced by $S$ is connected (see [10]). The minimum cardinality of a connected dominating set $S$ is called the *connected domination number*, and is denoted by $\gamma_c(G)$; such a set $S$ is called a $\gamma_c$-*set*. Condition (i) alone defines the notion of *dominating set*, the minimum cardinality of which is called *domination number* and is denoted by $\gamma(G)$.

Bounds on $\bar{\chi}_{3CC}(G)$ involving $\gamma_c$ and $\gamma$ are as follows.

**Theorem 3.** *For any connected graph $G$, $\bar{\chi}_{3CC}(G) \leq \gamma_c(G) + 1$ holds, moreover $\bar{\chi}_{3CC}(G) \leq 2\gamma(G)$.*

***Proof.*** If $S$ is a connected dominating set of $G$, its vertices have an ordering $v_1, \dots, v_{|S|}$ such that $(N[v_i] \cap \bigcup_{1 \leq j < i} N[v_j]) \neq \emptyset$ for all $2 \leq i \leq |S|$. Similarly to the proof of the first part of Theorem 2, it can be proved that $\bar{\chi}_{3CC}(G) \leq |S| + 1$. Hence, choosing $S$ to be a $\gamma_c$-set, the first inequality follows.

Since the closed neighborhood of each vertex contains at most two colors in any 3CC-coloring, and the closed neighborhoods of the vertices in a dominating set cover the entire vertex set, the second upper bound also holds. ∎

A relation between $\bar{\chi}_{3CC}(G)$ and the chromatic number $\chi(G)$ is as follows: It is known that for a connected graph $G$ of order $p$, $\gamma_c(G) \leq p - \Delta(G)$, where $\Delta(G)$ is the maximum degree of a vertex in $G$ (cf. [5]) and $\chi(G) \leq \Delta(G) + 1$. Therefore we have the following:

**Corollary 2.** *For any connected graph $G$ of order $p \geq 3$, $\bar{\chi}_{3CC}(G) \leq p - \chi(G) + 2$.*

**Theorem 4.** *For a unicyclic graph of order $p \geq 3$,*

$$\alpha_0(G) - 1 \leq \bar{\chi}_{3CC}(G) \leq \alpha_0(G) + 1.$$

***Proof.*** In view of Corollary 1 we need to establish only the lower bound. Let $C$ be the cycle in $G$ and $e$ be an edge of $C$. Then $G-e$ is a tree and, again by Corollary 1, $\bar{\chi}_{3CC}(G-e)=\alpha_0(G-e)+1$. Since $G$ is unicyclic, either $\alpha_0(G-e)=\alpha_0(G)-1$, or $\alpha_0(G-e)=\alpha_0(G)$. Therefore, $\alpha_0(G-e)\geq \alpha_0(G)-1$. Also, $\bar{\chi}_{3CC}(G)=\bar{\chi}_{3CC}(G-e)$ or $\bar{\chi}_{3CC}(G)=\bar{\chi}_{3CC}(G-e)-1$ and hence $\bar{\chi}_{3CC}(G)\geq\bar{\chi}_{3CC}(G-e)-1$. Thus, $\bar{\chi}_{3CC}(G)\geq\alpha_0(G)-1$. ■

The upper bound is attained e.g. if $G$ is a cycle with exactly one pendant edge at each of its vertices. The lower bound is attained e.g. if $G$ is an odd cycle of length at least 5.

## 3. Characterizations

### 3.1. Three-colorability

**Theorem 5.** *A connected graph $G=(V,E)$ has a 3-consecutive C-coloring with exactly three colors — that is, $\bar{\chi}_{3CC}(G)\geq 3$ — if and only if its diameter is at least* 3.

***Proof.*** To prove necessity by a contradiction, assume a connected graph $G$ with diameter at most 2 and its proper 3CC-coloring $\varphi$ using exactly 3 colors.

Since $G$ is connected, there exist two adjacent vertices $x$ and $y$ having different colors, say $\varphi(x)=1$ and $\varphi(y)=2$. Moreover, consider a vertex $z$ colored differently from each of them: $\varphi(z)=3$. There cannot occur multicolored $P_3$ and hence, vertex $z$ is adjacent neither with $x$ nor with $y$. Since the diameter equals 2, there exist common neighbors $x'$ and $y'$ for the vertex pairs $x,z$ and $y,z$, respectively. Now, consider the $P_3$ subgraphs $x'xy$ and $xx'z$. The former one forces that $x'$ does not have color 3, whilst the latter forbids color 2. Therefore, the color of $x'$ should be 1, and by a similar argument we obtain that $y'$ has color 2. (This implies that $x'$ and $y'$ are different vertices.) But in this case the forbidden multicolored $P_3$ $x'zy'$ would appear. Consequently, if $\bar{\chi}_{3CC}(G)\geq 3$, then the diameter is at least 3.

Proving the opposite direction, we consider a graph $G$ containing two vertices $x$ and $y$ at distance 3 apart. Let the vertex coloring $\varphi$ assign color 1 to $x$, color 3 to $y$, and all the remaining vertices receive color 2. Since $x$ and $y$ cannot belong to a common $P_3$, every three consecutive vertices can get at most two different colors. Therefore, $\varphi$ is a proper 3CC-coloring and $\bar{\chi}_{3CC}(G) \geq 3$ holds. ∎

### 3.2. Four-colorability

Here we characterize the graphs admitting a 3-consecutive C-coloring with at least four colors. In the proof, the following notion will be used.

**Definition 1.** Let $\varphi$ be a vertex coloring of a connected graph $G$. The *color-graph of $G$* with respect to coloring $\varphi$, denoted by $C_\varphi(G)$, has the colors occurring in $\varphi$ as its vertices — called *color-vertices* — and two distinct color-vertices are adjacent in $C_\varphi(G)$ if and only if there exist two adjacent vertices in $G$ having the corresponding two colors.

**Theorem 6.** *A connected graph $G = (V, E)$ admits a 3-consecutive C-coloring with exactly four colors — that is, $\bar{\chi}_{3CC}(G) \geq 4$ — if and only if it satisfies at least one of the following properties:*

(i) *There exist three vertices $x, y, z \in V$ such that any two of them are at distance at least 3 apart.*

(ii) *$G$ has diameter at least 5.*

(iii) *There exists a cycle $C$ of length eight in $G$ such that, for each vertex $v \in V$, there exists a vertex $u$ in $C$ for which $d_G(u, v) = 4$ holds.*

***Proof.*** ($\Rightarrow$) To prove necessity, we assume a 3CC-coloring $\varphi$ of $G$ with exactly four colors. In this case, $C_\varphi(G)$ has vertex set $\{1, 2, 3, 4\}$. Since $G$ is connected and all the four colors are used in $\varphi$, the graph $C_\varphi(G)$ is connected, too. We distinguish three cases on the basis of vertex degrees occurring in $C_\varphi(G)$.

1. First, we assume that there exists a color-vertex in $C_\varphi(G)$ whose degree is 3. We suppose without loss of generality that this color-vertex is 1.

That is, there exist three vertices $x, y$ and $z$ in $G$ with colors 2, 3 and 4, respectively, such that each of them is adjacent to a vertex from color class 1. We will prove that any two of the vertices $x, y, z$ have distance at least 3. In a 3CC-coloring the closed neighborhood of any vertex can contain at most two different colors. This implies that $x$ and $y$ cannot have a common neighbor with color 1 or 4. On the other hand, if their common neighbor had color 2 (or 3) then in $N[y]$ (or in $N[x]$) there would occur three colors 1, 2, and 3. Therefore, the distance between $x$ and $y$ cannot be smaller than 3. The analogous statement is true for the pairs $(x, z)$ and $(y, z)$ as well. Hence, condition (i) is fulfilled. In the sequel, we refer to three vertices with this property as *three distant vertices.*

2. Second, we suppose that the color-graph $C_\varphi(G)$ is a path, where we assume the order $1, 2, 3, 4$.

Choose a vertex $x \in V$ with color 1 and a vertex $y \in V$ with color 4. Due to the assumed structure of $C_\varphi(G)$, every path connecting $x$ and $y$ contains vertices with colors 2 and 3 as well. Moreover, there occur at least two vertices with color 2 in it. Indeed, assuming only one vertex from color-class 2 in this path, this would have neighbors colored with 1 and also with 3, yielding a forbidden multicolored $P_3$. Hence, every $x$–$y$ path has at least two internal vertices from color class 2 and, similarly, there exist at least two internal vertices with color 3. Consequently, $d_G(x, y) \geq 5$, complying with condition (ii).

3. In the remaining cases, every vertex of $C_\varphi(G)$ has degree two; that is, $C_\varphi(G)$ is a cycle. We assume the cyclic order 1-2-3-4-1 of colors, and all additions concerning them will be taken modulo 4. Also in this case, (i) and/or (ii) may be satisfied. But we assume throughout that none of the first two properties is valid for $G$, and then prove that (iii) necessarily holds under this assumption.

We can partition each color class $\alpha$ ($1 \leq \alpha \leq 4$) into two parts:

- $V_{\alpha,\alpha+1}$ contains the vertices colored with $\alpha$ and having a neighbor of color $\alpha + 1$.
- $V_{\alpha,\alpha-1}$ contains the vertices colored with $\alpha$ and having a neighbor of color $\alpha - 1$.

Both $V_{\alpha,\alpha-1}$ and $V_{\alpha,\alpha+1}$ are nonempty, but $V_{\alpha,\alpha-1} \cap V_{\alpha,\alpha+1} = \emptyset$. Now, suppose for a contradiction that there exists a vertex $v$ of color $\alpha$ which is not contained in $V_{\alpha,\alpha-1} \cup V_{\alpha,\alpha+1}$. Choose a vertex $u \in V_{\alpha+1,\alpha+2}$. All the neighbors of $v$ have color $\alpha$, therefore any $v$–$u$ path contains an internal vertex with color $\alpha$, while $u$ has no neighbor of color $\alpha$, forcing one more internal vertex. Consequently, $d_G(v, u) \geq 3$. Similarly, any vertex $w \in V_{\alpha-1,\alpha-2}$ has distance at least 3 from both vertices $v$ and $u$. This would mean three distant vertices complying with property (i), but this contradicts our present assumption.

Hence, $\bigcup_{\alpha=1}^{4}(V_{\alpha,\alpha-1} \cup V_{\alpha,\alpha+1}) = V$ holds. In other words, the vertex set of $G$ is partitioned into eight nonempty disjoint sets admitting a cyclic order.

For the sake of simpler discussion, let us introduce the notation $Q_1 = V_{1,2}$, $Q_2 = V_{2,1}$, $Q_3 = V_{2,3}$, $Q_4 = V_{3,2}$, $Q_5 = V_{3,4}$, $Q_6 = V_{4,3}$, $Q_7 = V_{4,1}$, $Q_8 = V_{1,4}$. Subscripts of the sets $Q_i$ will be considered modulo 8.

Every edge of $G$ must have its endpoints either in the same class $Q_i$ or in two cyclically consecutive classes. Thus, any two vertices $x \in Q_i$ and $y \in Q_j$ $(i < j)$ are at distance at least $\min\{j - i, 8 - (j - i)\}$ apart. On the other hand, we claim that any two vertices from the same or from two consecutive classes have distance at most 2. Indeed, if $d_G(x, y) \geq 3$ for some $x \in Q_i$ and $y \in Q_j$, where $j = i$ or $j = i + 1$, then $x$ and $y$ together with any vertex from $Q_{j+3}$ would be three distant vertices, what does not meet the present requirements.

Now, choose one vertex $q_i$ from each class $Q_i$. Due to the previous observation, any two vertices $q_i$ and $q_{i+1}$ are adjacent or have a common neighbor which received the color of $q_i$ or $q_{i+1}$. Hence, joining every two consecutive vertices by a shortest path, we obtain a cycle (or a closed walk), where all the four colors occur and, for each $1 \leq \alpha \leq 4$, the vertices having color $\alpha$ form a connected arc (subpath). Hence, we can consider a shortest cycle with this property. This cycle $C$ contains some vertex from each class $Q_i$; thus, its length is at least 8. By minimality and the structure of $C_\varphi(G)$, for any two $x, y \in C$, the equality $d_C(x, y) = d_G(x, y)$ holds. Consequently, if cycle $C$ had nine or more vertices, we could choose three distant vertices, and they would have distances at least 3 not only in $C$ but also in $G$. Hence, under the assumed conditions, $C \cong C_8$ and it involves exactly one vertex $r_i$ from each class $Q_i$.

Any vertex $v \in V$ is contained in a uniquely determined class $Q_j$. Since property (ii) is not valid for $G$, we have $d_G(v, r_{j+4}) \leq 4$. On the other hand, every path from $Q_j$ to $Q_{j+4}$ has to involve vertices either from $Q_{j+1}, Q_{j+2}$ and $Q_{j+3}$, or from $Q_{j-1}, Q_{j-2}$ and $Q_{j-3}$, hence we obtain $d_G(v, r_{j+4}) = 4$, what completes the proof of necessity.

$(\Leftarrow)$ To prove sufficiency, we will construct appropriate 3CC-colorings for graphs $G$ having property (i) or (ii), and also for graphs $G$ satisfying only (iii) from the three constraints.

$(I)$ Assume that there exist three vertices $x, y, z$ having mutual distances at least 3. By this property, there is no $P_3$ involving at least two of them. Therefore, we can assign colors $1, 2$ and 3 to the vertices $x, y$ and $z$, respectively. If all the remaining vertices receive color 4, the assignment obtained is a proper 3CC-coloring with four colors, hence $\bar{\chi}_{3CC}(G) \geq 4$.

(*II*) Assuming property (ii), there exist two vertices $x$ and $y$ at distance 5 apart. Consider the following coloring $\varphi$:

- $\varphi(x) = 1; \quad \varphi(y) = 4;$
- $\varphi(v) = 2$ if $1 \le d_G(x, v) \le 2;$
- $\varphi(v) = 3$ if $d_G(x, v) > 2$ and $v \ne y$.

This yields a proper 3CC-coloring, since color 1 can occur together only with color 2 in a $P_3$, and similarly, color 4 appears only with color 3. Consequently, no multicolored $P_3$ can arise.

(*III*) Assume that the graph $G$ satisfies (iii) but none of the conditions (i) and (ii). Let the cycle corresponding to (iii) be $C_8 = r_1r_2r_3r_4r_5r_6r_7r_8$.

Consider a vertex $v \in V$. By condition (iii), there exists a vertex $r_i$ in the cycle whose distance from $v$ equals 4. Since $4 = d_G(v, r_i) \le d_G(v, r_{i+1}) + 1$, we obtain $d_G(v, r_{i+1}) \ge 3$ and, similarly, $d_G(v, r_{i-1}) \ge 3$ must hold for the other neighbor of $r_i$, too. Let us assume that there exists a further vertex $r_j$ in the cycle whose distance from $v$ is at least 3. In this case $r_{i-1}, r_j, v$ or $r_{i+1}, r_j, v$ would be three distant vertices. Since (i) is supposed to be not valid, this is a contradiction. Therefore, for all vertices $r_k$ distinct from $r_{i-1}, r_i$ and $r_{i+1}$, the inequality $d_G(v, r_k) \le 2$ holds. Taking into account that the relations $d_G(v, r_i) \ge 4$ and $d_G(v, r_{i\pm2}) \le 2$ imply $d_G(v, r_{i\pm1}) = 3$, the above argument also yields that for every $v$ there exists precisely one $r_i \in C_8$ at distance 4.

This uniqueness makes it possible to define the partition of the vertex set into eight disjoint classes (subscript addition taken modulo 8):

$$v \in Q_j \iff d_G(v, r_{j+4}) = 4, \qquad \text{for all } 1 \le j \le 8.$$

Since $r_j \in Q_j$ holds for every $j$, none of the partition classes is empty. Summarizing the previous observations:

If $v \in Q_j$,

- $d_G(v, r_{j+4}) = 4;$
- $d_G(v, r_{j+3}) = d_G(v, r_{j+5}) = 3;$
- $d_G(v, r_k) \le 2$ otherwise.

Next, we prove that any two adjacent vertices $x$ and $y$ belong either to the same $Q_i$ or to two consecutive partition classes.

Assume $x \in Q_i$, $y \in Q_j$, and $d_G(x,y) = 1$. By the properties of the distance function we obtain

$$\begin{aligned} d_G(x,y) + d_G(y, r_{i+4}) &\geq d_G(x, r_{i+4}) = 4, \\ d_G(y, r_{i+4}) &\geq 3. \end{aligned}$$

The inequalities can be fulfilled only if $i+4$ equals either $j+4$ or $j+3$ or $j+5$. These correspond to the cases where $i=j$, $i=j-1$ or $i=j+1$; that is, when $x$ and $y$ belong either to the same or two consecutive partition classes.

Now, we can define an appropriate 3CC-coloring $\varphi$ with four colors:

- $\varphi(v) = k \quad \text{if} \quad v \in Q_{2k-1} \cup Q_{2k}, \qquad \text{for all} \quad 1 \leq k \leq 4.$

As we have shown, there occur edges only between consecutive partition classes and inside one class, hence no multicolored $P_3$ can arise. This proves the assertion for the last case. ■

**Remark 1.** As it can be read out from the proof, in Theorem 6 the prescribed property (iii) can be replaced by other statements without changing validity:

- $C_8$ can be assumed to be an induced subgraph of $G$.
- We can prescribe that for every vertex $v \in V$ there exists precisely one vertex $u \in C_8$ at distance 4.

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