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# FALL COLORING OF GRAPHS I

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### Abstract

A fall coloring of a graph G is a proper coloring of the vertex set of G such that every vertex of G is a color dominating vertex in G (that is, it has at least one neighbor in each of the other color classes). The fall coloring number  $\chi_f(G)$  of G is the minimum size of a fall color partition of G (when it exists). Trivially, for any graph G,  $\chi(G) \leq \chi_f(G)$ . In this paper, we show the existence of an infinite family of graphs G with prescribed values for  $\chi(G)$  and  $\chi_f(G)$ . We also obtain the smallest non-fall colorable graphs with a given minimum degree  $\delta$  and determine their number. These answer two of the questions raised by Dunbar *et al.* 

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## 1. INTRODUCTION

Let G = (V, E) be a simple connected undirected graph. A proper coloring of a graph G is a partition  $\Pi = \{V_1, V_2, \ldots, V_k\}$  of the vertex set V of Ginto independent subsets of V. Each  $V_i$  is called a color class of  $\Pi$ . A vertex  $v \in V_i$  is a *color dominating vertex* (c.d.v.) with respect to  $\Pi$ , if it is adjacent to at least one vertex in each color class  $V_j, j \neq i$ . A kcoloring  $\Pi = \{V_1, V_2, \ldots, V_k\}$  of G is a *fall coloring* of G if each vertex of

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G is a c.d.v. with respect to  $\Pi$ . In this case,  $\Pi$  is called a *k*-fall coloring of G. The least positive integer k for which G has a k-fall coloring is the fall chromatic number of G and denoted by  $\chi_f(G)$ . A graph G may or may not have a fall coloring. For example, the cycle  $C_n$  has a fall coloring if and only if n is multiple of 3 or even [3]. Trivially,  $\chi_f(K_n) = n$  and hence all complete graphs are fall colorable. Clearly, if G is fall colorable,  $\chi(G) \leq \chi_f(G) \leq \delta(G) + 1$ , where  $\delta(G)$  is the minimum degree of G.

In Sections 2 and 3, we answer two of the questions raised by Dunbar et al. — one relating to the existence of graphs with prescribed chromatic and fall chromatic numbers and the other relating to the determination of all smallest non-fall colorable graphs with prescribed minimum degree. Notation and terminology not mentioned here can be found in [2].

# 2. Existence of Graphs G with Prescribed Values for $\chi$ and $\chi_f$

In this section, we show that given any two positive integers a and b with 2 < a < b, there exists an infinite sequence of graphs  $\{H_i\}$  with  $\chi(H_i) = a$  and  $\chi_f(H_i) = b$ . First we define a new graph  $G^*$  from a given graph G.

Let  $V(G) = \{x_1, x_2, \dots, x_n\}$ , and let  $G^*$  be the graph with vertex set  $V(G^*) = V(G) \cup V'(G)$ , where  $V'(G) = \{y_i : x_i \in V(G)\}$ ,  $V(G) \cap V'(G) = \emptyset$ , and edge set  $E(G^*) = E(G) \cup \{x_i y_j : i \neq j\}$ .

Lemma 2.1 brings out the relation between the chromatic numbers of  $G^*$  and G. The proof is straightforward.

**Lemma 2.1.** If G is not complete, then  $\chi(G^*) = \chi(G) + 1$ .

The following remarks will be used to determine, for any graph G, the fall chromatic number of  $G^*$ .

**Remark 2.2.** Let G be a graph having a fall coloring. Then G has a universal vertex if and only if any fall color partition of G contains at least one singleton color class.

**Remark 2.3.** Consider the partition  $\{x_i, y_i\}$ , i = 1, 2, ..., |V(G)| of  $V(G^*)$ . Clearly this partition is a fall color partition of  $G^*$ . Thus the graph  $G^*$  is fall colorable irrespective of G being fall colorable or not. Moreover,  $\chi_f(G^*) \leq |V(G)|$ . **Remark 2.4.** In any fall coloring of  $G^*$ , all the vertices of V'(G) either receive the same color or else receive distinct colors. Also, V'(G) is an independent subset of  $G^*$ .

**Theorem 2.5.** If G has fall coloring and G has no universal vertex, then  $\chi_f(G^*) = \chi_f(G) + 1.$ 

**Proof.** As G has no universal vertex, by Remark 2.2, in any fall color partition of G, each color class contains at least two vertices. Consequently, if  $k = \chi_f(G)$ , then G has a k-fall color partition with each color class containing at least two vertices. Give a new color k + 1 to all vertices of V'(G) which yields a (k + 1)-fall coloring of  $G^*$ . Thus  $\chi_f(G^*) \leq \chi_f(G) + 1$ .

Suppose  $\chi_f(G^*) \leq \chi_f(G)$ . If  $l = \chi_f(G^*)$ , then l < n, where n = |V(G)|. By Remark 2.4, all vertices of V'(G) receive the same color, say, l. Then the remaining (l-1) colors must appear in G and this coloring induces a (l-1)-fall coloring of G and hence  $\chi_f(G) \leq l-1$ , contradiction to the assumption that  $\chi_f(G^*) \leq \chi_f(G)$ . Therefore  $\chi_f(G^*) = \chi_f(G) + 1$ .

**Theorem 2.6.** For any graph G,  $\chi_f(G^*) = |V(G)|$  if and only if

- (i) G has no fall coloring or
- (ii) G has a fall coloring and contains a universal vertex.

**Proof.** Suppose  $\chi_f(G^*) = |V(G)|$ . If G has no fall coloring, then we are done. If not, G has a fall coloring. Suppose G has no universal vertex, then by Theorem 2.5,  $\chi_f(G^*) = \chi_f(G) + 1$  and by Remark 2.2, in any fall color partition of G, each color class contains at least two vertices. Thus  $|V(G)| \geq 2\chi_f(G)$  and  $|V(G)| \geq 4$ . Therefore,  $\chi_f(G^*) \leq \frac{|V(G)|}{2} + 1$ , a contradiction to the fact that  $\chi_f(G^*) = |V(G)|$ .

Conversely, assume (i) so that G has no fall coloring and  $k = \chi_f(G^*) < |V(G)|$ . Then by Remark 2.4, if  $\Pi$  is a k-fall coloring of  $G^*$ , then V'(G) will be a color class receiving the same color, say, k of  $\Pi$ . Now it is clear that in a fall coloring of a graph H, the union S of any subset of color classes will induce a fall coloring on the subgraph of H induced by S. Therefore,  $\Pi - V'(G)$  will be a fall coloring of G, a contradiction.

Now assume (ii) so that G has a fall coloring and that G has a universal vertex. By Remark 2.2, any fall color partition of G contains at least one singleton color class. Suppose  $k = \chi_f(G^*) < |V(G)|$ . By Remark 2.4, in any k-fall color partition of  $G^*$ , all vertices of V'(G) receive the same color, say, k, and the remaining (k-1)-colors are present in G. These (k-1) colors

induce a (k-1)-fall coloring of G, say  $\Pi$ . By our assumption,  $\Pi$  contains at least one singleton color class, say,  $V_i = \{x\}$ , then its corresponding vertex y in V'(G) is not adjacent to the vertex x (the only vertex of color i), a contradiction.

**Corollary 2.7.** For any positive integers a, b with  $3 \le a < b$ , there is an infinite sequence of graphs  $\{H_i\}$  with  $\chi(H_i) = a$  and  $\chi_f(H_i) = b$ .

**Proof.** Let  $G_{a,b}$  be a graph obtained by attaching b-a+1 pendant edges at a vertex of  $K_{a-1}$ . Then  $|V(G_{a,b})| = b$ . If a = 3, then  $G_{a,b}$  has a fall coloring and being a star it has a universal vertex. If  $a \ge 4$ , then  $G_{a,b}$  has no fall coloring (as the condition  $\chi \le \delta + 1$  is violated). Therefore by Theorem 2.6,  $\chi_f(G^*) = b$ .

Since  $G_{a,b}$  is not complete and by Lemma 2.1,  $\chi(G_{a,b}^*) = a$  (as  $\chi(G_{a,b}) = a - 1$ ).

This construction can be used to generate an infinite sequence  $\mathcal{H}_{a,b} = \{H_i\}$  of graphs with  $\chi = a$  and  $\chi_f = b$  as follows:

Start with  $G_{a,b}$  and get  $H_1 = G_{a,b}^*$ . Form  $H_2$  by concatenating a copy of  $G_{a,b}^*$  at a vertex of  $H_1$ , and in general, form  $H_i$  by concatenating a copy of  $G_{a,b}^*$  at a vertex of  $H_{i-1}$  (Recall that a concatenation of a graph G with a graph H is the graph got by linking G and H by the identification of a vertex of G with a vertex of H). Each graph in  $\mathcal{H}_{a,b} = \{H_i\}$  has  $\chi(H_i) = a$ and  $\chi_f(H_i) = b$ .

# 3. Smallest Non-Fall Colorable Graphs with Given Minimum Degree

In this section, we determine the smallest (with respect to both order and size) non-fall colorable graphs with given minimum degree  $\delta$ .

**Theorem 3.1.** The graph  $G = \overline{C_{p_1} \cup C_{p_2} \cup \cdots \cup C_{p_l}}$ , (where  $\cup$  stands for disjoint union), has no fall coloring if and only if for at least one i,  $p_i$  is odd and  $p_i \geq 5$ .

**Proof.** Assume that G has no fall coloring and that no  $p_i$  is odd and greater than or equal to 5 (that is, if  $p_i$  is odd, then  $p_i = 3$ ). Without loss of generality, let  $p_1, \ldots, p_r$  be even and  $p_{r+1}, \ldots, p_l$  be odd. Then it is easy to give a fall color partition of G as follows: Just pair off the consecutive

vertices of  $C_{p_i}$  for each  $i, 1 \leq i \leq r$ , and treat each such part as a color class (for instance, for  $C_{2k}$ , color the vertices consecutively by  $1, 1; 2, 2; \ldots; k, k$ ), and in the case when  $j \geq r + 1$ , we can treat each of  $V(C_{p_j}) = V(C_3)$  as a color class. Thus, we get a contradiction.

Conversely, assume that for at least one  $i, p_i \ge 5$  and odd. Then G has no fall coloring, the reason being some vertex of  $C_{p_i}$  cannot be a c.d.v. in G.

**Theorem 3.2.** Any graph G with  $|V(G)| \leq \delta(G) + 2$ , where  $\delta(G)$  is the minimum degree of G, has a fall coloring.

# **Proof.** There are only two cases to consider.

(i)  $|V(G)| = \delta(G) + 1$ . In this case  $G = K_{\delta(G)+1}$  and hence G has a fall coloring.

(ii)  $|V(G)| = \delta(G) + 2$ . Let  $S = \{x \in V(G) : d(x) = \delta(G)\}$  and T = V(G) - S. Then  $\langle T \rangle$ , the subgraph induced by T, is a clique in G and for every  $x \in S$ , there exists a unique vertex  $y \neq x$  in S such that  $xy \notin E(G)$ . Thus |S| must be even and there are exactly  $\frac{|S|}{2}$  pairs of nonadjacent vertices in G. For  $1 \leq i \leq r := \frac{|S|}{2}$ , let  $S_i$  be the pair  $\{x_i, y_i\}$  of vertices in S such that  $x_iy_i \notin E(G)$ . Let  $T = \{u_1, u_2, \ldots, u_k\}$ .

Define  $c: V(G) \to \{1, 2, ..., r, r+1, ..., r+k\}$  by

$$c(v) = \begin{cases} i & \text{if } v \in S_i, \\ r+j & \text{if } v = u_j \text{ for some } j, \ 1 \le j \le k. \end{cases}$$

Clearly c is a proper coloring of G and every vertex of G is a c.d.v.. Thus G has a fall coloring.

Hence a smallest non-fall colorable graph of minimum degree  $\delta$  must be of order at least  $\delta + 3$  and size at least  $\frac{\delta(\delta+3)}{2}$ .

Naturally, any such graph G must be  $\delta$ -regular graph and order  $\delta + 3$  and hence its complement must be a disjoint union of cycles.

We can take  $G = \overline{C_{p_1} \cup C_{p_2} \cup \cdots \cup C_{p_l}}$ , where  $\sum_{i=1}^{l} p_i = \delta(G) + 3$ , all  $p_i \geq 3$  and at least one  $p_i$  is odd and  $p_i \geq 5$ . Then, clearly, G is a  $\delta(G)$ -regular graph and by Theorem 3.1, G has no fall coloring. This G is our required graph. Clearly, G is not unique if  $\delta \geq 6$  and unique if  $\delta = 5$ .

The smallest non-fall colorable graphs with  $\delta \leq 4$  have been determined earlier in [3]. The extremal graph, for  $\delta = 2$ , is  $\overline{C_5} \cong C_5$ , and for  $\delta = 4$ , it is  $\overline{C_7}$ . These coincide with the extremal graphs given in [3]. For  $\delta = 3$ , there are two smallest non-fall colorable graphs, namely,  $\overline{P_3 \cup K_3}$  and the wheel on 6 vertices and these are given in [3]. In this case, as  $\delta + 3 = 6$  does not have a partition in the way we required, we do not get the smallest non-fall colorable graphs by our result. However, if we treat  $\overline{C_5 \cup C_1}$  as a degenerate case, we get the wheel on 6 vertices. For  $\delta \ge 4$ , our result gives all the smallest non-fall colorable graphs. Their exact number (where  $\delta \ge 4$ ) can be obtained as follows: Let N(k) denote the number of partitions of k in which each part is of size at least 3 and one part is odd and of size at least 5. Then N(k) gives the number of smallest non-fall colorable graphs of order k (with minimum degree k - 3).

Let p(n) be the well-known partition function of n [1]. Sort each partition from smallest part to largest part. Then, p(n) - p(n-1) - p(n-2) + p(n-3) gives the number of partitions of n not beginning with a 1 or 2. Doubling each part of a partition of  $\frac{n}{2}$  gives an even partition of n, and so the number of even partitions which do not begin with 2 is  $p(\frac{n}{2}) - p(\frac{n}{2} - 1)$ . The remaining partitions to be excluded are those with smallest part equal to 3, whose remaining parts are even. Removing the first m copies of 3 (a fixed portion of the partition), the remaining even partitions can be given by  $p(\frac{n-3m}{2})$ , and to ensure that the even portion does not begin with two, we subtract  $p(\frac{(n-3m)}{2}+1)$ . Let p(n) = 0 if n is not an integer, and we have the following expression for N(k):

$$N(k) = (p(k) - p(k-1) - p(k-2) + p(k-3)) - \sum_{m=0}^{\lfloor k/3 \rfloor} \left( p\left(\frac{k-3m}{2}\right) - p\left(\frac{k-3m}{2} + 1\right) \right).$$

For example, N(8) = 1 and N(11) = 4. N(8) corresponds to the unique graph  $\overline{C_3 \cup C_5}$ , while N(11) corresponds to the four graphs  $\overline{C_{11}}$ ,  $\overline{C_4 \cup C_7}$ ,  $\overline{C_5 \cup C_6}$  and  $\overline{C_3 \cup C_3 \cup C_5}$ .

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