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FALL COLORING OF GRAPHS I

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Abstract

A fall coloring of a graph G is a proper coloring of the vertex set of G such that every vertex of G is a color dominating vertex in G (that is, it has at least one neighbor in each of the other color classes). The fall coloring number $\chi_f(G)$ of G is the minimum size of a fall color partition of G (when it exists). Trivially, for any graph G, $\chi(G) \leq \chi_f(G)$. In this paper, we show the existence of an infinite family of graphs G with prescribed values for $\chi(G)$ and $\chi_f(G)$. We also obtain the smallest non-fall colorable graphs with a given minimum degree δ and determine their number. These answer two of the questions raised by Dunbar *et al.*

Keywords: fall coloring of graphs, non-fall colorable graphs.

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1. Introduction

Let $G = (V, E)$ be a simple connected undirected graph. A proper coloring of a graph G is a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set V of G into independent subsets of V. Each V_i is called a color class of Π. A vertex $v \in V_i$ is a *color dominating vertex* (c.d.v.) with respect to Π, if it is adjacent to at least one vertex in each color class $V_j, j \neq i$. A k-coloring $\Pi = \{V_1, V_2, \ldots, V_k\}$ of G is a *fall coloring* of G if each vertex of

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G is a c.d.v. with respect to Π. In this case, Π is called a *k-fall coloring* of G. The least positive integer k for which G has a k-fall coloring is the *fall chromatic number* of G and denoted by $\chi_f(G)$. A graph G may or may not have a fall coloring. For example, the cycle C_n has a fall coloring if and only if n is multiple of 3 or even [3]. Trivially, $\chi_f(K_n) = n$ and hence all complete graphs are fall colorable. Clearly, if G is fall colorable, $\chi(G) \leq \chi_f(G) \leq \delta(G) + 1$, where $\delta(G)$ is the minimum degree of G.

In Sections 2 and 3, we answer two of the questions raised by Dunbar *et al.* — one relating to the existence of graphs with prescribed chromatic and fall chromatic numbers and the other relating to the determination of all smallest non-fall colorable graphs with prescribed minimum degree. Notation and terminology not mentioned here can be found in [2].

2. Existence of Graphs G with Prescribed Values for χ and χ_f

In this section, we show that given any two positive integers a and b with $2 < a < b$, there exists an infinite sequence of graphs $\{H_i\}$ with $\chi(H_i) = a$ and $\chi_f(H_i) = b$. First we define a new graph G^* from a given graph G.

Let $V(G) = \{x_1, x_2, \ldots, x_n\}$, and let G^* be the graph with vertex set $V(G^*) = V(G) \cup V'(G)$, where $V'(G) = \{y_i : x_i \in V(G)\}$, $V(G) \cap V'(G) = \emptyset$, and edge set $E(G^*) = E(G) \cup \{x_i y_j : i \neq j\}$.

Lemma 2.1 brings out the relation between the chromatic numbers of G^* and G. The proof is straightforward.

Lemma 2.1. *If G is not complete, then $\chi(G^*) = \chi(G) + 1$.*

The following remarks will be used to determine, for any graph G, the fall chromatic number of G^*.

Remark 2.2. Let G be a graph having a fall coloring. Then G has a universal vertex if and only if any fall color partition of G contains at least one singleton color class.

Remark 2.3. Consider the partition $\{x_i, y_i\}$, $i = 1, 2, \ldots, |V(G)|$ of $V(G^*)$. Clearly this partition is a fall color partition of G^*. Thus the graph G^* is fall colorable irrespective of G being fall colorable or not. Moreover, $\chi_f(G^*) \leq |V(G)|$.

Remark 2.4. In any fall coloring of G^*, all the vertices of $V'(G)$ either receive the same color or else receive distinct colors. Also, $V'(G)$ is an independent subset of G^*.

Theorem 2.5. *If G has fall coloring and G has no universal vertex, then $\chi_f(G^*) = \chi_f(G) + 1$.*

Proof. As G has no universal vertex, by Remark 2.2, in any fall color partition of G, each color class contains at least two vertices. Consequently, if $k = \chi_f(G)$, then G has a k-fall color partition with each color class containing at least two vertices. Give a new color $k+1$ to all vertices of $V'(G)$ which yields a $(k+1)$-fall coloring of G^*. Thus $\chi_f(G^*) \leq \chi_f(G) + 1$.

Suppose $\chi_f(G^*) \leq \chi_f(G)$. If $l = \chi_f(G^*)$, then $l < n$, where $n = |V(G)|$. By Remark 2.4, all vertices of $V'(G)$ receive the same color, say, l. Then the remaining $(l-1)$ colors must appear in G and this coloring induces a $(l-1)$-fall coloring of G and hence $\chi_f(G) \leq l-1$, contradiction to the assumption that $\chi_f(G^*) \leq \chi_f(G)$. Therefore $\chi_f(G^*) = \chi_f(G) + 1$. ■

Theorem 2.6. *For any graph G, $\chi_f(G^*) = |V(G)|$ if and only if*

(i) *G has no fall coloring or*

(ii) *G has a fall coloring and contains a universal vertex.*

Proof. Suppose $\chi_f(G^*) = |V(G)|$. If G has no fall coloring, then we are done. If not, G has a fall coloring. Suppose G has no universal vertex, then by Theorem 2.5, $\chi_f(G^*) = \chi_f(G) + 1$ and by Remark 2.2, in any fall color partition of G, each color class contains at least two vertices. Thus $|V(G)| \geq 2\chi_f(G)$ and $|V(G)| \geq 4$. Therefore, $\chi_f(G^*) \leq \frac{|V(G)|}{2} + 1$, a contradiction to the fact that $\chi_f(G^*) = |V(G)|$.

Conversely, assume (i) so that G has no fall coloring and $k = \chi_f(G^*) < |V(G)|$. Then by Remark 2.4, if Π is a k-fall coloring of G^*, then $V'(G)$ will be a color class receiving the same color, say, k of Π. Now it is clear that in a fall coloring of a graph H, the union S of any subset of color classes will induce a fall coloring on the subgraph of H induced by S. Therefore, $\Pi - V'(G)$ will be a fall coloring of G, a contradiction.

Now assume (ii) so that G has a fall coloring and that G has a universal vertex. By Remark 2.2, any fall color partition of G contains at least one singleton color class. Suppose $k = \chi_f(G^*) < |V(G)|$. By Remark 2.4, in any k-fall color partition of G^*, all vertices of $V'(G)$ receive the same color, say, k, and the remaining $(k-1)$-colors are present in G. These $(k-1)$ colors induce a $(k-1)$-fall coloring of G, say Π. By our assumption, Π contains at least one singleton color class, say, $V_i = \{x\}$, then its corresponding vertex y in $V'(G)$ is not adjacent to the vertex x (the only vertex of color i), a contradiction. ∎

Corollary 2.7. *For any positive integers a, b with $3 \leq a < b$, there is an infinite sequence of graphs $\{H_i\}$ with $\chi(H_i) = a$ and $\chi_f(H_i) = b$.*

Proof. Let $G_{a,b}$ be a graph obtained by attaching $b-a+1$ pendant edges at a vertex of K_{a-1}. Then $|V(G_{a,b})| = b$. If $a = 3$, then $G_{a,b}$ has a fall coloring and being a star it has a universal vertex. If $a \geq 4$, then $G_{a,b}$ has no fall coloring (as the condition $\chi \leq \delta + 1$ is violated). Therefore by Theorem 2.6, $\chi_f(G^*) = b$.

Since $G_{a,b}$ is not complete and by Lemma 2.1, $\chi(G^*_{a,b}) = a$ (as $\chi(G_{a,b}) = a-1$).

This construction can be used to generate an infinite sequence $\mathcal{H}_{a,b} = \{H_i\}$ of graphs with $\chi = a$ and $\chi_f = b$ as follows:

Start with $G_{a,b}$ and get $H_1 = G^*_{a,b}$. Form H_2 by concatenating a copy of $G^*_{a,b}$ at a vertex of H_1, and in general, form H_i by concatenating a copy of $G^*_{a,b}$ at a vertex of H_{i-1} (Recall that a concatenation of a graph G with a graph H is the graph got by linking G and H by the identification of a vertex of G with a vertex of H). Each graph in $\mathcal{H}_{a,b} = \{H_i\}$ has $\chi(H_i) = a$ and $\chi_f(H_i) = b$. ∎

3. Smallest Non-Fall Colorable Graphs with Given Minimum Degree

In this section, we determine the smallest (with respect to both order and size) non-fall colorable graphs with given minimum degree δ.

Theorem 3.1. *The graph $G = \overline{C_{p_1} \cup C_{p_2} \cup \cdots \cup C_{p_l}}$, (where $\cup$ stands for disjoint union), has no fall coloring if and only if for at least one i, p_i is odd and $p_i \geq 5$.*

Proof. Assume that G has no fall coloring and that no p_i is odd and greater than or equal to 5 (that is, if p_i is odd, then $p_i = 3$). Without loss of generality, let $p_1, \ldots, p_r$ be even and $p_{r+1}, \ldots, p_l$ be odd. Then it is easy to give a fall color partition of G as follows: Just pair off the consecutive vertices of C_{p_i} for each i, $1 \le i \le r$, and treat each such part as a color class (for instance, for C_{2k}, color the vertices consecutively by $1, 1; 2, 2; \dots ; k, k$), and in the case when $j \ge r + 1$, we can treat each of $V(C_{p_j}) = V(C_3)$ as a color class. Thus, we get a contradiction.

Conversely, assume that for at least one i, $p_i \ge 5$ and odd. Then G has no fall coloring, the reason being some vertex of C_{p_i} cannot be a c.d.v. in G. ■

Theorem 3.2. *Any graph G with $|V(G)| \le \delta(G) + 2$, where $\delta(G)$ is the minimum degree of G, has a fall coloring.*

Proof. There are only two cases to consider.

(i) $|V(G)| = \delta(G) + 1$. In this case $G = K_{\delta(G)+1}$ and hence G has a fall coloring.

(ii) $|V(G)| = \delta(G) + 2$. Let $S = \{x \in V(G) : d(x) = \delta(G)\}$ and $T = V(G) - S$. Then $\langle T \rangle$, the subgraph induced by T, is a clique in G and for every $x \in S$, there exists a unique vertex y $(\neq x)$ in S such that $xy \notin E(G)$. Thus $|S|$ must be even and there are exactly $\frac{|S|}{2}$ pairs of nonadjacent vertices in G. For $1 \le i \le r := \frac{|S|}{2}$, let S_i be the pair $\{x_i, y_i\}$ of vertices in S such that $x_i y_i \notin E(G)$. Let $T = \{u_1, u_2, \dots, u_k\}$.

Define $c : V(G) \to \{1, 2, \dots, r, r + 1, \dots, r + k\}$ by

$$c(v) = \begin{cases} i & \text{if } \ v \in S_i, \\ r + j & \text{if } \ v = u_j \ \text{ for some } j, \ 1 \le j \le k. \end{cases}$$

Clearly c is a proper coloring of G and every vertex of G is a c.d.v.. Thus G has a fall coloring. ■

Hence a smallest non-fall colorable graph of minimum degree δ must be of order at least $\delta + 3$ and size at least $\frac{\delta(\delta+3)}{2}$.

Naturally, any such graph G must be δ-regular graph and order $\delta + 3$ and hence its complement must be a disjoint union of cycles.

We can take $G = \overline{C_{p_1} \cup C_{p_2} \cup \cdots \cup C_{p_l}}$, where $\sum_{i=1}^{l} p_i = \delta(G) + 3$, all $p_i \ge 3$ and at least one p_i is odd and $p_i \ge 5$. Then, clearly, G is a $\delta(G)$-regular graph and by Theorem 3.1, G has no fall coloring. This G is our required graph. Clearly, G is not unique if $\delta \ge 6$ and unique if $\delta = 5$.

The smallest non-fall colorable graphs with $\delta \le 4$ have been determined earlier in [3]. The extremal graph, for $\delta = 2$, is $\overline{C_5} \cong C_5$, and for $\delta = 4$, it is $\overline{C_7}$. These coincide with the extremal graphs given in [3]. For $\delta = 3$, there are two smallest non-fall colorable graphs, namely, $\overline{P_3 \cup K_3}$ and the wheel on 6 vertices and these are given in [3]. In this case, as $\delta + 3 = 6$ does not have a partition in the way we required, we do not get the smallest non-fall colorable graphs by our result. However, if we treat $\overline{C_5 \cup C_1}$ as a degenerate case, we get the wheel on 6 vertices. For $\delta \geq 4$, our result gives all the smallest non-fall colorable graphs. Their exact number (where $\delta \geq 4$) can be obtained as follows: Let $N(k)$ denote the number of partitions of k in which each part is of size at least 3 and one part is odd and of size at least 5. Then $N(k)$ gives the number of smallest non-fall colorable graphs of order k (with minimum degree $k-3$).

Let $p(n)$ be the well-known partition function of n [1]. Sort each partition from smallest part to largest part. Then, $p(n) - p(n-1) - p(n-2) + p(n-3)$ gives the number of partitions of n not beginning with a 1 or 2. Doubling each part of a partition of $\frac{n}{2}$ gives an even partition of n, and so the number of even partitions which do not begin with 2 is $p(\frac{n}{2}) - p(\frac{n}{2} - 1)$. The remaining partitions to be excluded are those with smallest part equal to 3, whose remaining parts are even. Removing the first m copies of 3 (a fixed portion of the partition), the remaining even partitions can be given by $p\big(\frac{n-3m}{2}\big)$, and to ensure that the even portion does not begin with two, we subtract $p\big(\frac{(n-3m)}{2} + 1\big)$. Let $p(n) = 0$ if n is not an integer, and we have the following expression for $N(k)$:

$$\begin{aligned} N(k) \;=\; & (p(k) - p(k-1) - p(k-2) + p(k-3)) \\ & - \sum_{m=0}^{\lfloor k/3 \rfloor} \Big(p\Big(\frac{k-3m}{2}\Big) - p\Big(\frac{k-3m}{2} + 1\Big)\Big). \end{aligned}$$

For example, $N(8) = 1$ and $N(11) = 4$. $N(8)$ corresponds to the unique graph $\overline{C_3 \cup C_5}$, while $N(11)$ corresponds to the four graphs $\overline{C_{11}}$, $\overline{C_4 \cup C_7}$, $\overline{C_5 \cup C_6}$ and $\overline{C_3 \cup C_3 \cup C_5}$.

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References

[1] G.E. Andrews, The Theory of Partitions (Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998). Reprint of the 1976 original.

[2] R. Balakrishnan and K. Ranganathan. A Textbook of Graph Theory (Universitext, Springer-Verlag, New York, 2000).

[3] J.E. Dunbar, S.M. Hedetniemi, S.T. Hedetniemi, D.P. Jacobs, J. Knisely, R.C. Laskar and D.F. Rall, *Fall colorings of graphs*, J. Combin. Math. Combin. Comput. **33** (2000) 257–273. Papers in honour of Ernest J. Cockayne.

[4] R.C. Laskar and J. Lyle, *Fall coloring of bipartite graphs and cartesian products of graphs*, Discrete Appl. Math. **157** (2009) 330–338.

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