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THE EDGE C_4 GRAPH OF SOME GRAPH CLASSES

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Abstract

The edge C_4 graph of a graph G, $E_4(G)$ is a graph whose vertices are the edges of G and two vertices in $E_4(G)$ are adjacent if the corresponding edges in G are either incident or are opposite edges of some C_4. In this paper, we show that there exist infinitely many pairs of non isomorphic graphs whose edge C_4 graphs are isomorphic. We study the relationship between the diameter, radius and domination number of G and those of $E_4(G)$. It is shown that for any graph G without isolated vertices, there exists a super graph H such that $C(H) = G$ and $C(E_4(H)) = E_4(G)$. Also we give forbidden subgraph characterizations for $E_4(G)$ being a threshold graph, block graph, geodetic graph and weakly geodetic graph.

Keywords: edge C_4 graph, threshold graph, block graph, geodetic graph, weakly geodetic graph.

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1. Introduction

We consider the graph operator $E_4(G)$, whose vertices are the edges of G and two vertices in $E_4(G)$ are adjacent if the corresponding edges in G are either incident or are opposite edges of some C_4. This graph class is also known by the name edge graph in [11]. In $E_4(G)$ any two vertices are adjacent if the union of the corresponding edges in G induce any one of the graphs P_3, C_3, C_4, $K_4 - \{e\}$, K_4. If $a_1 - a_2$ is an edge in G, the corresponding vertex in $E_4(G)$ is denoted by a_1a_2. In [9], we obtained characterizations for $E_4(G)$ being connected, complete, bipartite etc and also some dynamical behaviour of $E_4(G)$ are studied. It was also proved that $E_4(G)$ has no forbidden subgraphs.

For a vertex $v \in V(G)$, $N(v)$ denotes the set of all vertices in G which are adjacent to v and $N[v] = N(v) \cup \{v\}$. A vertex x *dominates* a vertex y if $N(y) \subseteq N[x]$. If x dominates y or y dominates x, then x and y are *comparable.* Otherwise, they are *incomparable.* The *Dilworth number* of a graph G, $dilw(G)$ is the largest number of pairwise incomparable vertices of G. A vertex v is a *universal vertex* if it is adjacent to all the other vertices in G. A subset S of V is a *dominating set* if each vertex of G that is not in S is adjacent to at least one vertex of S. If S is a dominating set then $N[S] = V$. A dominating set of minimum cardinality is called a *minimum dominating set*, its cardinality is called the *domination number* of G and it is denoted by $\gamma(G)$. Many types of domination and its characteristics are discussed in [5]. In [4], it is observed that for graphs G without isolated vertices, $\gamma(G) \leq dilw(G)$.

All the graphs considered here are finite, undirected and simple. We denote by P_n (respectively C_n), a path (respectively cycle) on n vertices. The graph obtained by deleting any edge 'e' of K_n is denoted by $K_n - \{e\}$. The *join* of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is denoted by $G \vee H$ and has $V(G \vee H) = V_1 \cup V_2$ and $E(G \vee H) = E_1 \cup E_2 \cup \{(u, v) : u \in V_1$ and $v \in V_2\}$. A 'bow' is $K_1 \vee 2K_2$. The graph obtained by attaching a pendant vertex to any vertex of C_n, is called an 'n-pan' and a 'paw' is a 3-pan. The graph in Figure 1 is called a 'moth'.

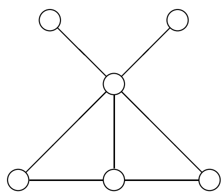


Figure 1

A graph G is H*-free* if G does not contain H as an induced subgraph. A graph H is a forbidden subgraph for a property P, if any graph G which satisfies the property P cannot have H as an induced subgraph. The *distance* between any two vertices u and v of a connected graph G, $d_G(u, v)$ is the length of a shortest path joining them. The *eccentricity* of a vertex $v \in V(G)$ is $e(v) = \max\{d(u,v) : u \in V(G)\}$. The *radius* and *diameter* of G are respectively $rad(G) = \min\{e(v) : v \in V(G)\}$, $diam(G) = \max\{e(v) : v \in V(G)\}$. A vertex v is called a *central* vertex of G if $e(v) = rad(G)$. The center, $C(G)$ of a connected graph G is the subgraph of G induced by its central vertices. The *girth* of G, $g(G)$ is the length of a shortest cycle in G. A *clique* in G is a complete subgraph of G. For all basic concepts and notations not mentioned in this paper we refer [13].

The *line graph* $L(G)$ of a graph G is a graph that has a vertex for every edge of G, and two vertices of $L(G)$ are adjacent if and only if they correspond to two edges of G with a common end vertex. In [8], it is shown that for any graph G without isolated vertices, there is a graph H such that $C(H) = G$ and $C(L(H)) = L(G)$. It is further proved that $diam(L(G)) \leq diam(G) + 1$ and $rad(L(G)) \leq rad(G) + 1$.

In [1], several graph classes and their forbidden subgraph characterizations for many properties are discussed in detail. We consider the graph classes — threshold graphs, cographs, block graphs, geodetic graphs and weakly geodetic graphs with regard to $E_4(G)$.

Threshold graphs were introduced by Chvátal and Hammer in [2]. It is known that a graph G is a threshold graph if and only if $dilw(G) = 1$ and that G is $\{2K_2, C_4, P_4\}$-free graph [2, 5].

In [7], it is proved that a connected graph G is a block graph if and only if every maximal 2-connected subgraph (block) is complete. A cycle C of G is a *b-cycle* of G if C is not contained in a clique of G. The *bulge* of G, $b(G)$ is the minimum length of a b-cycle in G if G contains a b-cycle and is ∞ otherwise. Also, G is a *block graph* if and only if $b(G) = \infty$ [6, 7].

A graph G is a *geodetic* graph [10] if any two vertices of G are joined by a unique shortest path and G is *weakly geodetic* if for every pair of vertices of distance two, there is a unique common neighbour [7]. A graph G is weakly geodetic if and only if $b(G) \geq 5$ [6, 7]. It is known that block graphs $\subseteq$ geodetic graphs $\subseteq$ weakly geodetic graphs [1].

P_4-free graphs are called *cographs* [3]. The domination number of cographs is at most two [12].

It is well known that $K_{1,3}$ and K_3 are the only non isomorphic graphs with isomorphic line graphs. Even though $L(G) \subseteq E_4(G)$, it is proved in this paper that there exist infinitely many pairs of non isomorphic graphs with isomorphic edge C_4 graphs. We study relations between $\gamma(G)$ and $\gamma(E_4(G))$, $diam(G)$ and $diam(E_4(G))$, and $rad(G)$ and $rad(E_4(G))$. We prove that for any graph G without isolated vertices, it is possible to construct a super graph H such that $C(H) = G$ and $C(E_4(H)) = E_4(G)$. We also obtain forbidden subgraph characterizations for $E_4(G)$ being threshold graph, block graph, geodetic graph and weakly geodetic graph.

2. Some Properties of $E_4(G)$

Theorem 1. *There exist infinitely many pairs of non isomorphic graphs whose edge C_4 graphs are isomorphic.*

Proof. Let $G = K_{1,n}$. If $n = 2k-1$, then take $H = K_2 \vee (k-1)K_1$ and if $n = 2k$, then take $H = 2K_1 \vee kK_1$. Clearly G and H are non isomorphic graphs. But $E_4(G) = E_4(H) = K_n$. ∎

Theorem 2. *For a connected graph G, $diam(G) - 1 \leq diam(E_4(G)) \leq diam(G) + 1$ and $rad(G) - 1 \leq rad(E_4(G)) \leq rad(G) + 1$.*

Proof. By the definition of $E_4(G)$ and $L(G)$, $diam(E_4(G)) \leq diam(L(G))$ and $rad(E_4(G)) \leq rad(L(G))$. But, $diam(L(G)) \leq diam(G) + 1$ and $rad(L(G)) \leq rad(G)+1$. Thus $diam(E_4(G)) \leq diam(G)+1$ and $rad(E_4(G)) \leq rad(G) + 1$.

Next let $diam(G) = k$. We want to prove that $diam(E_4(G)) \geq k - 1$. On the contrary, assume that $diam(E_4(G)) < k-1$. Let u and v be any two vertices in G and let $u - u'$, $v - v'$ be any two edges incident with u and v respectively. But $d_{E_4(G)}(uu', vv') < k - 1$. So $d_G(u, v) \leq d_{E_4(G)}(uu', vv') + 1 < k$, which is a contradiction to the fact that $diam(G) = k$.

Finally, let $rad(G) = k$. It is required to prove that $rad(E_4(G) \geq k-1$. On the contrary, suppose that $rad(E_4(G)) < k - 1$. Then there exists a vertex uu' in $E_4(G)$ such that $e(uu') < k - 1$. Consider the vertex u in G. Let v be any vertex in G and vv' be any edge incident with v. Then $d_G(u, v) \leq d_{E_4(G)}(uu', vv') + 1 < k$, and hence $e(u) < k$, which is a contradiction to the fact that $rad(G) = k$. ∎

Note 1. The bounds in Theorem 2 are strict.

If G is a bow, then $diam(G) = 2$, $diam(E_4(G)) = 3$, $rad(G) = 1$ and $rad(E_4(G)) = 2$.

If G is C_4, then $diam(G) = 2$, $diam(E_4(G)) = 1$, $rad(G) = 2$ and $rad(E_4(G)) = 1$. ∎

Theorem 3. *For any graph G without isolated vertices, there exists a super graph H such that $C(H) = G$ and $C(E_4(H)) = E_4(G)$.*

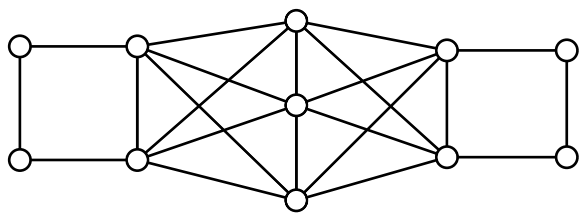
Proof. Consider $G \vee 2K_2$. Let the K_2's be $a - a'$ and $b - b'$. Attach $a'' - a'''$ to $a - a'$ such that a is adjacent to a''' and a' is adjacent to a''. Similarly attach $b'' - b'''$ to $b - b'$ such that b is adjacent to b''' and b' is adjacent to b''. The graph so obtained is H.

Claim 1. $C(H) = G$.
We prove that among the vertices in H, those vertices which are in G also have minimum eccentricity.
$e(u) = 2$, if $u \in V(G)$.
$= 3$, if $u \in \{a, a', b, b'\}$.
$= 4$, if $u \in \{a'', a''', b'', b'''\}$.
Hence Claim 1 is proved. □

Let $u_1, u_2, \ldots, u_m$ be the vertices in G. Consider $E_4(H))$. Let x be any vertex in $E_4(H)$.

Claim 2. $C(E_4(H)) = E_4(G)$.
$e(x) = 2$, if $x \in \{u_i u_j / u_i$ is adjacent to u_j in $G, i, j = 1, 2, \ldots, m, i \neq j$.
$= 3$, if $x \in \{aa', bb', au_i, a'u_i, bu_i, b'u_i\}$, $i = 1, 2, \ldots, m$.
$= 4$, if $x \in \{a'a'', aa''', b'b'', bb''', a''a''', b''b'''\}$. □

Illustration: Let $G = P_3$. Then H:



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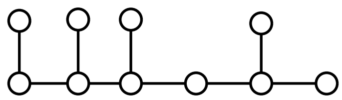
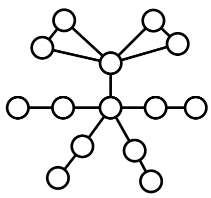
3. A BOUND ON THE DOMINATION NUMBER OF $E_4(G)$

Theorem 4. *For a connected graph G, $\gamma(G) \leq 2\gamma(E_4(G))$. Given any two integers a and b such that $a \leq 2b$, there exists a graph G such that $\gamma(G) = a$ and $\gamma(E_4(G)) = b$.*

Proof. Let $\gamma(E_4(G)) = b$ and let $\{e_1 = v_1v_1', e_2 = v_2v_2', \ldots, e_b = v_bv_b'\}$ dominate $E_4(G)$. Consider $S = \{v_1, v_1', v_2, v_2', \ldots, v_b, v_b'\}$. Then $S \subseteq V(G)$. Let w be any vertex in $V(G)$. Since G is a connected graph, w must be the end vertex of an edge $w - w'$. But the vertex ww' in $E_4(G)$ is dominated and hence is adjacent to at least one of the b vertices. Let e_i be adjacent to ww' in $E_4(G)$. Then in G, either e_i is incident with $w - w'$ or e_i and $w - w'$ are the opposite edges of some C_4. In both the cases, w is dominated by v_i or v_i'. Thus S is a dominating set of G and hence $\gamma(G) \leq 2\gamma(E_4(G))$.

Construction

		Construction	Illustration
Case 1	$b \leq a \leq 2b$	Consider $P_{2b} = \{v_1, v_2, \ldots, v_{2b}\}$. Attach a pendant vertex to each of v_{2i-1}, $i = 1, 2, \ldots, b$. Then to each of the v_{2i}'s, $i = 1, 2, \ldots, a - b$, attach a pendant vertex.	$a = 4$; $b = 3$ 
Case 2	$a < b$	Consider $K_{1,a}$. Replace a pendant vertex of $K_{1,a}$ by $K_1 \vee (b - a + 1)K_2$. To all the other pendant vertices of $K_{1,a}$, attach a pendant vertex.	$a = 5$; $b = 6$ 

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4. Some Theorems on Graph Classes

Theorem 5 [9]**.** *For a connected graph G, $E_4(G)$ is complete if and only if G is a complete multipartite graph.*

Theorem 6. *Let G be a connected graph such that $E_4(G)$ is a threshold graph. Then $\gamma(G) \leq 2$.*

Proof. We know that $E_4(G)$ is a threshold graph if and only if $dilw(E_4(G)) = 1$. Also $dilw(E_4(G)) \geq \gamma(E_4(G))$. Then the theorem follows from Theorem 4. ∎

The graph obtained from K_4 by attaching two pendant vertices to the same vertex of K_4 is denoted by H.

Theorem 7. *If G is a threshold graph then $E_4(G)$ is a threshold graph if and only if G is $\{moth, H\}$-free.*

Proof. Let G be a threshold graph. If G contains a moth graph or H as an induced sub graph, then $E_4(G)$ contains a $2K_2$ and hence it cannot be threshold.

Conversely, suppose that G is a {moth, H}-free threshold graph. Since G is threshold, $dilw(G) = 1$ and hence $\gamma(G) = 1$. So G must have a universal vertex u.

If at most two vertices in $N(u)$ are of degree greater than one, then $E_4(G)$ cannot contain an induced $2K_2$, C_4 or P_4.

Now let k, $k \geq 3$ vertices in $N(u)$ are of degree greater than one.

Claim: There exist three vertices u_1, u_2, u_3 such that the vertex u_2 is adjacent to u_1 and u_3.

If $k = 3$, this claim holds true. If $k > 3$, let u_1, u_2, u_3 and u_4 be four vertices of degree greater than one in $N(u)$ such that u_1 is adjacent to u_2 and u_3 is adjacent to u_4. Since G is threshold, it can not contain an induced $2K_2$ and hence u_3 or u_4 must be adjacent to u_1 or u_2. Let u_3 be adjacent to u_1. Then u_2, u_1, u_3, u_4 forms an induced P_4 which is not possible since G is threshold. In this case, if u_4 is adjacent to u_2, then G contains an induced C_4 which is again not possible. Hence the claim.

Further if u_1 and u_3 are adjacent, the vertex u can have at most one more neighbour since G is H-free. In this case also $E_4(G)$ is threshold since it is $\{2K_2, C_4, P_4\}$-free. On the other hand if u_1 and u_3 are not adjacent, then since G is moth-free, the vertex u can have at most one more neighbour. In this case also $E_4(G)$ is threshold. ■

Remark. Let G be a connected graph such that $E_4(G)$ is a cograph. Then $\gamma(G) \leq 4$, which follows from Theorem 4 and the fact that the domination number of cographs is at most two.

Theorem 8. *Let G be a connected graph. Then*

1. $E_4(G)$ *is a weakly geodetic graph if and only if* G *is* $\{$*paw,* 4-*pan*$\}$-*free.*
2. $E_4(G)$ *is a geodetic graph if and only if* G *is* $\{C_{2n} : n > 2\} \cup \{4$-*pan*$\} \cup \{2n-1 : n > 1\}$-*free.*
3. $E_4(G)$ *is a block graph if and only if* G *is* $\{$*paw,* 4-*pan*$\} \cup \{C_n : n \geq 5\}$-*free.*

Proof. 1. If G contains a paw in which $C_3 = (u_1, u_2, u_3)$ and a is a pendant vertex attached to u_1, then in $E_4(G)$, $d(au_1, u_2u_3) = 2$, but they have two common neighbours u_1u_2 and u_1u_3. Similarly if G contains a 4-pan in which $C_4 = (u_1, u_2, u_3, u_4)$ and a is a pendant vertex attached to u_1, then in $E_4(G)$, $d(au_1, u_3u_4) = 2$, but they have two neighbours u_1u_2 and u_1u_4.

Conversely, suppose that G is a {paw, 4-pan}-free graph. If G is an acyclic graph, there exists a unique shortest path joining any two vertices in $E_4(G)$. Thus $E_4(G)$ is weakly geodetic.

Next suppose that G contains cycles.

If $g(G) = 3$ then G contains a $C_3 = (u_1, u_2, u_3)$.

Claim. G is a cograph.

Suppose that G contains an induced $P_4 = (v_1, v_2, v_3, v_4)$. Let $u_1 \neq v_1$. Consider a shortest path $(u_1, a_1, a_2, \ldots, a_k, v_1)$ joining u_1 and v_1. Since G is paw free a_1 must be adjacent to at least one more u_i, $i = 2, 3$. Proceeding like this, v_1 and then v_2 must be adjacent to at least two u_i*'s*. This implies that v_1 and v_2 must have a common neighbour among the u_is. Let it be u_1. Then (v_1, u_1, v_2) form a C_3. Since G is paw-free, v_3 must be adjacent to at least one of v_1 and u_1. But, since (v_1, v_2, v_3, v_4) is an induced P_4, v_3 must be adjacent to u_1. Then (v_1, u_1, v_3) will form a C_3 in G. Again since G is paw-free, v_4 must be adjacent to u_1. Now, consider (v_1, u_1, v_2) with the edge $u_1 - v_4$. Since G is paw-free, v_4 must be adjacent to v_1 or v_2, which is a contradiction.

But a paw-free cograph is a complete multipartite graph and hence by Theorem 5, $E_4(G)$ is complete and thus weakly geodetic.

If $g(G) = 4$, then G contains a $C_4 = (u_1, u_2, u_3, u_4)$. If $G = C_4$, then $E_4(G) = K_4$. If there exists a vertex v_1 in G which is adjacent to u_1, v_1 must be adjacent to u_3 also since G is 4-pan-free. Similarly if there exists a vertex v_2 which is adjacent to u_2, v_2 must be adjacent to u_4. If there exists a vertex v_1' which is adjacent to v_1, it must be adjacent to both u_2 and u_4. Hence G is a complete bipartite graph. Since $g(G) = 4$, G is paw-free. Again by Theorem 5, $E_4(G)$ is complete, and hence G is a weakly geodetic graph.

Finally, Let $g(G) = k$, $k > 4$. Let $(u_1, u_2, u_3, \ldots, u_k)$ be a C_k in G. Then $E_4(G)$ also contains a C_k. This C_k is not a part of any clique in $E_4(G)$ and hence $b(E_4(G)) \leq k$. Since G does not contain any C_4, two vertices in $E_4(G)$ are adjacent if and only if the corresponding edges in G are adjacent. Thus $E_4(G)$ cannot contain a b-cycle of length less than k and so $b(E_4(G)) = k$ where $k > 4$. We know that a graph G is weakly geodetic if and only if $b(G) \geq 5$. Thus $E_4(G)$ is a weakly geodetic graph.

2. Let $E_4(G)$ be a geodetic graph. If G contains a 4-pan, there exists more than one shortest path joining two vertices in $E_4(G)$ as proved earlier. If G contains a $C_{2n} = (u_1, u_2, \ldots, u_{2n})$, then u_1u_2 and $u_{n+1}u_{n+2}$ in $E_4(G)$ are connected by more than one shortest path and hence $E_4(G)$ is not geodetic. If G contains a $(2n-1)$-pan in which $C_{2n-1} = (u_1, u_2, \ldots, u_{2n-1})$ and a is a pendant vertex attached to u_1, then au_1 and u_nu_{n+1} in $E_4(G)$ are connected by more than one shortest path and hence $E_4(G)$ is not geodetic.

Conversely, assume that G is $\{$4-pan, C_{2n}, $(2n-1)$-pan$\}$-free. If G is an acyclic graph there exists a unique shortest path joining any two vertices in $E_4(G)$ and hence is geodetic. So consider the graphs G containing cycles.

Let $g(G) = 3$. Since G is paw-free, $E_4(G)$ is complete and hence is geodetic. If $g(G) = 4$, $E_4(G)$ is complete since G is 4-pan-free and thus geodetic. If $g(G) = 2n-1, n > 2$, then G contains a $C_{2n-1} = (u_1, u_2, \ldots, u_{2n-1})$. If $G = C_{2n-1}$, then $E_4(G) = C_{2n-1}$ and hence geodetic. If a is a vertex attached to u_1, since G is $(2n-1)$-pan-free, a must be adjacent to at least one more u_i. But this is impossible since $g(G) = 2n-1$. Since G is C_{2n}-free, $g(G) \neq 2n$, $n > 2$. Hence in all the cases, it follows that $E_4(G)$ is geodetic.

3. Let $E_4(G)$ be a block graph. If G contains a paw in which $C_3 = u_1, u_2, u_3$ and a is the pendant vertex adjacent to u_1, then $E_4(G)$ contains a $C_4 = (au_1, u_1u_2, u_2u_3, u_3u_1)$ which is not a part of any clique. Thus $b(E_4(G)) \leq 4$. Similarly if G contains a 4-pan, in which $C_4 =$

$(u_1, u_2, u_3, u_4\}$ and a is a pendant vertex adjacent to u_1, then $E_4(G)$ contains a $C_4 = (au_1, u_1u_2, u_3u_4, u_4u_1)$ which is not a part of any clique and hence $b(E_4(G)) \leq 4$. If G contains a C_n, $n > 4$, then $E_4(G)$ also contains a C_n, $n > 4$. This C_n forms a b-cycle and hence $b(E_4(G)) \leq n$ and hence $E_4(G)$ is not a block graph.

Conversely, suppose that G is $\{$paw, 4-pan$\} \cup \{C_n : n > 4\}$-free. If G is an acyclic graph, then $E_4(G)$ cannot contain a b-cycle and hence is a block graph. Now, consider the graphs G containing cycles. Since G is $\{C_n : n \geq 5\}$-free, $g(G) = 3$ or 4. But since G is $\{$paw, 4-pan$\}$-free, $E_4(G)$ is complete as proved earlier and thus is a block graph. ■

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