

DECOMPOSITIONS OF MULTIGRAPHS INTO PARTS WITH THE SAME SIZE

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Abstract

Given a family \mathcal{F} of multigraphs without isolated vertices, a multigraph M is called \mathcal{F} -decomposable if M is an edge disjoint union of multigraphs each of which is isomorphic to a member of \mathcal{F} . We present necessary and sufficient conditions for existence of such decompositions if \mathcal{F} consists of all multigraphs of size q except for one. Namely, for a multigraph H of size q we find each multigraph M of size kq , such that every partition of the edge set of M into parts of cardinality q contains a part which induces a submultigraph of M isomorphic to H .

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1. INTRODUCTION

We consider finite undirected multigraphs without loops and isolated vertices. Given a family \mathcal{F} of multigraphs, an \mathcal{F} -decomposition of a multigraph M is a collection of submultigraphs which partition the edge set $E(M)$ of M and are all isomorphic to members of \mathcal{F} . If such a decomposition exists, M is called \mathcal{F} -decomposable; and also H -decomposable if H is the only member of \mathcal{F} .

If M is a multigraph, then $V(M)$ and $E(M)$ stand for the vertex set and edge set of M , respectively. Cardinalities of those sets, denoted by $v(M)$ and $e(M)$, are called the *order* and *size* of M , respectively. For $E \subset E(M)$,

$M[E]$ denotes the submultigraph of M induced by E . The number of edges incident to a vertex x in M , denoted by $\text{val}_M(x)$, is called the *valency* of x , whilst the number of neighbours of x in M , denoted by $\deg_M(x)$, is called the *degree* of x . As usual $\Delta(M)$ stands for the maximum valency among vertices of M . For any two vertices x, y of M , let $p_M(x, y)$ denote the number of edges joining x and y . We call $p_M(x, y)$ the *multiplicity* of an edge xy in M . $\Pi(M)$ stands for the maximum multiplicity among edges of M . Edges joining the same vertices are called *parallel edges* (if they are distinct).

The multipath of length k with edge multiplicities m_1, \dots, m_k is denoted by $P(m_1, \dots, m_k)$. Note that $P(1) = P_2 = K_2$ is the simple path on two vertices and $P(1, 1) = P_3$ is the simple path on three vertices. The multistar of order $k+1$ with edge multiplicities m_1, \dots, m_k is denoted by $S(m_1, \dots, m_k)$. Note that $S(1, 1, 1) = K_{1,3}$ is a simple 3-star. The multicycle of length k with edge multiplicities m_1, \dots, m_k is denoted by $C(m_1, \dots, m_k)$. Note that $C(1, 1, 1) = C_3$ is a 3-cycle. The multiplicity 0 is allowed in these cases. Note that $S(1, 0, 1)$ is a path. The union of two disjoint multigraphs M and H is denoted by $M \dot{\cup} H$ and the union of k disjoint copies of a multigraph H is denoted by kH . For a multigraph H , denote by H^{+e} the set of all multigraphs which we obtain from H by adding an edge. Note that $K_2^{+e} = \{C_2, P_3, 2K_2\}$.

Given a multigraph M , let $G(M)$ denote a graph which we obtain from M by removal of all edges of the maximal family of pairwise edge-disjoint copies of C_2 , i.e., $V(G(M)) := V(M)$ and $E(G(M)) := \{xy : p_M(x, y) \equiv 1 \pmod{2}\}$.

In [1] there are provided necessary and sufficient conditions for a multigraph M to be $\{H_1, H_2\}$ -decomposable, where H_1, H_2 are any two multigraphs out of C_2, P_3 and $2K_2$. One of the results follows.

Theorem 1 ([1]). *A multigraph M is $\{C_2, 2K_2\}$ -decomposable if and only if each of the following five conditions holds:*

- (1) $e(M) \equiv 0 \pmod{2}$;
- (2) $\text{val}_M(x) + \deg_{G(M)}(x) \leq e(M)$ for every $x \in V(M)$;
- (3) if $xy \in E(G(M))$ then $\text{val}_M(x) + \text{val}_M(y) - p_M(x, y) < e(M)$;
- (4) if $yx, xz \in E(G(M))$, then $1 + \text{val}_M(x) + p_M(y, z) < e(M)$;
- (5) M is different from each of the (forbidden) multigraphs shown in Figure 1.

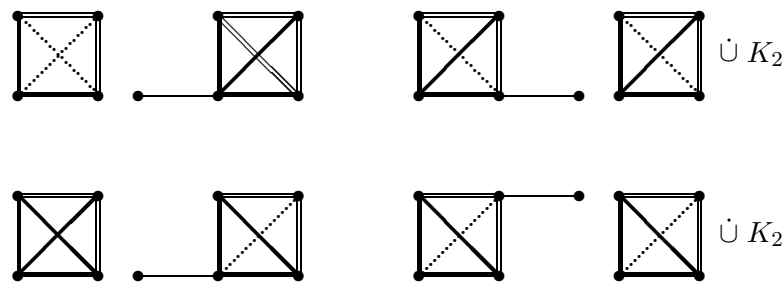


Figure 1. Forbidden multigraphs

Table 1. Codes in Figure 1.

edge :	bold	thin	doubled	dotted
multiplicity :	odd	1	even ≥ 2	even ≥ 0

As it can be seen the assertion is not trivial. So, the aim of this paper is to extend the previous result. For a positive integer k , let $\mathcal{M}(k)$ be the set of all mutually non-isomorphic multigraphs of size k . For a multigraph $H \in \mathcal{M}(k)$, denote by \tilde{H} the set $\mathcal{M}(k) - \{H\}$. As $\mathcal{M}(2) = \{C_2, P_3, 2K_2\}$, Theorem 1 characterizes \tilde{P}_3 -decomposable multigraphs. In the paper we characterize \tilde{H} -decomposable multigraphs where H is an arbitrary multigraph.

2. \tilde{H} -DECOMPOSABLE MULTIGRAPHS

For a given multigraph H we define the family $\mathcal{F}(H)$ as follows. A multigraph M belongs to $\mathcal{F}(H)$ iff $e(M) \equiv 0 \pmod{e(H)}$ and every partition of $E(M)$ into parts of cardinality $e(H)$ contains a part which induces a submultigraph of M isomorphic to H .

According to the definition of $\mathcal{F}(H)$ we have immediately

Proposition 1. *A multigraph M is \tilde{H} -decomposable if and only if*

$$e(M) \equiv 0 \pmod{e(H)} \quad \text{and} \quad M \notin \mathcal{F}(H).$$

The previous characterization of \tilde{H} -decomposable multigraphs may be useful only for a multigraph H whose forbidden set $\mathcal{F}(H)$ is described. Evidently,

$\mathcal{F}(K_2)$ includes all multigraphs. However, the forbidden sets of the others multigraphs are not so large, but a multigraph H is usually (except for some special cases) the only member of $\mathcal{F}(H)$. Next we describe the forbidden sets of exceptional multigraphs.

Theorem 2. *A multigraph M belongs to $\mathcal{F}(P(k))$ if and only if $e(M) \equiv 0 \pmod{k}$ and $\Pi(M) > \frac{k-1}{k}e(M)$.*

Proof. The condition $e(M) \equiv 0 \pmod{k}$ is obvious. If there is an edge xy of a multigraph M with multiplicity $p_M(x, y) > \frac{k-1}{k}e(M)$ then any partition of $E(M)$ into parts of cardinality k contains a part consisting of k parallel edges joining x and y . It induces a submultigraph of M isomorphic to $P(k)$. On the other hand, if $\Pi(M) \leq \frac{k-1}{k}e(M)$ then there is a partition of $E(M)$ into parts of cardinality k such that any part contains at most $k-1$ parallel edges. So, $M \notin \mathcal{F}(P(k))$. ■

Theorem 3. *A multigraph M belongs to $\mathcal{F}(kK_2)$ if and only if $e(M) \equiv 0 \pmod{k}$ and the number of odd size components of M is greater than $\frac{k-2}{k}e(M)$.*

Proof. The condition $e(M) \equiv 0 \pmod{k}$ is obvious. It is proved in [1] that a multigraph is $\{C_2, P_3\}$ -decomposable if and only if each its component has an even number of edges. Thus, if we remove an appropriate edge from each odd size component of a multigraph M , we get a $\{C_2, P_3\}$ -decomposable multigraph. Therefore, the maximum number of mutually edge-disjoint pairs of adjacent edges in M is $\frac{1}{2}(e(M) - c)$, where c denotes the number of odd size components. If $c > \frac{k-2}{k}e(M)$, then M contains less than $e(M)/k$ edge-disjoint pairs of adjacent edges. So, any partition of $E(M)$ into parts of cardinality k contains a part consisting of k edges lying in distinct components. It induces a submultigraph of M isomorphic to kK_2 . On the other hand, if $c \leq \frac{k-2}{k}e(M)$ then there is a partition of $E(M)$ into parts of cardinality k such that any part contains two adjacent edges. Thus, $M \notin \mathcal{F}(kK_2)$. ■

Theorem 4. *A multigraph M belongs to $\mathcal{F}(K_{1,2})$ if and only if $e(M)$ is even and at least one of the following four conditions holds:*

- (1) *there is $x \in V(M)$ such that $\text{val}_M(x) + \deg_{G(M)}(x) > e(M)$;*
- (2) *there is $xy \in E(G(M))$ such that $\text{val}_M(x) + \text{val}_M(y) - p_M(x, y) = e(M)$;*

- (3) there are $yx, xz \in E(G(M))$ such that $1 + \text{val}_M(x) + p_M(y, z) \geq e(M)$;
- (4) M is one of the multigraphs shown in Figure 1.

For $k \geq 3$, $M \in \mathcal{F}(K_{1,k})$ if and only if $e(M) \equiv 0 \pmod{k}$ and there is a vertex $x \in V(M)$ such that $\text{val}_M(x) + \deg_{G(M)}(x) > 2\frac{k-1}{k}e(M)$.

Proof. The first equivalence follows immediately from Theorem 1.

The condition $e(M) \equiv 0 \pmod{k}$ is obvious. If $M \notin \mathcal{F}(K_{1,k})$, then there is a partition of $E(M)$ into $e(M)/k$ parts of cardinality k such that no part induces a submultigraph of M isomorphic to $K_{1,k}$. Hence, for any vertex x , every part contains either an edge not incident to x or two parallel edges incident to x . Therefore, the sum of the number of edges not incident to x and the number of edge-disjoint pairs of parallel edges incident to this vertex is at least $e(M)/k$, i.e., $(e(M) - \text{val}_M(x)) + (\text{val}_M(x) - \deg_{G(M)}(x))/2 \geq e(M)/k$. This implies the inequality $\text{val}_M(x) + \deg_{G(M)}(x) \leq 2\frac{k-1}{k}e(M)$.

On the other hand, assume that M is a multigraph of size kt such that $\text{val}_M(x) + \deg_{G(M)}(x) \leq 2\frac{k-1}{k}e(M)$ for every $x \in V(M)$. Evidently, M is not isomorphic to $K_{1,k}$ if $t = 1$. For $t \geq 2$, consider a multigraph $H := M \dot{\cup} mK_2$, where $m = (k-2)t$. Clearly, $e(H) = e(M) + m = 2(k-1)t$. By Theorem 1, one can easily check that H is a $\{C_2, 2K_2\}$ -decomposable multigraph. Therefore, there exist $t = (k-1)t - m$ edge-disjoint pairs of edges $e_i^1, e_i^2 \in E(M)$ such that $M[\{e_i^1, e_i^2\}]$ is isomorphic to either C_2 , or $2K_2$, for every $i \in \{1, \dots, t\}$. Thus, there is a partition of $E(M)$ into parts of cardinality k such that the i -th part contains edges e_i^1 and e_i^2 . Clearly, none of these parts induces a submultigraph isomorphic to $K_{1,k}$ and so $M \notin \mathcal{F}(K_{1,k})$. ■

In the next proofs we will use an induction, so the following description of forbidden sets will be very useful.

For a given multigraph H and each positive integer n we define the family $\mathcal{F}_n(H)$ recursively as follows. H is the only member of $\mathcal{F}_1(H)$. For $n > 1$ a multigraph M belongs to $\mathcal{F}_n(H)$ iff $e(M) = ne(H)$ and for every subset $E \subset E(M)$ of cardinality $|E| = e(H)$ it holds either $M[E]$ is isomorphic to H or $M[E(M) - E]$ is isomorphic to a member of $\mathcal{F}_{n-1}(H)$. According to the definition of $\mathcal{F}(H)$ we have immediately

Lemma 1. Let H and M be multigraphs such that $e(M) = ne(H)$. The multigraph M belongs to $\mathcal{F}(H)$ if and only if it belongs to $\mathcal{F}_n(H)$.

Put $\mathcal{F}^*(H) := \cup_{i \geq 2} \mathcal{F}_i(H)$. Then $\mathcal{F}(H) = \cup_{i \geq 1} \mathcal{F}_i(H) = \mathcal{F}_1(H) \cup \mathcal{F}^*(H)$. Note that the conditions $\mathcal{F}_2(H) \neq \emptyset$, $\mathcal{F}^*(H) \neq \emptyset$, $|\mathcal{F}(H)| > 1$ are mutually equivalent.

We will often use the following auxiliary assertion.

Lemma 2. *Let H and M be multigraphs such that $M \in \mathcal{F}_2(H)$. Let E be a subset of $E(M)$ such that $M[E]$ is isomorphic to no submultigraph of H .*

If $|E| = e(H) - 1$, then there is a multigraph $H^ = M[E(M) - E] \in H^{+e}$ such that all edges of H^* have the same multiplicities and pairs of degrees of their end vertices.*

If $|E| < e(H) - 1$, then $H \in \{kK_2, K_{1,k}, P(k)\}$.

Proof. Let H^* be a submultigraph of M induced by $E(M) - E$. If we remove any edge from H^* (in the case $|E| = e(H) - 1$), we get a multigraph isomorphic to H . So, all edges of H^* are equivalent and H^* has the required properties. Similarly, for $|E| < e(H) - 1$, if we remove any $e(H) - |E| \geq 2$ edges from H^* , we get a multigraph isomorphic to H . Thus, all pairs of edges of H^* are equivalent in this case. Evidently, $H^* \in \{P(t), K_{1,t}, tK_2\}$, where $t = e(H^*)$. As H is a submultigraph of H^* , we get the assertion. ■

Theorem 5. $\mathcal{F}^*(P(3, 2)) = \{P(7, 3)\}$ and $\mathcal{F}^*(P(3) \dot{\cup} P(2)) = \{P(7) \dot{\cup} P(3)\}$.

Proof. Suppose that $M \in \mathcal{F}_2(P(3, 2))$. Then $e(M) = 10$ and $\Pi(M) \geq 5$ because otherwise there is a partition of $E(M)$ into parts E_1, E_2 of cardinality five such that $\Pi(M[E_i]) \leq 2$, $i \in \{1, 2\}$, i.e., $M[E_i]$ is not isomorphic to $P(3, 2)$, a contradiction. According to Lemma 2, we get a multigraph H^* isomorphic to $P(3, 3)$, if we remove four parallel edges from M . It is easy to see that $M = P(7, 3)$, i.e., $\mathcal{F}_2(P(3, 2)) = \{P(7, 3)\}$.

Now suppose that $M \in \mathcal{F}_3(P(3, 2))$. Then $e(M) = 15$ and $\Pi(M) \geq 7$ because $P(7, 3)$ is a submultigraph of M . Let xy be an edge of M such that $p_M(xy) = \Pi(M)$. If we remove five parallel edges joining x and y from M , we get a multigraph isomorphic to $P(7, 3)$. Thus, $M \in \{P(12, 3), P(8, 7)\}$. However, it is easy to see that neither $P(12, 3)$ nor $P(8, 7)$ belongs to $\mathcal{F}_3(P(3, 2))$. Therefore, $\mathcal{F}_3(P(3, 2)) = \emptyset$ and consequently $\mathcal{F}_i(P(3, 2)) = \emptyset$ for every $i \geq 3$.

The second equality can be proved in the same manner, details are left to the reader. ■

Theorem 6. $\mathcal{F}^*(P_4) = \{C(3, 1, 1, 1)\}$.

Proof. Assume that $M \in \mathcal{F}_2(P_4)$. As M contains just six edges it is not difficult to check that M is not a simple graph (i.e., $\Pi(M) > 1$). According to Lemma 2, a multigraph which is obtained from M by deleting two parallel edges is a 4-cycle. Now it is easy to see that $C(3, 1, 1, 1)$ is the only member of $\mathcal{F}_2(P_4)$.

Suppose that $M \in \mathcal{F}_3(P_4)$. As $C(3, 1, 1, 1)$ is a submultigraph of M , $\Pi(M) \geq 3$. If we remove any triple of parallel edges from M , we must obtain a multigraph isomorphic to $C(3, 1, 1, 1)$. So, there are two edge-disjoint triples of parallel edges in M . The only multigraph satisfying the previous two conditions is $C(6, 1, 1, 1)$, but it does not belong to $\mathcal{F}_3(P_4)$. Therefore, $\mathcal{F}_3(P_4) = \emptyset$ and consequently $\mathcal{F}_i(P_4) = \emptyset$ for every $i \geq 3$. ■

Theorem 7. *For the multigraph $P(k) \dot{\cup} K_2$ it holds:*

$$\begin{aligned} \mathcal{F}(P(2) \dot{\cup} K_2) &= \{P(r) \dot{\cup} P(s) \dot{\cup} P(t) : 0 \leq r \equiv 0, s \equiv 1, t \equiv 2 \pmod{3}\} \cup \\ &\quad \{P(r, 1) \dot{\cup} P(t) : 0 \leq r \equiv 0, t \equiv 2 \pmod{3}\}, \\ \mathcal{F}(P(3) \dot{\cup} K_2) &= \{P(r) \dot{\cup} P(s) : r \equiv 1, s \equiv 3 \pmod{4}\} \text{ and} \\ \mathcal{F}(P(k) \dot{\cup} K_2) &= \{P(r) \dot{\cup} K_2 : 3 \leq r \equiv -1 \pmod{k+1}\}, \text{ if } k \geq 4. \end{aligned}$$

Proof. Suppose that $M \in \mathcal{F}_n(P(2) \dot{\cup} K_2)$ for $n \geq 2$. Then $e(M) = 3n$ and $\Pi(M) \geq 1 + n$ because otherwise there is a partition of $E(M)$ into disjoint parts E_1, \dots, E_n of cardinality three such that $\Pi(M[E_i]) \leq 1$, $i \in \{1, \dots, n\}$, i.e., $M[E_i]$ is not isomorphic to $P(2) \dot{\cup} K_2$, a contradiction. If we remove any triple of parallel edges (of multiplicity $\Pi(M)$) from M , we must obtain a multigraph isomorphic to a member of $\mathcal{F}_{n-1}(P(2) \dot{\cup} K_2)$. Thus, for $n = 2$, $M \in \{P(1) \dot{\cup} P(5), P(4) \dot{\cup} P(2), P(1, 3, 2), P(3, 2) \dot{\cup} P(1), P(3, 1) \dot{\cup} P(2), P(3) \dot{\cup} P(1) \dot{\cup} P(2)\}$. Now, it is not difficult to check that $\mathcal{F}_2(P(2) \dot{\cup} K_2) = \{P(3) \dot{\cup} P(1) \dot{\cup} P(2), P(1) \dot{\cup} P(5), P(4) \dot{\cup} P(2), P(3, 1) \dot{\cup} P(2)\}$. Similarly, using induction for $n \geq 3$, we get the assertion.

The other equalities can be proved in the same manner, details are left to the reader. ■

Theorem 8. *For the multigraph $P(k, 1)$ it holds:*

$$\begin{aligned} \mathcal{F}(P(2, 1)) &= \{C(r, s, t) : r \equiv 1, s \equiv 2, 0 \leq t \equiv 0 \pmod{3}\} \cup \\ &\quad \{S(r, s, t) : r \equiv 1, s \equiv 2, 0 \leq t \equiv 0 \pmod{3}\} \cup \\ &\quad \{P(1, s, t) : s \equiv 2, 0 \leq t \equiv 0 \pmod{3}\}, \\ \mathcal{F}(P(3, 1)) &= \{P(r, s) : r \equiv 1, s \equiv 3 \pmod{4}\} \text{ and} \\ \mathcal{F}(P(k, 1)) &= \{P(r, 1) : 4 \leq r \equiv -1 \pmod{k+1}\}, \text{ if } k \geq 4. \end{aligned}$$

Proof. Suppose that $M \in \mathcal{F}_n(P(2,1))$ for $n \geq 2$. Then $e(M) = 3n$ and $\Pi(M) \geq 1 + n$ because otherwise there is a partition of $E(M)$ into disjoint parts E_1, \dots, E_n of cardinality three such that $\Pi(M[E_i]) \leq 1$, $i \in \{1, \dots, n\}$, i.e., $M[E_i]$ is not isomorphic to $P(2,1)$, a contradiction. If we remove any triple of parallel edges (of multiplicity $\Pi(M)$) from M , we must obtain a multigraph isomorphic to a member of $\mathcal{F}_{n-1}(P(2,1))$. Thus, for $n = 2$, $M \in \{P(5,1), P(4,2), P(3) \dot{\cup} P(2,1), P(3,2,1), P(2,1,3), S(3,2,1), C(1,2,3)\}$. Now, it is not difficult to check that $\mathcal{F}_2(P(2,1)) = \{P(5,1), P(4,2), P(3,2,1), S(3,2,1), C(1,2,3)\}$. Similarly, using induction for $n \geq 3$, we get the assertion.

The other equalities can be proved in the same manner, details are left to the reader. ■

Theorem 9. $\mathcal{F}^*(K_{1,2} \dot{\cup} K_{1,3}) = \{K_{1,3} \dot{\cup} K_{1,7}\}$.

Proof. Suppose that $M \in \mathcal{F}_2(K_{1,2} \dot{\cup} K_{1,3})$. Then $e(M) = 10$ and by Lemma 2 M contains no parallel edges, no triangle and no 3-matching. Assume that $\Delta(M) \leq 3$. Then there is an equitable 4-edge-coloring of M (see [2]) and so there is a partition of $E(M)$ into parts E_1, E_2 of cardinality five (E_i consists of edges having two distinct colors) such that $\Delta(M[E_i]) \leq 2$, $i \in \{1, 2\}$, i.e., $M[E_i]$ is not isomorphic to $K_{1,2} \dot{\cup} K_{1,3}$, a contradiction. Therefore, $\Delta(M) \geq 4$. According to Lemma 2, a multigraph which is obtained from M by deleting four edges incident to a maximum degree vertex is isomorphic to $2K_{1,3}$. It is easy to see that $M = K_{1,3} \dot{\cup} K_{1,7}$, i.e., $\mathcal{F}_2(K_{1,2} \dot{\cup} K_{1,3}) = \{K_{1,3} \dot{\cup} K_{1,7}\}$.

Now suppose that $M \in \mathcal{F}_3(K_{1,2} \dot{\cup} K_{1,3})$. Then $e(M) = 15$ and $\Delta(M) \geq 7$ because $K_{1,3} \dot{\cup} K_{1,7}$ is a submultigraph of M . Let x be a vertex of M such that $\deg_M(x) = \Delta(M)$. If we remove five edges incident to x from M , we get a multigraph isomorphic to $K_{1,3} \dot{\cup} K_{1,7}$. Thus, M is a disjoint union of two multistars. However, it is easy to see that none of such multigraphs belongs to $\mathcal{F}_3(K_{1,2} \dot{\cup} K_{1,3})$. Therefore, $\mathcal{F}_3(K_{1,2} \dot{\cup} K_{1,3}) = \emptyset$ and consequently, for every $i \geq 3$, $\mathcal{F}_i(K_{1,2} \dot{\cup} K_{1,3}) = \emptyset$. ■

Lemma 3. Let M be a connected multigraph of size at least 4. Then there are edges $e_1, e_2, e_3 \in E(M)$ satisfying

- (1) $M[\{e_1, e_2, e_3\}]$ is either a matching or a connected submultigraph of M ;
- (2) $M[E(M) - \{e_1, e_2, e_3\}]$ is a connected multigraph.

Proof. Suppose that M is a counterexample. Let T be a spanning tree of M . Evidently, $M[E(M) - (E \cup \{e\})]$ is a connected multigraph for any pendant edge e of T and any set $E \subseteq E(M) - E(T)$. The multigraph M satisfies the following conditions.

A. *There is no pendant edge of T adjacent to two distinct edges of $E(M) - E(T)$.*

Suppose to the contrary that $e_1, e_2 \in E(M) - E(T)$ are two distinct edges adjacent to a pendant edge e_3 of T . Clearly, the multigraphs $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected, a contradiction.

B. *There is no edge of $E(M) - E(T)$ adjacent to two pendant edges of T .* Assume that $e_1 \in E(M) - E(T)$ is an edge adjacent to two pendant edges e_2, e_3 of T . Moreover, assume that e_1 is incident to a pendant vertex of T (if there exists such edge). If $M[E(M) - \{e_1, e_2, e_3\}]$ is a disconnected multigraph then there is an edge e_4 whose end vertices are pendant vertices of T (end vertices of e_2 and e_3). Thus, edges e_1 and e_4 are adjacent to a pendant edge of T , contrary to **A**. Therefore, $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected multigraphs, a contradiction.

C. *There is no 3-matching consisting of pendant edges of T .*

Suppose that e_1, e_2, e_3 are three independent pendant edges of T . By **B**, there is no edge of $E(M) - E(T)$ adjacent to any two pendant edges of $\{e_1, e_2, e_3\}$. Thus, $M[E(M) - \{e_1, e_2, e_3\}]$ is a connected multigraph and $M[\{e_1, e_2, e_3\}]$ is a matching, a contradiction.

D. *There are not three mutually adjacent pendant edges of T .*

Assume to the contrary that e_1, e_2, e_3 are three adjacent pendant edges of T . By **B**, there is no edge of $E(M) - E(T)$ adjacent to any two pendant edges of $\{e_1, e_2, e_3\}$. Therefore, $M[E(M) - \{e_1, e_2, e_3\}]$ is a connected multigraph. The multigraph $M[\{e_1, e_2, e_3\}]$ is also connected in this case, a contradiction.

E. *There are not two adjacent pendant edges of T .*

Suppose that e_1, e_2 are two adjacent pendant edges of T . By **B**, there is no edge of $E(M) - E(T)$ adjacent to e_1 and e_2 . If e_3 is another edge of T adjacent to both of e_1, e_2 , then $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected multigraphs, a contradiction.

F. *There is no edge of $E(M) - E(T)$ adjacent to a pendant edge of T .*

According to **C** and **E**, the tree T is a path. Assume that $e_1 \in E(M) - E(T)$ is an edge adjacent to a pendant edge e_2 of T . By **A**, there is no other edge of $E(M) - E(T)$ adjacent to e_2 . If e_3 is the edge of T adjacent to e_2 , then $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected multigraphs, a contradiction.

By **A** – **F**, the tree T is a path and there is no edge of $E(M) - E(T)$ adjacent to a pendant edge of T . If e_1 is any pendant edge of T and edges e_1, e_2, e_3 induce a subpath of T , then $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected multigraphs, a contradiction. Therefore, there is no counterexample of the assertion. ■

Theorem 10. *For the multigraph $K_{1,k} \dot{\cup} K_2$ it holds:*

$$\begin{aligned} \mathcal{F}(K_{1,2} \dot{\cup} K_2) = & \{K_{1,r} \dot{\cup} K_{1,s} \dot{\cup} K_{1,t} : r \equiv 1, s \equiv 2, t \equiv 0 \pmod{3}\} \cup \\ & \{K_{1,r} \dot{\cup} K_{1,s} \dot{\cup} K_3 : r \equiv 1, s \equiv 2 \pmod{3}\} \cup \\ & \{K_{1,r} \dot{\cup} C_5 : r \equiv 1 \pmod{3}\} \cup \\ & \{H \dot{\cup} K_{1,s} : H \in K_{1,t}^{+e}, t \equiv 0, s \equiv 2 \pmod{3}\} \cup \\ & \{H \dot{\cup} K_{1,s} : H \in K_3^{+e}, s \equiv 2 \pmod{3}\} \cup \\ & \{S(3, 1, \dots, 1) \dot{\cup} K_{1,s} : e(S(3, 1, \dots, 1)) \equiv 1, s \equiv 2 \pmod{3}\} \cup \\ & \{P(2, 1, 1) \dot{\cup} K_{1,s} : s \equiv 2 \pmod{3}\}, \\ \mathcal{F}(K_{1,3} \dot{\cup} K_2) = & \{K_{1,r} \dot{\cup} K_{1,s} : r \equiv 1, s \equiv 3 \pmod{4}\} \text{ and} \\ \mathcal{F}(K_{1,k} \dot{\cup} K_2) = & \{K_{1,r} \dot{\cup} K_2 : 4 \leq r \equiv -1 \pmod{k+1}\}, \text{ if } k \geq 4. \end{aligned}$$

Proof. Suppose that $M \in \mathcal{F}_n(K_{1,2} \dot{\cup} K_2)$. If $n = 2$, then $e(M) = 6$ and by Lemma 3 M is disconnected. If $\Pi(M) > 1$, then according to Lemma 2, a multigraph which is obtained from M by deleting two parallel edges is isomorphic to $2K_{1,2}$. Now it is easy to check that $M \in \{S(3, 1) \dot{\cup} K_{1,2}, C(2, 1, 1) \dot{\cup} K_{1,2}, S(2, 1, 1) \dot{\cup} K_{1,2}, P(2, 1, 1) \dot{\cup} K_{1,2}\}$. If $\Pi(M) = 1$, then M has at most three components because otherwise there is a partition of $E(M)$ into parts E_1, E_2 of cardinality three such that $M[E_1]$ is a matching and $M[E_2]$ is either a matching or a component of M , a contradiction. If M has three components, then by Lemma 3 the size of each component is at most three. Now, it is not difficult to check that M is isomorphic to either $K_2 \dot{\cup} K_{1,2} \dot{\cup} K_{1,3}$ or $K_2 \dot{\cup} K_{1,2} \dot{\cup} K_3$ in this case. If M has two components, then there is a component of size at most two because otherwise the components decompose M into two connected submultigraphs of size three. If M has a component of size 1, then verifying twenty possible graphs it is not difficult to check that M is isomorphic to either $K_{1,5} \dot{\cup} K_2$ or $C_5 \dot{\cup} K_2$. Similarly, if M has a component of size two, then it is not difficult to check that M is a graph belonging to $\{H \dot{\cup} K_{1,2} : H \in K_{1,3}^{+e}\}$. Thus, the assertion holds in this case.

If $n \geq 3$, then it is not difficult to see (using induction and Lemma 3) that M includes a vertex of degree at least three. Assume that $x \in V(M)$ is a vertex of maximum degree. If we remove three (non-parallel) edges

incident to x from M , we get a multigraph belonging to $\mathcal{F}_{n-1}(K_{1,2} \dot{\cup} K_2)$. Using induction it is not difficult to check that the assertion holds.

Now suppose that $M \in \mathcal{F}_n(K_{1,3} \dot{\cup} K_2)$. If $n = 2$, then $e(M) = 8$. According to Lemma 2 M contains no parallel edges, no 3-matching, no triangle and no path of length three. Now, it is not difficult to check that M is isomorphic to either $K_2 \dot{\cup} K_{1,7}$ or $K_{1,3} \dot{\cup} K_{1,5}$. Evidently, $\Delta(M) > 4$ for $n \geq 3$. If we remove four edges incident to a maximum degree vertex from M , we get a multigraph belonging to $\mathcal{F}_{n-1}(K_{1,3} \dot{\cup} K_2)$. Using induction we get the assertion.

The last equality can be proved in the same manner, details are left to the reader. \blacksquare

We conclude this paper with the following result.

Theorem 11. *Let H be a multigraph. $|\mathcal{F}(H)| > 1$ if and only if H is one of the following multigraphs:*

- (1) $P(k)$, for every positive integer k ;
- (2) $P(k, 1)$, for every positive integer k ;
- (3) $P(k) \dot{\cup} K_2$, for every positive integer k ;
- (4) $P(3, 2)$;
- (5) $P(3) \dot{\cup} P(2)$;
- (6) kK_2 , for every positive integer k ;
- (7) $K_{1,k}$, for every positive integer k ;
- (8) $K_{1,k} \dot{\cup} K_2$, for every positive integer k ;
- (9) $K_{1,2} \dot{\cup} K_{1,3}$;
- (10) P_4 .

Proof. According to previous theorems, $|\mathcal{F}(H)| > 1$ for every multigraph H of the list (1)–(10).

On the other hand, let us assume to the contrary that H is a multigraph such that $|\mathcal{F}(H)| > 1$ and it does not belong to the list (1)–(10). So, there is a multigraph $M \in \mathcal{F}_2(H)$. Consider the following cases.

A. $\Pi(H) = \Pi > 1$. As the multigraph H is not belonging to the list (1)–(10), $4 \leq e(H) \geq \Pi + 2$. Moreover, $\Pi(M) \geq 2\Pi - 1$ because otherwise there is a partition of $E(M)$ into parts E_1, E_2 of cardinality $e(H)$ such that $\Pi(M[E_i]) \leq \Pi - 1$, $i \in \{1, 2\}$, i.e., $M[E_i]$ is not isomorphic to H , a contradiction. The multigraph H contains no $\Pi + 1$ parallel edges

and so according to Lemma 2, $e(H) = \Pi + 2$ and there is a multigraph $H^* \in H^{+e} \cap \{P(3, 3), P(3) \dot{\cup} P(3)\}$. Therefore, $H \in \{P(3, 2), P(3) \dot{\cup} P(2)\}$, i.e., H appears in the list, a contradiction.

B. $\Pi(H) = 1$. Thus, H is a simple graph. If M contains parallel edges then according to Lemma 2, $e(H) = 3$ and there is a simple graph $H^* \in H^{+e}$ such that all its edges have the same pairs of degrees of their end vertices. Therefore, $H^* \in \{K_{1,4}, C_4, 2K_{1,2}, 4K_2\}$. Hence, $H \in \{K_{1,3}, P_3, K_{1,2} \dot{\cup} K_2, 3K_2\}$, a contradiction. Therefore, M is also a simple graph. Consider the following subcases.

B1. $\Delta(H) = e(H) - 1$. Then H is either K_3 or a connected graph belonging to $K_{1,k}^{+e}$, where $k \geq 3$. If $H = K_3$, then according to Lemma 2, M contains no 2-matching. However, $K_{1,6}$ does not belong to $\mathcal{F}_2(K_3)$ and so $\mathcal{F}_2(K_3) = \emptyset$. If H is a connected graph belonging to $K_{1,k}^{+e}$, then according to Lemma 2, M contains no 3-matching. Thus, M is a supergraph of H having no 3-matching. It is not difficult to check that none of such graphs belongs to $\mathcal{F}_2(H)$ and so $\mathcal{F}_2(H) = \emptyset$, a contradiction.

B2. $\Delta(H) < e(H) - 1$ and $\Delta(M) > \Delta(H)$. According to Lemma 2, $e(H) = \Delta(H) + 2$ and there is a graph $H^* \in H^{+e} \cap \{2K_{1,3}, K_{2,3}, K_4, C_5\}$. It is not difficult to check that $\mathcal{F}_2(K_{2,3} - e) = \emptyset$, $\mathcal{F}_2(K_4 - e) = \emptyset$ and $\mathcal{F}_2(P_5) = \emptyset$. Thus, $H = K_{1,2} \dot{\cup} K_{1,3}$ appears in the list, a contradiction.

B3. $\Delta(H) < e(H) - 1$ and $\Delta(M) = \Delta(H)$. For $\Delta(H) \geq 3$, there is a positive integer k such that $\Delta(H) + 1 \leq 2k$ and $k < \Delta(H)$. Then there is an equitable $2k$ -edge-coloring of M (see [2]) and so there is a partition of $E(M)$ into parts E_1, E_2 of cardinality $e(H)$ (E_i consists of edges having k distinct colors) such that $\Delta(M[E_i]) \leq k$, $i \in \{1, 2\}$, i.e., $M[E_i]$ is not isomorphic to H , a contradiction. Thus, $\Delta(H) = 2$ and $e(H) \geq 4$. As $\Delta(M) = 2$ there is a partition of $E(M)$ into parts E_1, E_2 of cardinality $e(H)$ such that the size of each component of $M[E_i]$, $i \in \{1, 2\}$, is at most two. Therefore, each component of H has at most two edges, i.e., P_4 is not a subgraph of H . According to Lemma 2 there is no appropriate graph M , a contradiction. ■

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REFERENCES

- [1] J. Ivančo, M. Meszka and Z. Skupień, *Decompositions of multigraphs into parts with two edges*, Discuss. Math. Graph Theory **22** (2002) 113-121.
- [2] W. Wang and K. Zhang, *Equitable colorings of line graphs and complete r -partite graphs*, Systems Science and Math. Sciences **13** (2000) 190-194.

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