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DECOMPOSITIONS OF MULTIGRAPHS INTO PARTS WITH THE SAME SIZE

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Abstract

Given a family $\mathcal{F}$ of multigraphs without isolated vertices, a multigraph M is called *$\mathcal{F}$-decomposable* if M is an edge disjoint union of multigraphs each of which is isomorphic to a member of $\mathcal{F}$. We present necessary and sufficient conditions for existence of such decompositions if $\mathcal{F}$ consists of all multigraphs of size q except for one. Namely, for a multigraph H of size q we find each multigraph M of size kq, such that every partition of the edge set of M into parts of cardinality q contains a part which induces a submultigraph of M isomorphic to H.

Keywords: edge decompositions, multigraphs.

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1. Introduction

We consider finite undirected multigraphs without loops and isolated vertices. Given a family $\mathcal{F}$ of multigraphs, an *$\mathcal{F}$-decomposition* of a multigraph M is a collection of submultigraphs which partition the edge set $E(M)$ of M and are all isomorphic to members of $\mathcal{F}$. If such a decomposition exists, M is called *$\mathcal{F}$-decomposable*; and also *H-decomposable* if H is the only member of $\mathcal{F}$.

If M is a multigraph, then $V(M)$ and $E(M)$ stand for the vertex set and edge set of M, respectively. Cardinalities of those sets, denoted by $v(M)$ and $e(M)$, are called the *order* and *size* of M, respectively. For $E \subset E(M)$,

$M[E]$ denotes the submultigraph of M induced by E. The number of edges incident to a vertex x in M, denoted by $\mathrm{val}_M(x)$, is called the *valency* of x, whilst the number of neighbours of x in M, denoted by $\deg_M(x)$, is called the *degree* of x. As usual $\Delta(M)$ stands for the maximum valency among vertices of M. For any two vertices x, y of M, let $\mathrm{p}_M(x,y)$ denote the number of edges joining x and y. We call $\mathrm{p}_M(x,y)$ the *multiplicity* of an edge xy in M. $\Pi(M)$ stands for the maximum multiplicity among edges of M. Edges joining the same vertices are called *parallel edges* (if they are distinct).

The multipath of length k with edge multiplicities $m_1,\dots,m_k$ is denoted by $P(m_1,\dots,m_k)$. Note that $P(1)=P_2=K_2$ is the simple path on two vertices and $P(1,1)=P_3$ is the simple path on three vertices. The multistar of order $k+1$ with edge multiplicities $m_1,\dots,m_k$ is denoted by $S(m_1,\dots,m_k)$. Note that $S(1,1,1)=K_{1,3}$ is a simple 3-star. The multicycle of length k with edge multiplicities $m_1,\dots,m_k$ is denoted by $C(m_1,\dots,m_k)$. Note that $C(1,1,1)=C_3$ is a 3-cycle. The multiplicity 0 is allowed in these cases. Note that $S(1,0,1)$ is a path. The union of two disjoint multigraphs M and H is denoted by $M\dot\cup H$ and the union of k disjoint copies of a multigraph H is denoted by kH. For a multigraph H, denote by H^{+e} the set of all multigraphs which we obtain from H by adding an edge. Note that $K_2^{+e}=\{C_2,P_3,2K_2\}$.

Given a multigraph M, let $G(M)$ denote a graph which we obtain from M by removal of all edges of the maximal family of pairwise edge-disjoint copies of C_2, i.e., $V(G(M)):=V(M)$ and $E(G(M)):=\{xy:\ \mathrm{p}_M(x,y)\equiv 1 \pmod 2\}$.

In [1] there are provided necessary and sufficient conditions for a multigraph M to be $\{H_1,H_2\}$-decomposable, where H_1, H_2 are any two multigraphs out of C_2, P_3 and $2K_2$. One of the results follows.

Theorem 1 ([1]). *A multigraph M is $\{C_2,\,2K_2\}$-decomposable if and only if each of the following five conditions holds:*

(1) $e(M)\equiv 0 \pmod 2$;

(2) $\mathrm{val}_M(x)+\deg_{G(M)}(x)\le e(M)$ *for every* $x\in V(M)$;

(3) *if* $xy\in E(G(M))$ *then* $\mathrm{val}_M(x)+\mathrm{val}_M(y)-\mathrm{p}_M(x,y)<e(M)$;

(4) *if* $yx,xz\in E(G(M))$, *then* $1+\mathrm{val}_M(x)+\mathrm{p}_M(y,z)<e(M)$;

(5) *M is different from each of the (forbidden) multigraphs shown in Figure* 1.

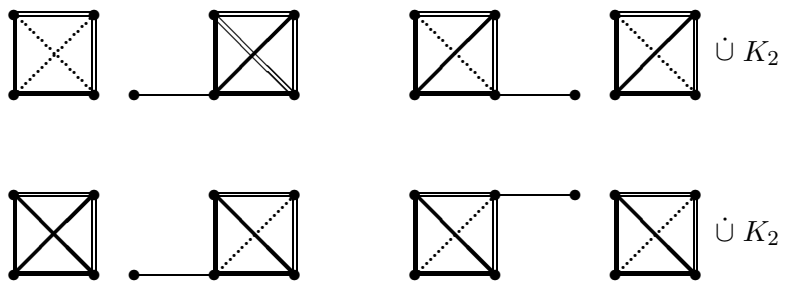

Figure 1. Forbidden multigraphs

Table 1. Codes in Figure 1.

edge :	bold	thin	doubled	dotted
multiplicity :	odd	1	even ≥ 2	even ≥ 0

As it can be seen the assertion is not trivial. So, the aim of this paper is to extend the previous result. For a positive integer k, let $\mathcal{M}(k)$ be the set of all mutually non-isomorphic multigraphs of size k. For a multigraph $H \in \mathcal{M}(k)$, denote by $\tilde{H}$ the set $\mathcal{M}(k) - \{H\}$. As $\mathcal{M}(2) = \{C_2, P_3, 2K_2\}$, Theorem 1 characterizes $\tilde{P}_3$-decomposable multigraphs. In the paper we characterize $\tilde{H}$-decomposable multigraphs where H is an arbitrary multigraph.

2. $\tilde{H}$-Decomposable Multigraphs

For a given multigraph H we define the family $\mathcal{F}(H)$ as follows. A multigraph M belongs to $\mathcal{F}(H)$ iff $e(M) \equiv 0 \pmod{e(H)}$ and every partition of $E(M)$ into parts of cardinality $e(H)$ contains a part which induces a submultigraph of M isomorphic to H.

According to the definition of $\mathcal{F}(H)$ we have immediately

Proposition 1. *A multigraph M is $\tilde{H}$-decomposable if and only if*

$$e(M) \equiv 0 \pmod{e(H)} \quad \textit{and} \quad M \notin \mathcal{F}(H).$$

The previous characterization of $\tilde{H}$-decomposable multigraphs may be useful only for a multigraph H whose forbidden set $\mathcal{F}(H)$ is described. Evidently,

$\mathcal{F}(K_2)$ includes all multigraphs. However, the forbidden sets of the others multigraphs are not so large, but a multigraph H is usually (except for some special cases) the only member of $\mathcal{F}(H)$. Next we describe the forbidden sets of exceptional multigraphs.

Theorem 2. *A multigraph M belongs to $\mathcal{F}(P(k))$ if and only if $e(M) \equiv 0 \pmod{k}$ and $\Pi(M) > \frac{k-1}{k}e(M)$.*

Proof. The condition $e(M) \equiv 0 \pmod{k}$ is obvious. If there is an edge xy of a multigraph M with multiplicity $\mathrm{p}_M(x,y) > \frac{k-1}{k}e(M)$ then any partition of $E(M)$ into parts of cardinality k contains a part consisting of k parallel edges joining x and y. It induces a submultigraph of M isomorphic to $P(k)$. On the other hand, if $\Pi(M) \leq \frac{k-1}{k}e(M)$ then there is a partition of $E(M)$ into parts of cardinality k such that any part contains at most $k-1$ parallel edges. So, $M \notin \mathcal{F}(P(k))$. ■

Theorem 3. *A multigraph M belongs to $\mathcal{F}(kK_2)$ if and only if $e(M) \equiv 0 \pmod{k}$ and the number of odd size components of M is greater than $\frac{k-2}{k}e(M)$.*

Proof. The condition $e(M) \equiv 0 \pmod{k}$ is obvious. It is proved in [1] that a multigraph is $\{C_2, P_3\}$-decomposable if and only if each its component has an even number of edges. Thus, if we remove an appropriate edge from each odd size component of a multigraph M, we get a $\{C_2, P_3\}$-decomposable multigraph. Therefore, the maximum number of mutually edge-disjoint pairs of adjacent edges in M is $\frac{1}{2}(e(M) - c)$, where c denotes the number of odd size components. If $c > \frac{k-2}{k}e(M)$, then M contains less than $e(M)/k$ edge-disjoint pairs of adjacent edges. So, any partition of $E(M)$ into parts of cardinality k contains a part consisting of k edges lying in distinct components. It induces a submultigraph of M isomorphic to kK_2. On the other hand, if $c \leq \frac{k-2}{k}e(M)$ then there is a partition of $E(M)$ into parts of cardinality k such that any part contains two adjacent edges. Thus, $M \notin \mathcal{F}(kK_2)$. ■

Theorem 4. *A multigraph M belongs to $\mathcal{F}(K_{1,2})$ if and only if $e(M)$ is even and at least one of the following four conditions holds:*

(1) *there is $x \in V(M)$ such that $\mathrm{val}_M(x) + \deg_{G(M)}(x) > e(M)$;*

(2) *there is $xy \in E(G(M))$ such that $\mathrm{val}_M(x) + \mathrm{val}_M(y) - \mathrm{p}_M(x,y) = e(M)$;*

(3) *there are* $yx, xz \in E(G(M))$ *such that* $1 + \mathrm{val}_M(x) + \mathrm{p}_M(y,z) \geq e(M)$*;*

(4) M *is one of the multigraphs shown in Figure* 1.

For $k \geq 3$, $M \in \mathcal{F}(K_{1,k})$ *if and only if* $e(M) \equiv 0 \pmod{k}$ *and there is a vertex* $x \in V(M)$ *such that* $\mathrm{val}_M(x) + \deg_{G(M)}(x) > 2\frac{k-1}{k}e(M)$.

Proof. The first equivalence follows immediately from Theorem 1.

The condition $e(M) \equiv 0 \pmod{k}$ is obvious. If $M \notin \mathcal{F}(K_{1,k})$, then there is a partition of $E(M)$ into $e(M)/k$ parts of cardinality k such that no part induces a submultigraph of M isomorphic to $K_{1,k}$. Hence, for any vertex x, every part contains either an edge not incident to x or two parallel edges incident to x. Therefore, the sum of the number of edges not incident to x and the number of edge-disjoint pairs of parallel edges incident to this vertex is at least $e(M)/k$, i.e., $(e(M) - \mathrm{val}_M(x)) + (\mathrm{val}_M(x) - \deg_{G(M)}(x))/2 \geq e(M)/k$. This implies the inequality $\mathrm{val}_M(x) + \deg_{G(M)}(x) \leq 2\frac{k-1}{k}e(M)$.

On the other hand, assume that M is a multigraph of size kt such that $\mathrm{val}_M(x) + \deg_{G(M)}(x) \leq 2\frac{k-1}{k}e(M)$ for every $x \in V(M)$. Evidently, M is not isomorphic to $K_{1,k}$ if $t = 1$. For $t \geq 2$, consider a multigraph $H := M \dot\cup mK_2$, where $m = (k-2)t$. Clearly, $e(H) = e(M) + m = 2(k-1)t$. By Theorem 1, one can easily check that H is a $\{C_2, 2K_2\}$-decomposable multigraph. Therefore, there exist $t = (k-1)t - m$ edge-disjoint pairs of edges $e_i^1, e_i^2 \in E(M)$ such that $M[\{e_i^1, e_i^2\}]$ is isomorphic to either C_2, or $2K_2$, for every $i \in \{1, \ldots, t\}$. Thus, there is a partition of $E(M)$ into parts of cardinality k such that the i-th part contains edges e_i^1 and e_i^2. Clearly, none of these parts induces a submultigraph isomorphic to $K_{1,k}$ and so $M \notin \mathcal{F}(K_{1,k})$. ■

In the next proofs we will use an induction, so the following description of forbidden sets will be very useful.

For a given multigraph H and each positive integer n we define the family $\mathcal{F}_n(H)$ recursively as follows. H is the only member of $\mathcal{F}_1(H)$. For $n > 1$ a multigraph M belongs to $\mathcal{F}_n(H)$ iff $e(M) = ne(H)$ and for every subset $E \subset E(M)$ of cardinality $|E| = e(H)$ it holds either $M[E]$ is isomorphic to H or $M[E(M) - E]$ is isomorphic to a member of $\mathcal{F}_{n-1}(H)$. According to the definition of $\mathcal{F}(H)$ we have immediately

Lemma 1. *Let* H *and* M *be multigraphs such that* $e(M) = ne(H)$*. The multigraph* M *belongs to* $\mathcal{F}(H)$ *if and only if it belongs to* $\mathcal{F}_n(H)$*.*

Put $\mathcal{F}^*(H) := \cup_{i\geq 2}\mathcal{F}_i(H)$. Then $\mathcal{F}(H) = \cup_{i\geq 1}\mathcal{F}_i(H) = \mathcal{F}_1(H) \cup \mathcal{F}^*(H)$. Note that the conditions $\mathcal{F}_2(H) \neq \emptyset$, $\mathcal{F}^*(H) \neq \emptyset$, $|\mathcal{F}(H)| > 1$ are mutually equivalent.

We will often use the following auxiliary assertion.

Lemma 2. *Let H and M be multigraphs such that $M \in \mathcal{F}_2(H)$. Let E be a subset of $E(M)$ such that $M[E]$ is isomorphic to no submultigraph of H.*

If $|E| = e(H)-1$, then there is a multigraph $H^* = M[E(M)-E] \in H^{+e}$ such that all edges of H^* have the same multiplicities and pairs of degrees of their end vertices.

If $|E| < e(H) - 1$, then $H \in \{kK_2, K_{1,k}, P(k)\}$.

Proof. Let H^* be a submultigraph of M induced by $E(M) - E$. If we remove any edge from H^* (in the case $|E| = e(H) - 1$), we get a multigraph isomorphic to H. So, all edges of H^* are equivalent and H^* has the required properties. Similarly, for $|E| < e(H) - 1$, if we remove any $e(H) - |E| \geq 2$ edges from H^*, we get a multigraph isomorphic to H. Thus, all pairs of edges of H^* are equivalent in this case. Evidently, $H^* \in \{P(t), K_{1,t}, tK_2\}$, where $t = e(H^*)$. As H is a submultigraph of H^*, we get the assertion. ■

Theorem 5. $\mathcal{F}^*(P(3,2)) = \{P(7,3)\}$ *and* $\mathcal{F}^*(P(3)\dot\cup P(2)) = \{P(7)\dot\cup P(3)\}$.

Proof. Suppose that $M \in \mathcal{F}_2(P(3,2))$. Then $e(M) = 10$ and $\Pi(M) \geq 5$ because otherwise there is a partition of $E(M)$ into parts E_1, E_2 of cardinality five such that $\Pi(M[E_i]) \leq 2$, $i \in \{1,2\}$, i.e., $M[E_i]$ is not isomorphic to $P(3,2)$, a contradiction. According to Lemma 2, we get a multigraph H^* isomorphic to $P(3,3)$, if we remove four parallel edges from M. It is easy to see that $M = P(7,3)$, i.e., $\mathcal{F}_2(P(3,2)) = \{P(7,3)\}$.

Now suppose that $M \in \mathcal{F}_3(P(3,2))$. Then $e(M) = 15$ and $\Pi(M) \geq 7$ because $P(7,3)$ is a submultigraph of M. Let xy be an edge of M such that $\mathrm{p}_M(xy) = \Pi(M)$. If we remove five parallel edges joining x and y from M, we get a multigraph isomorphic to $P(7,3)$. Thus, $M \in \{P(12,3), P(8,7)\}$. However, it is easy to see that neither $P(12,3)$ nor $P(8,7)$ belongs to $\mathcal{F}_3(P(3,2)$. Therefore, $\mathcal{F}_3(P(3,2)) = \emptyset$ and consequently $\mathcal{F}_i(P(3,2)) = \emptyset$ for every $i \geq 3$.

The second equality can be proved in the same manner, details are left to the reader. ■

Theorem 6. $\mathcal{F}^*(P_4) = \{C(3,1,1,1)\}$.

Proof. Assume that $M \in \mathcal{F}_2(P_4)$. As M contains just six edges it is not difficult to check that M is not a simple graph (i.e., $\Pi(M) > 1$). According to Lemma 2, a multigraph which is obtained from M by deleting two parallel edges is a 4-cycle. Now it is easy to see that $C(3,1,1,1)$ is the only member of $\mathcal{F}_2(P_4)$.

Suppose that $M \in \mathcal{F}_3(P_4)$. As $C(3,1,1,1)$ is a submultigraph of M, $\Pi(M) \geq 3$. If we remove any triple of parallel edges from M, we must obtain a multigraph isomorphic to $C(3,1,1,1)$. So, there are two edge-disjoint triples of parallel edges in M. The only multigraph satisfying the previous two conditions is $C(6,1,1,1)$, but it does not belong to $\mathcal{F}_3(P_4)$. Therefore, $\mathcal{F}_3(P_4) = \emptyset$ and consequently $\mathcal{F}_i(P_4) = \emptyset$ for every $i \geq 3$. ■

Theorem 7. *For the multigraph $P(k)\dot{\cup}K_2$ it holds:*

$$\begin{aligned}
\mathcal{F}(P(2)\dot{\cup}K_2) &= \{P(r)\dot{\cup}P(s)\dot{\cup}P(t) : 0 \leq r \equiv 0, s \equiv 1, t \equiv 2 \pmod 3\}\cup \\
&\qquad \{P(r,1)\dot{\cup}P(t) : 0 \leq r \equiv 0, t \equiv 2 \pmod 3\}, \\
\mathcal{F}(P(3)\dot{\cup}K_2) &= \{P(r)\dot{\cup}P(s) : r \equiv 1, s \equiv 3 \pmod 4\} \quad \textit{and} \\
\mathcal{F}(P(k)\dot{\cup}K_2) &= \{P(r)\dot{\cup}K_2 : 3 \leq r \equiv -1 \pmod{k+1}\}, \textit{ if } k \geq 4.
\end{aligned}$$

Proof. Suppose that $M \in \mathcal{F}_n(P(2)\dot{\cup}K_2)$ for $n \geq 2$. Then $e(M) = 3n$ and $\Pi(M) \geq 1 + n$ because otherwise there is a partition of $E(M)$ into disjoint parts $E_1, \ldots, E_n$ of cardinality three such that $\Pi(M[E_i]) \leq 1$, $i \in \{1, \ldots, n\}$, i.e., $M[E_i]$ is not isomorphic to $P(2)\dot{\cup}K_2$, a contradiction. If we remove any triple of parallel edges (of multiplicity $\Pi(M)$) from M, we must obtain a multigraph isomorphic to a member of $\mathcal{F}_{n-1}(P(2)\dot{\cup}K_2)$. Thus, for $n = 2$, $M \in \{P(1)\dot{\cup}P(5),\ P(4)\dot{\cup}P(2),\ P(1,3,2),\ P(3,2)\dot{\cup}P(1),\ P(3,1)\dot{\cup}P(2),\ P(3)\dot{\cup}P(1)\dot{\cup}P(2)\}$. Now, it is not difficult to check that $\mathcal{F}_2(P(2)\dot{\cup}K_2) = \{P(3)\dot{\cup}P(1)\dot{\cup}P(2),\ P(1)\dot{\cup}P(5),\ P(4)\dot{\cup}P(2),\ P(3,1)\dot{\cup}P(2)\}$. Similarly, using induction for $n \geq 3$, we get the assertion.

The other equalities can be proved in the same manner, details are left to the reader. ■

Theorem 8. *For the multigraph $P(k,1)$ it holds:*

$$\begin{aligned}
\mathcal{F}(P(2,1)) &= \{C(r,s,t) : r \equiv 1, s \equiv 2, 0 \leq t \equiv 0 \pmod 3\}\cup \\
&\qquad \{S(r,s,t) : r \equiv 1, s \equiv 2, 0 \leq t \equiv 0 \pmod 3\}\cup \\
&\qquad \{P(1,s,t) : s \equiv 2, 0 \leq t \equiv 0 \pmod 3\}, \\
\mathcal{F}(P(3,1)) &= \{P(r,s) : r \equiv 1, s \equiv 3 \pmod 4\} \quad \textit{and} \\
\mathcal{F}(P(k,1)) &= \{P(r,1) : 4 \leq r \equiv -1 \pmod{k+1}\}, \textit{ if } k \geq 4.
\end{aligned}$$

Proof. Suppose that $M \in \mathcal{F}_n(P(2,1))$ for $n \geq 2$. Then $e(M) = 3n$ and $\Pi(M) \geq 1 + n$ because otherwise there is a partition of $E(M)$ into disjoint parts $E_1, \ldots, E_n$ of cardinality three such that $\Pi(M[E_i]) \leq 1$, $i \in \{1, \ldots, n\}$, i.e., $M[E_i]$ is not isomorphic to $P(2,1)$, a contradiction. If we remove any triple of parallel edges (of multiplicity $\Pi(M)$) from M, we must obtain a multigraph isomorphic to a member of $\mathcal{F}_{n-1}(P(2,1))$. Thus, for $n = 2$, $M \in \{P(5,1), P(4,2), P(3)\dot{\cup}P(2,1), P(3,2,1), P(2,1,3), S(3,2,1), C(1,2,3)\}$. Now, it is not difficult to check that $\mathcal{F}_2(P(2,1)) = \{P(5,1), P(4,2), P(3,2,1), S(3,2,1), C(1,2,3)\}$. Similarly, using induction for $n \geq 3$, we get the assertion.

The other equalities can be proved in the same manner, details are left to the reader. ■

Theorem 9. $\mathcal{F}^*(K_{1,2}\dot{\cup}K_{1,3}) = \{K_{1,3}\dot{\cup}K_{1,7}\}$.

Proof. Suppose that $M \in \mathcal{F}_2(K_{1,2}\dot{\cup}K_{1,3})$. Then $e(M) = 10$ and by Lemma 2 M contains no parallel edges, no triangle and no 3-matching. Assume that $\Delta(M) \leq 3$. Then there is an equitable 4-edge-coloring of M (see [2]) and so there is a partition of $E(M)$ into parts E_1, E_2 of cardinality five (E_i consists of edges having two distinct colors) such that $\Delta(M[E_i]) \leq 2$, $i \in \{1,2\}$, i.e., $M[E_i]$ is not isomorphic to $K_{1,2}\dot{\cup}K_{1,3}$, a contradiction. Therefore, $\Delta(M) \geq 4$. According to Lemma 2, a multigraph which is obtained from M by deleting four edges incident to a maximum degree vertex is isomorphic to $2K_{1,3}$. It is easy to see that $M = K_{1,3}\dot{\cup}K_{1,7}$, i.e., $\mathcal{F}_2(K_{1,2}\dot{\cup}K_{1,3}) = \{K_{1,3}\dot{\cup}K_{1,7}\}$.

Now suppose that $M \in \mathcal{F}_3(K_{1,2}\dot{\cup}K_{1,3})$. Then $e(M) = 15$ and $\Delta(M) \geq 7$ because $K_{1,3}\dot{\cup}K_{1,7}$ is a submultigraph of M. Let x be a vertex of M such that $\deg_M(x) = \Delta(M)$. If we remove five edges incident to x from M, we get a multigraph isomorphic to $K_{1,3}\dot{\cup}K_{1,7}$. Thus, M is a disjoint union of two multistars. However, it is easy to see that none of such multigraphs belongs to $\mathcal{F}_3(K_{1,2}\dot{\cup}K_{1,3})$. Therefore, $\mathcal{F}_3(K_{1,2}\dot{\cup}K_{1,3}) = \emptyset$ and consequently, for every $i \geq 3$, $\mathcal{F}_i(K_{1,2}\dot{\cup}K_{1,3}) = \emptyset$. ■

Lemma 3. *Let M be a connected multigraph of size at least 4. Then there are edges $e_1, e_2, e_3 \in E(M)$ satisfying*

(1) *$M[\{e_1, e_2, e_3\}]$ is either a matching or a connected submultigraph of M;*

(2) *$M[E(M) - \{e_1, e_2, e_3\}]$ is a connected multigraph.*

Proof. Suppose that M is a counterexample. Let T be a spanning tree of M. Evidently, $M[E(M) - (E \cup \{e\})]$ is a connected multigraph for any pendant edge e of T and any set $E \subseteq E(M) - E(T)$. The multigraph M satisfies the following conditions.

A. *There is no pendant edge of T adjacent to two distinct edges of $E(M) - E(T)$.*
Suppose to the contrary that $e_1, e_2 \in E(M) - E(T)$ are two distinct edges adjacent to a pendant edge e_3 of T. Clearly, the multigraphs $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected, a contradiction.

B. *There is no edge of $E(M)$–$E(T)$ adjacent to two pendant edges of T.*
Assume that $e_1 \in E(M) - E(T)$ is an edge adjacent to two pendant edges e_2, e_3 of T. Moreover, assume that e_1 is incident to a pendant vertex of T (if there exists such edge). If $M[E(M) - \{e_1, e_2, e_3\}]$ is a disconnected multigraph then there is an edge e_4 whose end vertices are pendant vertices of T (end vertices of e_2 and e_3). Thus, edges e_1 and e_4 are adjacent to a pendant edge of T, contrary to **A**. Therefore, $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected multigraphs, a contradiction.

C. *There is no 3-matching consisting of pendant edges of T.*
Suppose that e_1, e_2, e_3 are three independent pendant edges of T. By **B**, there is no edge of $E(M) - E(T)$ adjacent to any two pendant edges of $\{e_1, e_2, e_3\}$. Thus, $M[E(M) - \{e_1, e_2, e_3\}]$ is a connected multigraph and $M[\{e_1, e_2, e_3\}]$ is a matching, a contradiction.

D. *There are not three mutually adjacent pendant edges of T.*
Assume to the contrary that e_1, e_2, e_3 are three adjacent pendant edges of T. By **B**, there is no edge of $E(M) - E(T)$ adjacent to any two pendant edges of $\{e_1, e_2, e_3\}$. Therefore, $M[E(M) - \{e_1, e_2, e_3\}]$ is a connected multigraph. The multigraph $M[\{e_1, e_2, e_3\}]$ is also connected in this case, a contradiction.

E. *There are not two adjacent pendant edges of T.*
Suppose that e_1, e_2 are two adjacent pendant edges of T. By **B**, there is no edge of $E(M) - E(T)$ adjacent to e_1 and e_2. If e_3 is another edge of T adjacent to both of e_1, e_2, then $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected multigraphs, a contradiction.

F. *There is no edge of $E(M) - E(T)$ adjacent to a pendant edge of T.*
According to **C** and **E**, the tree T is a path. Assume that $e_1 \in E(M) - E(T)$ is an edge adjacent to a pendant edge e_2 of T. By **A**, there is no other edge of $E(M) - E(T)$ adjacent to e_2. If e_3 is the edge of T adjacent to e_2, then $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected multigraphs, a contradiction.

By **A** – **F**, the tree T is a path and there is no edge of $E(M)-E(T)$ adjacent to a pendant edge of T. If e_1 is any pendant edge of T and edges e_1, e_2, e_3 induce a subpath of T, then $M[\{e_1, e_2, e_3\}]$ and $M[E(M) - \{e_1, e_2, e_3\}]$ are connected multigraphs, a contradiction. Therefore, there is no counterexample of the assertion. ■

Theorem 10. *For the multigraph $K_{1,k}\dot{\cup}K_2$ it holds:*

$$\begin{aligned}
\mathcal{F}(K_{1,2}\dot{\cup}K_2) = {} & \{K_{1,r}\dot{\cup}K_{1,s}\dot{\cup}K_{1,t} : r \equiv 1, s \equiv 2, t \equiv 0 \pmod 3\}\cup \\
& \{K_{1,r}\dot{\cup}K_{1,s}\dot{\cup}K_3 : r \equiv 1, s \equiv 2 \pmod 3\}\cup \\
& \{K_{1,r}\dot{\cup}C_5 : r \equiv 1 \pmod 3\}\cup \\
& \{H\dot{\cup}K_{1,s} : H \in K_{1,t}^{+e}, t \equiv 0, s \equiv 2 \pmod 3\}\cup \\
& \{H\dot{\cup}K_{1,s} : H \in K_3^{+e}, s \equiv 2 \pmod 3\}\cup \\
& \{S(3,1,\ldots,1)\dot{\cup}K_{1,s} : e(S(3,1,\ldots,1)) \equiv 1, s \equiv 2 \pmod 3\}\cup \\
& \{P(2,1,1)\dot{\cup}K_{1,s} : s \equiv 2 \pmod 3\}, \\
\mathcal{F}(K_{1,3}\dot{\cup}K_2) = {} & \{K_{1,r}\dot{\cup}K_{1,s} : r \equiv 1, s \equiv 3 \pmod 4\} \quad \textit{and} \\
\mathcal{F}(K_{1,k}\dot{\cup}K_2) = {} & \{K_{1,r}\dot{\cup}K_2 : 4 \le r \equiv -1 \pmod{k+1}\}, \textit{ if } k \ge 4.
\end{aligned}$$

Proof. Suppose that $M \in \mathcal{F}_n(K_{1,2}\dot{\cup}K_2)$. If $n = 2$, then $e(M) = 6$ and by Lemma 3 M is disconnected. If $\Pi(M) > 1$, then according to Lemma 2, a multigraph which is obtained from M by deleting two parallel edges is isomorphic to $2K_{1,2}$. Now it is easy to check that $M \in \{S(3,1)\dot{\cup}K_{1,2}$, $C(2,1,1)\dot{\cup}K_{1,2}$, $S(2,1,1)\dot{\cup}K_{1,2}$, $P(2,1,1)\dot{\cup}K_{1,2}\}$. If $\Pi(M) = 1$, then M has at most three components because otherwise there is a partition of $E(M)$ into parts E_1, E_2 of cardinality three such that $M[E_1]$ is a matching and $M[E_2]$ is either a matching or a component of M, a contradiction. If M has three components, then by Lemma 3 the size of each component is at most three. Now, it is not difficult to check that M is isomorphic to either $K_2\dot{\cup}K_{1,2}\dot{\cup}K_{1,3}$ or $K_2\dot{\cup}K_{1,2}\dot{\cup}K_3$ in this case. If M has two components, then there is a component of size at most two because otherwise the components decompose M into two connected submultigraphs of size three. If M has a component of size 1, then verifying twenty possible graphs it is not difficult to check that M is isomorphic to either $K_{1,5}\dot{\cup}K_2$ or $C_5\dot{\cup}K_2$. Similarly, if M has a component of size two, then it is not difficult to check that M is a graph belonging to $\{H\dot{\cup}K_{1,2} : H \in K_{1,3}^{+e}\}$. Thus, the assertion holds in this case.

If $n \ge 3$, then it is not difficult to see (using induction and Lemma 3) that M includes a vertex of degree at least three. Assume that $x \in V(M)$ is a vertex of maximum degree. If we remove three (non-parallel) edges

incident to x from M, we get a multigraph belonging to $\mathcal{F}_{n-1}(K_{1,2}\dot{\cup}K_2)$. Using induction it is not difficult to check that the assertion holds.

Now suppose that $M \in \mathcal{F}_n(K_{1,3}\dot{\cup}K_2)$. If $n = 2$, then $e(M) = 8$. According to Lemma 2 M contains no parallel edges, no 3-matching, no triangle and no path of length three. Now, it is not difficult to check that M is isomorphic to either $K_2\dot{\cup}K_{1,7}$ or $K_{1,3}\dot{\cup}K_{1,5}$. Evidently, $\Delta(M) > 4$ for $n \geq 3$. If we remove four edges incident to a maximum degree vertex from M, we get a multigraph belonging to $\mathcal{F}_{n-1}(K_{1,3}\dot{\cup}K_2)$. Using induction we get the assertion.

The last equality can be proved in the same manner, details are left to the reader. ■

We conclude this paper with the following result.

Theorem 11. *Let H be a multigraph. $|\mathcal{F}(H)| > 1$ if and only if H is one of the following multigraphs:*

(1) $P(k)$, *for every positive integer* k;

(2) $P(k, 1)$, *for every positive integer* k;

(3) $P(k)\dot{\cup}K_2$, *for every positive integer* k;

(4) $P(3, 2)$;

(5) $P(3)\dot{\cup}P(2)$;

(6) kK_2, *for every positive integer* k;

(7) $K_{1,k}$, *for every positive integer* k;

(8) $K_{1,k}\dot{\cup}K_2$, *for every positive integer* k;

(9) $K_{1,2}\dot{\cup}K_{1,3}$;

(10) P_4.

Proof. According to previous theorems, $|\mathcal{F}(H)| > 1$ for every multigraph H of the list (1)–(10).

On the other hand, let us assume to the contrary that H is a multigraph such that $|\mathcal{F}(H)| > 1$ and it does not belong to the list (1)–(10). So, there is a multigraph $M \in \mathcal{F}_2(H)$. Consider the following cases.

A. $\Pi(H) = \Pi > 1$. As the multigraph H is not belonging to the list (1)–(10), $4 \leq e(H) \geq \Pi + 2$. Moreover, $\Pi(M) \geq 2\Pi - 1$ because otherwise there is a partition of $E(M)$ into parts E_1, E_2 of cardinality $e(H)$ such that $\Pi(M[E_i]) \leq \Pi - 1$, $i \in \{1, 2\}$, i.e., $M[E_i]$ is not isomorphic to H, a contradiction. The multigraph H contains no $\Pi + 1$ parallel edges

and so according to Lemma 2, $e(H) = \Pi + 2$ and there is a multigraph $H^* \in H^{+e} \cap \{P(3,3), P(3)\dot\cup P(3)\}$. Therefore, $H \in \{P(3,2), P(3)\dot\cup P(2)\}$, i.e., H appears in the list, a contradiction.

B. $\Pi(H) = 1$. Thus, H is a simple graph. If M contains parallel edges then according to Lemma 2, $e(H) = 3$ and there is a simple graph $H^* \in H^{+e}$ such that all its edges have the same pairs of degrees of their end vertices. Therefore, $H^* \in \{K_{1,4}, C_4, 2K_{1,2}, 4K_2\}$. Hence, $H \in \{K_{1,3}, P_3, K_{1,2}\dot\cup K_2, 3K_2\}$, a contradiction. Therefore, M is also a simple graph. Consider the following subcases.

B1. $\Delta(H) = e(H) - 1$. Then H is either K_3 or a connected graph belonging to $K_{1,k}^{+e}$, where $k \geq 3$. If $H = K_3$, then according to Lemma 2, M contains no 2-matching. However, $K_{1,6}$ does not belong to $\mathcal{F}_2(K_3)$ and so $\mathcal{F}_2(K_3) = \emptyset$. If H is a connected graph belonging to $K_{1,k}^{+e}$, then according to Lemma 2, M contains no 3-matching. Thus, M is a supergraph of H having no 3-matching. It is not difficult to check that none of such graphs belongs to $\mathcal{F}_2(H)$ and so $\mathcal{F}_2(H) = \emptyset$, a contradiction.

B2. $\Delta(H) < e(H) - 1$ and $\Delta(M) > \Delta(H)$. According to Lemma 2, $e(H) = \Delta(H) + 2$ and there is a graph $H^* \in H^{+e} \cap \{2K_{1,3}, K_{2,3}, K_4, C_5\}$. It is not difficult to check that $\mathcal{F}_2(K_{2,3} - e) = \emptyset$, $\mathcal{F}_2(K_4 - e) = \emptyset$ and $\mathcal{F}_2(P_5) = \emptyset$. Thus, $H = K_{1,2}\dot\cup K_{1,3}$ appears in the list, a contradiction.

B3. $\Delta(H) < e(H) - 1$ and $\Delta(M) = \Delta(H)$. For $\Delta(H) \geq 3$, there is a positive integer k such that $\Delta(H) + 1 \leq 2k$ and $k < \Delta(H)$. Then there is an equitable $2k$-edge-coloring of M (see [2]) and so there is a partition of $E(M)$ into parts E_1, E_2 of cardinality $e(H)$ (E_i consists of edges having k distinct colors) such that $\Delta(M[E_i]) \leq k$, $i \in \{1,2\}$, i.e., $M[E_i]$ is not isomorphic to H, a contradiction. Thus, $\Delta(H) = 2$ and $e(H) \geq 4$. As $\Delta(M) = 2$ there is a partition of $E(M)$ into parts E_1, E_2 of cardinality $e(H)$ such that the size of each component of $M[E_i]$, $i \in \{1,2\}$, is at most two. Therefore, each component of H has at most two edges, i.e., P_4 is not a subgraph of H. According to Lemma 2 there is no appropriate graph M, a contradiction. ■

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