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ON A FAMILY OF CUBIC GRAPHS CONTAINING THE FLOWER SNARKS

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Abstract

We consider cubic graphs formed with $k \geq 2$ disjoint claws $C_i \sim K_{1,3}$ $(0 \leq i \leq k-1)$ such that for every integer i modulo k the three vertices of degree 1 of C_i are joined to the three vertices of degree 1 of C_{i-1} and joined to the three vertices of degree 1 of C_{i+1}. Denote by t_i the vertex of degree 3 of C_i and by T the set $\{t_1, t_2, \ldots, t_{k-1}\}$. In such a way we construct three distinct graphs, namely $FS(1,k)$, $FS(2,k)$ and $FS(3,k)$. The graph $FS(j,k)$ $(j \in \{1,2,3\})$ is the graph where the set of vertices $\cup_{i=0}^{i=k-1} V(C_i) \setminus T$ induce j cycles (note that the graphs $FS(2, 2p+1)$, $p \geq 2$, are the flower snarks defined by Isaacs [8]). We determine the number of perfect matchings of every $FS(j,k)$. A cubic graph G is said to be 2-*factor hamiltonian* if every 2-factor of G is a hamiltonian cycle. We characterize the graphs $FS(j,k)$ that are 2-factor hamiltonian (note that $FS(1,3)$ is the "Triplex Graph" of Robertson, Seymour and Thomas [15]). A *strong matching* M in a graph G is a matching M such that there is no edge of $E(G)$ connecting any two edges of M. A cubic graph having a perfect matching union of two strong matchings is said to be a *Jaeger's graph*. We characterize the graphs $FS(j,k)$ that are Jaeger's graphs.

Keywords: cubic graph, perfect matching, strong matching, counting, hamiltonian cycle, 2-factor hamiltonian.

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1. INTRODUCTION

The complete bipartite graph $K_{1,3}$ is called, as usually, a *claw*. Let k be an integer ≥ 2 and let G be a cubic graph on $4k$ vertices formed with k disjoint claws $C_i = \{x_i, y_i, z_i, t_i\}$ $(0 \leq i \leq k-1)$ where t_i (the *center* of C_i) is joined to the three independent vertices x_i, y_i and z_i (the *external* vertices of C_i). For every integer i modulo k C_i has three neighbours in C_{i-1} and three neighbours in C_{i+1}. For any integer $k \geq 2$ we shall denote the set of integers modulo k as $\mathbf{Z}_k$. In the sequel of this paper indices i of claws C_i belong to $\mathbf{Z}_k$.

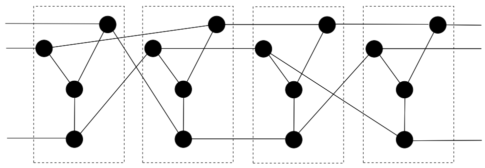

Figure 1. Four consecutive claws.

By renaming some external vertices of claws we can suppose, without loss of generality, that $\{x_i x_{i+1}, y_i y_{i+1}, z_i z_{i+1}\}$ are edges for any i distinct from $k-1$. That is to say the subgraph induced on $X = \{x_0, x_1, \dots, x_{k-1}\}$ (respectively $Y = \{y_0, y_1, \dots, y_{k-1}\}$, $Z = \{z_0, z_1, \dots, z_{k-1}\}$) is a path or a cycle (as induced subgraph of G). Denote by T the set of the internal vertices $\{t_0, t_1, \dots, t_{k-1}\}$.

Up to isomorphism, the matching joining the external vertices of C_{k-1} to those of C_0 (also called, for $k \geq 3$, *edges between* C_{k-1} *and* C_0) determines the graph G. In this way we construct essentially three distinct graphs, namely $FS(1,k)$, $FS(2,k)$ and $FS(3,k)$. The graph $FS(j,k)$ $(j \in \{1,2,3\})$ is the graph where the set of vertices $\cup_{i=0}^{i=k-1}\{C_i \setminus \{t_i\}\}$ induces j cycles. For $k \geq 3$ and any $j \in \{1,2,3\}$ the graph $FS(j,k)$ is a simple cubic graph. When k is odd, the $FS(2,k)$ are the graphs known as the flower snarks [8]. We note that $FS(3,2)$ and $FS(2,2)$ are multigraphs, and that $FS(1,2)$ is isomorphic to the cube. For $k=2$ the notion of "edge between C_{k-1} and C_0" is ambiguous, so we must define it precisely. For two parallel edges having one end in C_0 and the other in C_1, for instance two parallel edges having x_0 and x_1 as endvertices, we denote one edge by x_0x_1 and the other by x_1x_0. An edge in $\{x_1x_0, x_1y_0, x_1z_0, y_1x_0, y_1y_0, y_1z_0, z_1x_0, z_1y_0, z_1z_0\}$, if it exists, is an *edge between* C_1 *and* C_0. We will say that x_0x_1, y_0y_1 and z_0z_1 are *edges between* C_0 *and* C_1.

By using an ad hoc translation of the indices of claws (and of their vertices) and renaming some external vertices of claws, we see that for any reasoning about a sequence of $h \geq 3$ consecutive claws $(C_i, C_{i+1}, C_{i+2}, \ldots, C_{i+h-1})$ there is no loss of generality to suppose that $0 \leq i < i+h-1 \leq k-1$. For a sequence of claws $(C_p, \ldots, C_r)$ with $0 \leq p < r \leq k-1$, since 0 is a possible value for subscript p and since $k-1$ is a possible value for subscript r, it will be useful from time to time to denote by x'_{p-1} the neighbour in C_{p-1} of the vertex x_p of C_p (recall that $x'_{p-1} \in \{x_{k-1}, y_{k-1}, z_{k-1}\}$ if $p = 0$), and to denote by x'_{r+1} the neighbour in C_{r+1} of the vertex x_r of C_r (recall that $x'_{r+1} \in \{x_0, y_0, z_0\}$ if $r = k-1$). We shall make use of analogous notations for neighbours of y_p, z_p, y_r and z_r.

We shall prove in the following lemma that there are essentially two types of perfect matchings in $FS(j,k)$.

Lemma 1. *Let $G \in \{FS(j,k), j \in \{1,2,3\}, k \geq 2\}$ and let M be a perfect matching of G. Then the 2-factor $G \setminus M$ induces a path of length 2 and an isolated vertex in each claw C_i $(i \in \mathbf{Z}_k)$ and M fulfils one (and only one) of the three following properties:*

(i) *For every i in $\mathbf{Z}_k$ M contains exactly one edge joining the claw C_i to the claw C_{i+1},*

(ii) *For every even i in $\mathbf{Z}_k$ M contains exactly two edges between C_i and C_{i+1} and none between C_{i-1} and C_i,*

(iii) *For every odd i in $\mathbf{Z}_k$ M contains exactly two edges between C_i and C_{i+1} and none between C_{i-1} and C_i.*

Moreover, when k is odd M satisfies only item (i).

Proof. Let M be a perfect matching of $G = FS(j,k)$ for some $j \in \{1,2,3\}$. Since M contains exactly one edge of each claw, it is obvious that $G \setminus M$ induces a path of length 2 and an isolated vertex in each claw C_i.

For each claw C_i of G the vertex t_i must be saturated by an edge of M whose end (distinct from t_i) is in $\{x_i, y_i, z_i\}$. Hence there are exactly two edges of M having one end in C_i and the other in $C_{i-1} \cup C_{i+1}$.

If there are two edges of M between C_i and C_{i+1} then there is no edge of M between C_{i-1} and C_i. If there are two edges of M between C_{i-1} and C_i then there is no edge of M between C_i and C_{i+1}. Hence, we get (ii) or (iii) and we must have an even number k of claws in G.

Assume now that there is only one edge of M between C_{i-1} and C_i. Then there exists exactly one edge between C_i and C_{i+1} and, extending this trick to each claw of G, we get (i) when k is even or odd. ■

Definition 2. We say that a perfect matching M of $FS(j,k)$ is of *type* 1 in Case (i) of Lemma 1 and of *type* 2 in Cases (ii) and (iii). If neccessary, to distinguish Case (ii) from Case (iii) we shall say *type* 2.0 in Case (ii) and *type* 2.1 in Case (iii). We note that the numbers of perfect matchings of type 2.0 and of type 2.1 are equal.

Notation. The length of a path P (respectively a cycle Γ) is denoted by $l(P)$ (respectively $l(\Gamma)$).

2. COUNTING PERFECT MATCHINGS OF $FS(j,k)$

We shall say that a vertex v of a cubic graph G is *inflated* into a triangle when we construct a new cubic graph G' by deleting v and adding three new vertices inducing a triangle and joining each vertex of the neighbourhood $N(v)$ of v to a single vertex of this new triangle. We say also that G' is obtained from G by a *triangular extension.* The converse operation is the *contraction* or *reduction* of the triangle. The number of perfect matchings of G is denoted by $\mu(G)$.

Lemma 3. *Let G be a bipartite cubic graph and let $\{V_1, V_2\}$ be the bipartition of its vertex set. Assume that each vertex in some subset $W_1 \subseteq V_1$ is inflated into a triangle and let G' be the graph obtained in that way. Then $\mu(G) = \mu(G')$.*

Proof. Note that $\{V_1, V_2\}$ is a balanced bipartition and, by König's Theorem, the graph G is a cubic 3-edge colourable graph. So, G' is also a cubic 3-edge colourable graph(hence, G and G' have perfect matchings). Let M be a perfect matching of G'. Each vertex of $V_1 \setminus W_1$ is saturated by an edge whose second end vertex is in V_2. Let $A \subseteq V_2$ be the set of vertices so saturated in V_2. Assume that some triangle of G' is such that the three vertices are saturated by three edges having one end in the triangle and the second one in V_2. Then we need to have at least $|W_1| + 2$ vertices in $V_2 \setminus A$, a contradiction. Hence, M must have exactly one edge in each triangle and the contraction of each triangle in order to get back G transforms M in a perfect matching of G. Conversely, each perfect matching of G leads to a unique perfect matching of G' and we obtain the result. ■

Let us denote by $\mu(j,k)$ the number of perfect matchings of $FS(j,k)$, $\mu_1(j,k)$ its number of perfect matchings of type 1 and $\mu_2(j,k)$ its number of perfect matchings of type 2.

Lemma 4. *We have*

- $\mu(1,3) = \mu_1(1,3) = 9$,
- $\mu(2,3) = \mu_1(2,3) = 8$,
- $\mu(3,3) = \mu_1(3,3) = 6$,
- $\mu(1,2) = 9$, $\mu_1(1,2) = 3$,
- $\mu(2,2) = 10$, $\mu_1(2,2) = 4$,
- $\mu(3,2) = 12$, $\mu_1(3,2) = 6$.

Proof. The cycle containing the external vertices of the claws of the graph $FS(1,3)$ is $x_0, x_1, x_2, y_0, y_1, y_2, z_0, z_1, z_2, x_0$. Consider a perfect matching M containing the edge t_0x_0. There are two cases: (i) $x_1x_2 \in M$ and (ii) $x_1t_1 \in M$. In Case (i) we must have $y_0y_1, t_1z_1, t_2z_2, z_0y_2 \in M$. In Case (ii) there are two sub-cases: (ii).a $x_2y_0 \in M$ and (ii).b $x_2t_2 \in M$. In Case (ii).a we must have $y_1y_2, t_2z_2, z_0z_1 \in M$ and in Case (ii).b we must have $y_0y_1, y_2z_0, z_1z_2 \in M$. Thus, there are exactly 3 distinct perfect matching containing t_0x_0. By symmetry, there are 3 distinct perfect matchings containing t_0y_0, and 3 distinct matchings containing t_0z_0, therefore $\mu(1,3) = 9$.

It is well known that the Petersen graph has exactly 6 perfect matchings. Since $FS(2,3)$ is obtained from the Petersen graph by inflating a vertex into a triangle these 6 perfect matchings lead to 6 perfect matchings of $FS(2,3)$. We have two new perfect matchings when considering the three edges connected to this triangle (we have two ways to include these edges into a perfect matching). Hence $\mu(2,3) = 8$.

$FS(3,3)$ is obtained from $K_{3,3}$ by inflating three vertices in the same colour of the bipartition. Since $K_{3,3}$ has six perfect matchings, applying Lemma 3 we get immediately the result for $\mu(3,3)$.

Is is a routine matter to obtain the values for $FS(j,2)$ ($j \in \{1,2,3\}$). ■

Theorem 5. *The numbers* $\mu_1(j,k)$ *of perfect matchings of type* 1 *of* $FS(j,k)$ ($j \in \{1,2,3\}$) *are given by:*

- $\mu_1(1,k) = 2^k - (-1)^k$,

- $\mu_1(2,k) = 2^k$,
- $\mu_1(3,k) = 2^k + 2(-1)^k$.

Proof. By Lemma 4, when $k = 2$ or $k = 3$ $\mu_1(j,k)$ fulfils our requirement. We want to compute $\mu_1(j,k)$ by induction on k and we assume that the property holds for $FS(j,k-2)$ with $k \geq 4$ ($j \in \{1,2,3\}$).

The following trick will be helpful. Let us consider the four consecutive claws C_{k-2}, C_{k-1}, C_0 and C_1 of $FS(j,k)$ ($j \in \{1,2,3\}$). We can delete C_{k-1} and C_0 and join the three external vertices of C_{k-2} to the three external vertices of C_1 by a matching (to choose) in such a way that the resulting graph is a graph $FS(j',k-2)$, with $j' \in \{1,2,3\}$ (it must be clear that j and j' may be different). In order to count the number of perfect matchings of type 1 of $FS(j,k)$ we need to consider nine numbers, each of them counts the number of perfect matchings of type 1 of $FS(j,k)$ using two edges, one being an edge between C_{k-2} and C_{k-1} and the other being an edge between C_1 and C_0.

Let us define $\nu(e,e')$ as the number of perfect matchings of type 1 containing the two edges e and e'. Then we set

$$a_1 = \nu(x_{k-2}x_{k-1}, x_0x_1),\ a_2 = \nu(x_{k-2}x_{k-1}, y_0y_1),\ a_3 = \nu(x_{k-2}x_{k-1}, z_0z_1),$$
$$a_4 = \nu(y_{k-2}y_{k-1}, x_0x_1),\ a_5 = \nu(y_{k-2}y_{k-1}, y_0y_1),\ a_6 = \nu(y_{k-2}y_{k-1}, z_0z_1),$$
$$a_7 = \nu(z_{k-2}z_{k-1}, x_0x_1),\ a_8 = \nu(z_{k-2}z_{k-1}, y_0y_1),\ a_9 = \nu(z_{k-2}z_{k-1}, z_0z_1).$$

Obviously we have $\mu_1(j,k) = \sum_{i=1}^{9} a_i$.

Let us delete the vertices of C_{k-1} and C_0 and denote by H the resulting induced subgraph.

Claim 1. When $j = 1$, we have

$$a_2 + a_6 + a_7 = 2\mu_1(1,k-2),$$
$$a_1 + a_5 + a_9 = \mu_1(3,k-2),$$
$$a_3 + a_4 + a_8 = \mu_1(1,k-2).$$

Proof. Without loss of generality we can consider that $x_{k-1}y_0$, $y_{k-1}z_0$ and $z_{k-1}x_0$ are edges of $FS(1,k)$.

In order to evaluate $a_2 + a_6 + a_7$ we add the edges $x_{k-2}y_1$, $y_{k-2}z_1$ and $z_{k-2}x_1$ to H. In other words we set $H_1 = (V(H), E(H) \cup \{x_{k-2}y_1, y_{k-2}z_1, z_{k-2}x_1\})$, we get hence $FS(1,k-2)$. Each perfect matching of type 1 of

H_1 containing $x_{k-2}y_1$ gives two perfect matchings of type 1 of $FS(1,k)$ as well as each perfect matching of type 1 of H_1 containing $y_{k-2}z_1$ and each perfect matching of type 1 containing the edge $z_{k-2}x_1$. If follows that $a_2 + a_6 + a_7 = 2\mu_1(1, k-2)$.

Let us now set $H_2 = (V(H), E(H) \cup \{x_{k-2}x_1, y_{k-2}y_1, z_{k-2}z_1\})$. The graph H_2 is isomorphic to $FS(3, k-2)$. Each perfect matching of type 1 of H_2 containing $x_{k-2}x_1$ gives one perfect matching of type 1 of $FS(1,k)$ as well as each perfect matching of type 1 of H_2 containing $y_{k-2}y_1$ and each perfect matching of type 1 of H_2 containing $z_{k-2}z_1$. Consequently $a_1 + a_5 + a_9 = \mu_1(3, k-2)$.

For computing $a_3+a_4+a_8$ we set $H_3 = (V(H), E(H) \cup \{x_{k-2}z_1, y_{k-2}x_1, z_{k-2}y_1\})$, that is a graph isomorphic to $FS(1, k-2)$. Each perfect matching of type 1 of H_3 containing $x_{k-2}z_1$ gives one perfect matching of type 1 of $FS(1,k)$ as well as each perfect matching of type 1 of H_3 containing $y_{k-2}x_1$ and each perfect matching of type 1 of H_3 containing $z_{k-2}y_1$. Thus $a_3 + a_4 + a_8 = \mu_1(1, k-2)$ as claimed. □

Claim 2. When $j = 2$, we have

$$a_1 + a_6 + a_8 = 2\mu_1(2, k-2),$$
$$a_2 + a_4 + a_9 = \mu_1(2, k-2),$$
$$a_3 + a_5 + a_7 = \mu_1(2, k-2).$$

Proof. Without loss of generality we can consider that $x_{k-1}x_0$, $y_{k-1}z_0$ and $z_{k-1}y_0$ are edges of $FS(2,k)$.

In order to evaluate $a_1 + a_6 + a_8$ we add the edges $x_{k-2}x_1$, $y_{k-2}z_1$ and $z_{k-2}y_1$ to H. In other words we set $H_1 = (V(H), E(H) \cup \{x_{k-2}x_1, y_{k-2}z_1, z_{k-2}y_1\})$, we get hence $FS(2, k-2)$. Each perfect matching of type 1 of H_1 leads to precisely 2 perfect matchings of type 1 of $FS(2,k)$, thus $a_1+a_6+a_8 = 2\mu_1(2, k-2)$.

We consider now the graph $H_2 = (V(H), E(H) \cup \{x_{k-2}y_1, y_{k-2}x_1, z_{k-2}z_1\})$ isomorphic to $FS(2, k-2)$. Each perfect matching of type 1 of H_2 gives one perfect matching of type 1 of $FS(2,k)$, consequently $a_2+a_4+a_9 = \mu_1(2, k-2)$.

Let $H_3 = (V(H), E(H) \cup \{x_{k-2}z_1, y_{k-2}y_1, z_{k-2}x_1\})$, a graph isomorphic to $FS(2, k-2)$. Each perfect matching of type 1 of H_3 leads to precisely one perfect matching of type 1 of $FS(2,k)$, consequently $a_3 + a_5 + a_7 = \mu_1(2, k-2)$. □

Claim 3. When $j = 3$, we have

$$a_1 + a_5 + a_9 = 2\mu_1(3, k-2),$$
$$a_2 + a_6 + a_7 = \mu_1(1, k-2),$$
$$a_3 + a_4 + a_8 = \mu_1(1, k-2).$$

Proof. Suppose that $x_{k-1}x_0$, $y_{k-1}y_0$, $z_{k-1}z_0$ are edges of $FS(3, k)$.

In order to evaluate $a_1 + a_5 + a_9$ we add the edges $x_{k-2}x_1$, $y_{k-2}y_1$ and $z_{k-2}z_1$ to H. In other words we set $H_1 = (V(H), E(H) \cup \{x_{k-2}x_1, y_{k-2}y_1, z_{k-2}z_1\})$, we get hence $FS(3, k-2)$. Each perfect matching of type 1 of H_1 can be extended into 2 perfect matchings of type 1 of $FS(3, k)$, thus $a_1 + a_5 + a_9 = 2\mu_1(3, k-2)$.

Let H_2 be isomorphic to $FS(1, k-2)$ with the edges $x_{k-2}y_1$, $y_{k-2}z_1$, $z_{k-2}x_1$. A perfect matching of type 1 of H_2 can precisely be extended into one perfect matching of type 1 of $FS(3, k)$, then $a_2 + a_6 + a_7 = \mu_1(1, k-2)$.

Let $H_3 = (V(H), E(H) \cup \{x_{k-2}z_1, y_{k-2}x_1, z_{k-2}y_1\})$, a graph isomorphic to $FS(1, k-2)$. Each perfect matching of type 1 of H_3 gives one perfect matching of type 1 of $FS(3, k)$, consequently $a_3 + a_4 + a_8 = \mu_1(1, k-2)$. □

Recall that $\mu_1(j, k) = \sum_{i=1}^{9} a_i$. Then it follows from Claims 1, 2 and 3:

$$\begin{aligned}
\mu_1(1, k) &= 3\mu_1(1, k-2) + \mu_1(3, k-2),\\
\mu_1(2, k) &= 4\mu_1(2, k-2),\\
\mu_1(3, k) &= 2\mu_1(3, k-2) + 2\mu_1(1, k-2).
\end{aligned}$$

By induction,

$$\begin{aligned}
\mu_1(1, k-2) &= 2^{k-2} - (-1)^k,\\
\mu_1(2, k-2) &= 2^{k-2},\\
\mu_1(3, k-2) &= 2^{k-2} + 2(-1)^k.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mu_1(1, k) &= 3(2^{k-2} - (-1)^k) + 2^{k-2} + 2(-1)^k = 2^k - (-1)^k,\\
\mu_1(2, k) &= 4(2^{k-2}) = 2^k,\\
\mu_1(3, k) &= 2(2^{k-2} + 2(-1)^k) + 2(2^{k-2} - (-1)^k) = 2^k + 2(-1)^k.
\end{aligned}$$

This ends the proof. ■

Theorem 6. *The number* $\mu_2(j,k)$ *of perfect matchings of* $FS(j,k)$ $(j \in \{1,2,3\})$ *verifies:* $\mu_2(j,k) = 0$ *when* k *is odd and* $\mu_2(j,k) = 2 \times 3^{\frac{k}{2}}$ *otherwise.*

Proof. When k is odd, we have $\mu_2(j,k) = 0$ by Lemma 1.

When k is even, let M be a perfect matching of type 2 of $FS(j,k)$. For every two consecutive claws C_i and C_{i+1}, by Lemma 1, we have either two edges of M joining the external vertices of C_i to those of C_{i+1} or none. We have 3 ways to choose 2 edges between C_i and C_{i+1}, each choice of these two edges can be completed in a unique way in a perfect matching of the subgraph $C_i \cup C_{i+1}$. Hence we get easily that the number of perfect matchings of type 2 in $FS(j,k)$ $(j \in \{1,2,3\})$ is $\mu_2(j,k) = 2 \times 3^{\frac{k}{2}}$. ■

From Theorem 5 and Theorem 6 we deduce:

Corollary 7. *The numbers* $\mu(i,k)$ *of perfect matchings of* $FS(i,k)$ $(i \in \{1,2,3\})$ *are given by:*

When k *is odd*

- $\mu(2,k) = 2^k$,
- $\mu(1,k) = 2^k + 1$,
- $\mu(3,k) = 2^k - 2$,

When k *is even*

- $\mu(2,k) = 2 \times 3^{\frac{k}{2}} + 2^k$,
- $\mu(1,k) = 2 \times 3^{\frac{k}{2}} + 2^k - 1$,
- $\mu(3,k) = 2 \times 3^{\frac{k}{2}} + 2^k + 2$.

3. Some Structural Results About Perfect Matchings of $FS(j,k)$

3.1. Perfect matchings of type 1

Lemma 8. *Let* M *be a perfect matching of type* 1 *of* $G = FS(j,k)$. *Then the* 2*-factor* $G \setminus M$ *has exactly one or two cycles and each cycle of* $G \setminus M$ *has at least one vertex in each claw* C_i $(i \in \mathbf{Z}_k)$.

Proof. Let M be a perfect matching of type 1 in G. Let us consider the claw C_i for some i in $\mathbf{Z}_k$. Assume without loss of generality that the edge of M contained in C_i is t_ix_i. The cycle of $G \setminus M$ visiting x_i comes from C_{i-1}, crosses C_i by using the vertex x_i and goes to C_{i+1}. By Lemma 1, the path

$y_it_iz_i$ is contained in a cycle of $G \setminus M$. The two edges incident to y_i and z_i joining C_i to C_{i-1} (as well as those joining C_i to C_{i+1}) are not contained both in M (since M has type 1). Thus, the cycle of $G \setminus M$ containing $y_it_iz_i$ comes from C_{i-1}, crosses C_i and goes to C_{i+1}. Thus, we have at most two cycles in $G \setminus M$, as claimed, and we can note that each claw must be visited by these cycles. ■

Definition 9. Let us suppose that M is a perfect matching of type 1 in $G = FS(j,k)$ such that the 2-factor $G \setminus M$ has exactly two cycles Γ_1 and Γ_2. A claw C_i intersected by three vertices of Γ_1 (respectively Γ_2) is said to be Γ_1-*major* (respectively Γ_2-*major*).

Lemma 10. *Let M be a perfect matching of type 1 of $G = FS(j,k)$ such that the 2-factor $G \setminus M$ has exactly two cycles. Then, the lengths of these two cycles have the same parity as k, and those lengths are distinct when k is odd.*

Proof. Let Γ_1 and Γ_2 be the two cycles of $G \setminus M$. By Lemma 8, for each i in $\mathbf{Z}_k$ these two cycles must cross the claw C_i. Let k_1 be the number of Γ_1-major claws and let k_2 be the number of Γ_2-major claws. We have $k_1 + k_2 = k$, $l(\Gamma_1) = 3k_1 + k_2$ and $l(\Gamma_2) = 3k_2 + k_1$. When k is odd, we must have either k_1 odd and k_2 even, or k_1 even and k_2 odd. Then Γ_1 and Γ_2 have distinct odd lengths. When k is even, we must have either k_1 and k_2 even, or k_2 and k_1 odd. Then Γ_1 and Γ_2 have even lengths. ■

Lemma 11. *Let M be a perfect matching of type 1 of $G = FS(j,k)$ such that the 2-factor $G \setminus M$ has exactly two cycles Γ_1 and Γ_2. Suppose that there are two consecutive Γ_1-major claws C_j and C_{j+1} with $j \in \mathbf{Z}_k \setminus \{k-1\}$. Then there is a perfect matching M' of type 1 such that the 2-factor $G \setminus M'$ has exactly two cycles Γ'_1 and Γ'_2 having the following properties:*

(a) *for $i \in \mathbf{Z}_k \setminus \{j, j+1\}$ C_i is Γ'_2-major if and only if C_i is Γ_2-major,*

(b) *$C_j$and C_{j+1} are Γ'_2-major,*

(c) *$l(\Gamma'_1) = l(\Gamma_1) - 4$ and $l(\Gamma'_2) = l(\Gamma_2) + 4$.*

Proof. Consider the claws C_j and C_{j+1}. Since C_j is a Γ_1-major claw suppose without loss of generality that t_jz_j belongs to M and that Γ_1 contains the path $x'_{j-1}x_jt_jy_jy_{j+1}$ where x'_{j-1} denotes the neighbour of x_j in C_{j-1} (then x_jx_{j+1} belongs to M). Since C_{j+1} is Γ_1-major and Γ_2 goes through

C_j and C_{j+1}, the cycle Γ_1 must contain the path $y_{j+1}t_{j+1}x_{j+1}x'_{j+2}$ where x'_{j+2} denotes the neighbour of x_{j+1} in C_{j+2} (then M contains $t_{j+1}z_{j+1}$ and $y_{j+1}y'_{j+2}$). Denote by P_1 the path $x'_{j-1}x_jt_jy_jy_{j+1}t_{j+1}x_{j+1}x'_{j+2}$. Note that Γ_2 contains the path $P_2 = z'_{j-1}z_jz_{j+1}z'_{j+1}$ where z'_{j-1} and z'_{j+1} are defined similarly. See to the left part of Figure 2.

Let us perform the following local transformation: delete x_jx_{j+1}, t_jz_j and $t_{j+1}z_{j+1}$ from M and add z_jz_{j+1}, t_jx_j and $t_{j+1}x_{j+1}$. Let M' be the resulting perfect matching. Then the subpath P_1 of Γ_1 is replaced by $P'_1 = x'_{j-1}x_jx_{j+1}x'_{j+2}$ and the subpath P_2 of Γ_2 is replaced by $P'_2 = z'_{j-1}z_jt_jy_jy_{j+1}t_{j+1}z_{j+1}z'_{j+2}$ (see Figure 2). We obtain a new 2-factor containing two new cycles Γ'_1 and Γ'_2. Note that C_j and C_{j+1} are Γ'_2-major claws and for i in $\mathbf{Z}_k \setminus \{j, j+1\}$ C_i is Γ'_2-major (respectively Γ'_1-major) if and only if C_i is Γ_2-major (respectively Γ_1-major). The length of Γ_1 (now Γ'_1) decreases of 4 units while the length of Γ_2 (now Γ'_2) increases of 4 units. ■

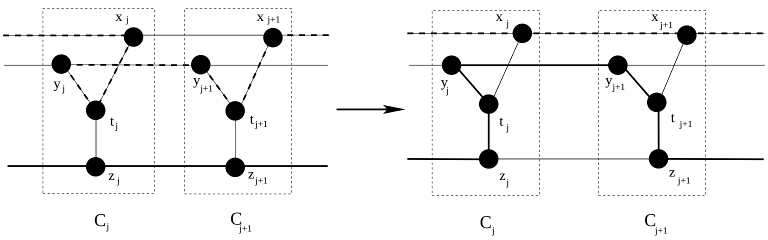

Figure 2. Local transformation of type 1.

The operation depicted in Lemma 11 above will be called a *local transformation of type* 1.

Lemma 12. *Let M be a perfect matching of type 1 of $G = FS(j,k)$ such that the 2-factor $G \setminus M$ has exactly two cycles Γ_1 and Γ_2. Suppose that there are three consecutive claws C_j, C_{j+1} and C_{j+2} with j in $\mathbf{Z}_k \setminus \{k-1, k-2\}$ such that C_j and C_{j+2} are Γ_1-major and C_{j+1} is Γ_2-major. Then there is a perfect matching M' of type 1 such that the 2-factor $G \setminus M'$ has exactly two cycles Γ'_1 and Γ'_2 having the following properties:*

(a) *for $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$ C_i is Γ'_2-major if and only if C_i is Γ_2-major,*

(b) *C_j and C_{j+2} are Γ'_2-major and C_{j+1} is Γ'_1-major,*

(c) *$l(\Gamma'_1) = l(\Gamma_1) - 2$ and $l(\Gamma'_2) = l(\Gamma_2) + 2$.*

Proof. Since C_j is Γ_1-major, as in the proof of Lemma 11 suppose that Γ_1 contains the path $x'_{j-1}x_jt_jy_jy_{j+1}$ (that is edges t_jz_j and x_jx_{j+1} belong to M). Since C_{j+1} is Γ_2-major the cycle Γ_1 contains the edge $y_{j+1}y_{j+2}$. Then we see that Γ_1 contains the path $Q_1 = x'_{j-1}x_jt_jy_jy_{j+1}y_{j+2}t_{j+2}z_{j+2}z'_{j+3}$ and that Γ_2 contains the path $Q_2 = z'_{j-1}z_jz_{j+1}t_{j+1}x_{j+1}x_{j+2}x'_{j+3}$. Note that $y_{j+1}t_{j+1}$, $z_{j+1}z_{j+2}$ and $t_{j+2}x_{j+2}$ belong to M.

Let us perform the following local transformation: delete t_jz_j, x_jx_{j+1}, $z_{j+1}z_{j+2}$ and $x_{j+2}t_{j+2}$ from M and add x_jt_j, z_jz_{j+1}, $x_{j+1}x_{j+2}$ and $z_{j+2}t_{j+2}$ to M. Let M' be the resulting perfect matching. Then the subpath Q_1 of Γ_1 is replaced by $Q'_1 = x'_{j-1}x_jx_{j+1}t_{j+1}z_{j+1}z_{j+2}z'_{j+3}$ and the subpath Q_2 of Γ_2 is replaced by $Q'_2 = z'_{j-1}z_jt_jy_jy_{j+1}y_{j+2}t_{j+2}x_{j+2}x'_{j+3}$ (see Figure 3). We obtain a new 2-factor containing two new cycles named Γ'_1 and Γ'_2. Note that C_j and C_{j+2} are now Γ'_2-major claws and C_{j+1} is Γ'_1-major. The length of Γ_1 decreases of 2 units while the length of Γ_2 increases of 2 units. It is clear that for $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$ C_i is Γ'_2-major (respectively Γ'_1-major) if and only if C_i is Γ_2-major (respectively Γ_1-major). ■

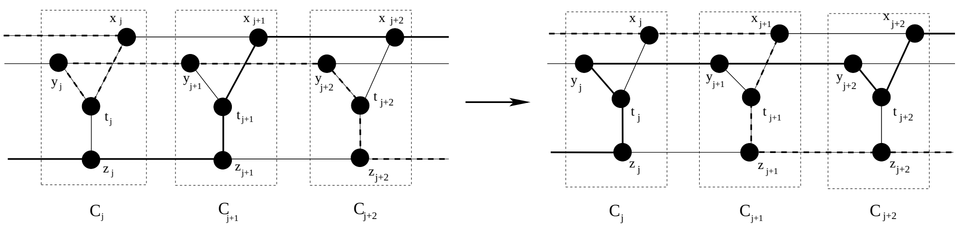

Figure 3. Local transformation of type 2.

The operation depicted in Lemma 12 above will be called a *local transformation of type* 2.

Lemma 13. *Let M be a perfect matching of type* 1 *of $G = FS(j, k)$ such that the* 2*-factor $G \setminus M$ has exactly two cycles Γ_1 and Γ_2. Suppose that there are three consecutive claws C_j, C_{j+1} and C_{j+2} with j in $\mathbf{Z}_k \setminus \{k-1, k-2\}$ such that C_{j+1} and C_{j+2} are Γ_2-major and C_j is Γ_1-major. Then there is a perfect matching M' of type* 1 *such that the* 2*-factor $G \setminus M'$ has exactly two cycles Γ'_1 and Γ'_2 having the following properties:*

(a) *for $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$ C_i is Γ'_2-major if and only if C_i is Γ_2-major,*

(b) *C_j and C_{j+1} are Γ'_2-major and C_{j+2} is Γ'_1-major,*

(c) *$l(\Gamma'_1) = l(\Gamma_1)$ and $l(\Gamma'_2) = l(\Gamma_2)$.*

Proof. Since C_j is Γ_1-major, as in the proof of Lemma 11 suppose that Γ_1 contains the path $x'_{j-1}x_jt_jy_jy_{j+1}$ (that is edges t_jz_j and x_jx_{j+1} belong to M). Since C_{j+1} and C_{j+2} are Γ_2-major, the unique vertex of C_{j+1} (respectively C_{j+2}) contained in Γ_1 is y_{j+1} (respectively y_{j+2}). Note that the perfect matching M contains the edges t_jz_j, x_jx_{j+1}, $t_{j+1}y_{j+1}$, $z_{j+1}z_{j+2}$ a nd $t_{j+2}y_{j+2}$. Then the path $R_1 = x'_{j-1}x_jt_jy_jy_{j+1}y_{j+2}y'_{j+3}$ is a subpath of Γ_1 and the path $R_2 = z'_{j-1}z_jz_{j+1}t_{j+1}x_{j+1}x_{j+2}t_{j+2}z_{j+2}z'_{j+3}$ is a subpath of Γ_2. See to the left part of Figure 4.

Let us perform the following local transformation: delete t_jz_j, x_jx_{j+1}, $t_{j+1}y_{j+1}$, $z_{j+1}z_{j+2}$ and $t_{j+2}y_{j+2}$ from M and add x_jt_j, z_jz_{j+1}, $t_{j+1}x_{j+1}$, $y_{j+1}y_{j+2}$ and $t_{j+2}z_{j+2}$. Let M' be the resulting perfect matching. Then the subpath R_1 of Γ_1 is replaced by $R'_1 = x'_{j-1}x_jx_{j+1}x_{j+2}t_{j+2}y_{j+2}y'_{j+3}$ and the subpath R_2 of Γ_2 is replaced by $R'_2 = z'_{j-1}z_jt_jy_jy_{j+1}t_{j+1}z_{j+1}z_{j+2}z'_{j+3}$. We obtain a new 2-factor containing two new cycles named Γ'_1 and Γ'_2 such that $l(\Gamma'_1) = l(\Gamma_1)$ and $l(\Gamma'_2) = l(\Gamma_2)$ (see Figure 4). It is clear that for $i \in \mathbf{Z}_k \setminus \{j, j+1, j+2\}$ C_i is Γ'_2-major (respectively Γ'_1-major) if and only if C_i is Γ_2-major (respectively Γ_1-major). Note that C_j and C_{j+1} are Γ'_2-major and C_{j+2} is Γ'_1-major. ■

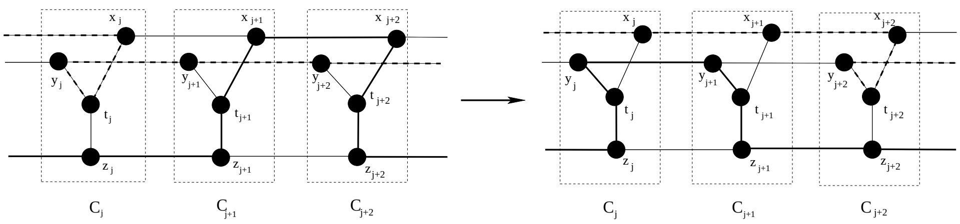

Figure 4. Local transformation of type 3.

The operation depicted in Lemma 13 above will be called a *local transformation of type* 3.

Lemma 14. *Let M be a perfect matching of type 1 of $G = FS(j, k)$ such that the 2-factor $G \setminus M$ has exactly two cycles Γ_1 and Γ_2 such that $l(\Gamma_1) \leq l(\Gamma_2)$ and $l(\Gamma_2)$ is as great as possible. Then there exists at most one Γ_1-major claw.*

Proof. Suppose, for the sake of contradiction, that there exist at least two Γ_1-major claws. Since $l(\Gamma_2)$ is maximum, by Lemma 11 these claws are not consecutive. Then consider two Γ_1-major claws C_i and C_{i+h+1} (with $h \geq 1$) such that the h consecutive claws $(C_{i+1}, \ldots, C_{i+h})$ are Γ_2-major. Since $l(\Gamma_2)$ is maximum, by Lemma 12 the number h is at least 2. Then by applying $r = \lfloor \frac{h}{2} \rfloor$ consecutive local transformations of type 3 (Lemma 13) we obtain a perfect matching $M^{(r)}$ such that the 2-factor $G \setminus M^{(r)}$ has exactly two cycles $\Gamma_1^{(r)}$ and $\Gamma_2^{(r)}$ with $l(\Gamma_1^{(r)}) = l(\Gamma_1)$ and $l(\Gamma_2^{(r)}) = l(\Gamma_2)$ and such that $C_{i+2\lfloor \frac{h}{2} \rfloor}$ and C_{i+h+1} are $\Gamma_1^{(r)}$-major. Since $l(\Gamma_2^{(r)})$ is maximum, we can conclude by Lemma 11 and by Lemma 12 that h is neither even nor odd, a contradiction. ■

3.2. Perfect matchings of type 2

We give here a structural result about perfect matchings of type 2 in $G = FS(j,k)$.

Lemma 15. *Let M be a perfect matching of type 2 of $G = FS(j,k)$ (with $k \geq 4$). Then the 2-factor $G \setminus M$ has exactly one cycle of even length $l \geq k$ and a set of p cycles of length 6 where $l + 6p = 4k$ (with $0 \leq p \leq \frac{k}{2}$).*

Proof. Let M be a perfect matching of type 2 in G. By Lemma 1 the number k of claws is even. Let i in $\mathbf{Z}_k$ such that there are two edges of M between C_{i-1} and C_i. There are no edges of M between C_i and C_{i+1} and two edges of M between C_{i+1} and C_{i+2}. We may consider that $0 \leq i < k-1$. For $j \in \{i, i+2, i+4, \ldots\}$ we denote by e_j the unique edge of $G \setminus M$ having one end vertex in C_{j-1} and the other in C_j. Let us denote by A the set $\{e_i, e_{i+2}, e_{i+4}, \ldots\}$. We note that $\mid A \mid = \frac{k}{2}$.
Assume without loss of generality that the two edges of M between C_{i-1} and C_i have end vertices in C_i which are x_i and y_i (then z_i is the end vertex of e_i in C_i). Two cases may now occur.

Case 1. The end vertices in C_{i+1} of the two edges of M between C_{i+1} and C_{i+2} are x_{i+1} and y_{i+1} (then z_{i+1} is the end vertex of e_{i+2} in C_{i+1}).

In that case the 2-factor $G \setminus M$ contains the cycle of length 6 $x_i x_{i+1} t_{i+1} y_{i+1} y_i t_i$ while the edge $z_i z_{i+1}$ of $G \setminus M$ connects e_i and e_{i+2}.

Case 2. The end vertices in C_{i+1} of the two edges of M between C_{i+1} and C_{i+2} are y_{i+1} and z_{i+1} (respectively x_{i+1} and z_{i+1}). Then x_{i+1} (respectively y_{i+1}) is the end vertex of e_{i+2} in C_{i+1}.

In that case the edges e_i and e_{i+2} are connected in $G \setminus M$ by the path $z_iz_{i+1}t_{i+1}y_{i+1}y_it_ix_ix_{i+1}$ (respectively $z_iz_{i+1}t_{i+1}x_{i+1}x_it_iy_iy_{i+1}$).

The same reasoning can be done for $\{e_{i+2}, e_{i+4}\}$, $\{e_{i+4}, e_{i+6}\}$, and so on. Then, we see that the set A is contained in a unique cycle Γ of $G \setminus M$ which crosses each claw. Thus, the length l of Γ is at least k. More precisely, each e_j in A contributes for 1 in l, in Case 1 the edge z_iz_{i+1} contributes for 1 in l and in Case 2 the path $z_iz_{i+1}t_{i+1}y_{i+1}y_it_ix_ix_{i+1}$ contributes for 7 in l. Let us suppose that Case 1 appears p times ($0 \leq p \leq \frac{k}{2}$), that is to say $G \setminus M$ contains p cycles of length 6. Since Case 2 appears $\frac{k}{2} - p$ times, the length of Γ is $l = \frac{k}{2} + p + 7(\frac{k}{2} - p) = 4k - 6p$. ■

Remark 16. If k is even then by Lemmas 8, 10 and 15 $FS(j,k)$ has an even 2-factor. That is to say $FS(j,k)$ is a cubic 3-edge colourable graph.

4. PERFECT MATCHINGS AND HAMILTONIAN CYCLES OF $F(j,k)$

4.1. Perfect matchings of type 1 and hamiltonicity

Theorem 17. *Let M be a perfect matching of type 1 of $G = FS(j,k)$. Then the 2-factor $G \setminus M$ is a hamiltonian cycle except for k odd and $j = 2$, and for k even and $j = 1$ or 3.*

Proof. Suppose that there exists a perfect matching M of type 1 of G such that $G \setminus M$ is not a hamiltonian cycle. By Lemma 8 and Lemma 10 the 2-factor $G \setminus M$ is made of exactly two cycles Γ_1 and Γ_2 whose lengths have the same parity as k. Without loss of generality we suppose that $l(\Gamma_1) \leq l(\Gamma_2)$. Assume moreover that among the perfect matchings of type 1 of G such that the 2-factor $G \setminus M$ is composed of two cycles, M has been chosen in such a way that the length of the longest cycle Γ_2 is as great as possible. By Lemma 14 there exists at most one Γ_1-major claw.

Case 1. There exists one Γ_1-major claw.
Without loss of generality, suppose that C_0 is intersected by Γ_1 into $\{y_0, t_0, x_0\}$ and that $y'_{k-1}y_0$ belongs to Γ_1. Since for every $i \neq 0$ the claw C_i is Γ_2-major, Γ_1 contains the vertices $y_0, t_0, x_0, x_1, x_2, \ldots, x_{k-1}$ which means that $x'_{k-1} = y_0$.

- If $k = 2r + 1$ with $r \geq 1$ then Γ_2 contains the path

$$z_0z_1t_1y_1y_2t_2z_2 \ldots z_{2r-1}t_{2r-1}y_{2r-1}y_{2r}t_{2r}z_{2r}.$$

Thus, y_0x_{k-1}, x_0y_{k-1}, z_0z_{k-1} are edges of G. This means that $\cup_{i=0}^{i=k-1}\{C_i \setminus \{t_i\}\}$ induces two cycles, that is to say $j = 2$ and $G = FS(2, k)$.

• If $k = 2r + 2$ with $r \geq 1$ then Γ_2 contains the path

$$z_0z_1t_1y_1y_2t_2z_2 \dots z_{2r-1}t_{2r-1}y_{2r-1}y_{2r}t_{2r}z_{2r}z_{2r+1}t_{2r+1}y_{2r+1}.$$

Thus, x_0z_{k-1}, y_0x_{k-1} and z_0y_{k-1} are edges. This means that $\cup_{i=0}^{i=k-1}\{C_i \setminus \{t_i\}\}$ induces one cycle, that is to say $j = 1$ and $G = FS(1, k)$.

Case 2. There is no Γ_1-major claw.
Suppose that x_0 belongs to Γ_1. Then, Γ_1 contains $x_0, x_1, \dots, x_{k-1}$.

• If $k = 2r + 1$ with $r \geq 1$ then Γ_2 contains the path

$$y_0t_0z_0z_1t_1y_1y_2 \dots z_{2r-1}t_{2r-1}y_{2r-1}y_{2r}t_{2r}z_{2r}.$$

Thus, x_0x_{k-1}, y_0z_{k-1} and z_0y_{k-1} are edges of G and the set $\cup_{i=0}^{i=k-1}\{C_i\setminus\{t_i\}\}$ induces two cycles,that is to say $j = 2$ and $G = FS(2, k)$.

• If $k = 2r + 2$ with $r \geq 1$ then Γ_2 contains the path

$$y_0t_0z_0z_1t_1y_1y_2 \dots y_{2r}t_{2r}z_{2r}z_{2r+1}t_{2r+1}y_{2r+1}.$$

Thus, x_0x_{k-1}, y_0y_{k-1} and z_0z_{k-1} are edges. This means that $\cup_{i=0}^{i=k-1}\{C_i \setminus \{t_i\}\}$ induces three cycles, that is to say $j = 3$ and $G = FS(3, k)$. ∎

Definition 18. A cubic graph G is said to be 2-*factor hamiltonian* [6] if every 2-factor of G is a hamiltonian cycle (or equivalently, if for every perfect matching M of G the 2-factor $G \setminus M$ is a hamiltonian cycle).

By Theorem 17 for any odd $k \geq 3$ and $j \in \{1, 3\}$ or for any even k and $j = 2$, and for every perfect matching M of type 1 in $FS(j, k)$ the 2-factor $FS(j, k) \setminus M$ is a hamiltonian cycle. By Lemma 15 $FS(2, k)$ ($k \geq 4$) may have a perfect matching M of type 2 such that the 2-factor $FS(2, k) \setminus M$ is not a hamiltonian cycle (it may contains cycles of length 6).

Then we have the following.

Corollary 19. *A graph* $G = FS(j, k)$ *is* 2*-factor hamiltonian if and only if* k *is odd and* $j = 1$ *or* 3.

We note that $FS(1,3)$ is the "Triplex Graph" of Robertson, Seymour and Thomas [15]. We shall examine other known results about 2-factor hamiltonian cubic graphs in Section 5..

Corollary 20. *The chromatic index of a graph $G = FS(j,k)$ is 4 if and only if $j = 2$ and k is odd.*

Proof. When $j = 2$ and k is odd, any 2-factor must have at least two cycles, by Theorem 17. Then Lemma 10 implies that any 2-factor is composed of two odd cycles. Hence G has chromatic index 4.

When $j = 1$ or 3 and k is odd by Theorem 17 $FS(j,k)$ is hamiltonian. If k is even then by Lemmas 8, 10 and 15 $FS(j,k)$ has an even 2-factor. ∎

4.2. Perfect matchings of type 2 and hamiltonicity

At this point of the discourse one may ask what happens for perfect matchings of type 2 in $FS(j,k)$ (k even). Can we characterize and count perfect matchings of type 2, complementary 2-factor of which is a hamiltonian cycle ? An affirmative answer shall be given.

Let us consider a perfect matching M of type 2 in $FS(j,2p)$ with $p \geq 2$. Suppose that there are no edges of M between C_{2i-1} and C_{2i} (for any $i \geq 1$), that is to say M is a matching of type 2.0 (see Definition 2). Consider two consecutive claws C_{2i} and C_{2i+1} ($0 \leq i \leq p-1$). There are three cases:

Case (x).
$\{y_{2i}y_{2i+1}, z_{2i}z_{2i+1}\} \subset M$ (then, $M \cap (C_{2i} \cup C_{2i+1}) = \{x_{2i}t_{2i}, x_{2i+1}t_{2i+1}\}$).

Case (y).
$\{x_{2i}x_{2i+1}, z_{2i}z_{2i+1}\} \subset M$ (then, $M \cap (C_{2i} \cup C_{2i+1}) = \{y_{2i}t_{2i}, y_{2i+1}t_{2i+1}\}$).

Case (z).
$\{x_{2i}x_{2i+1}, y_{2i}y_{2i+1}\} \subset M$ (then, $M \cap (C_{2i} \cup C_{2i+1}) = \{z_{2i}t_{2i}, z_{2i+1}t_{2i+1}\}$).

The subgraph induced on $C_{2i} \cup C_{2i+1}$ is called a ***block***. In Case (x) (respectively Case (y), Case (z)) a block is called a ***block of type*** X (respectively ***block of type*** Y, ***block of type*** Z). Then $FS(j,2p)$ with a perfect matching M of type 2.0 can be seen as a sequence of p blocks properly connected. In other words, a perfect matching M of type 2 in $FS(j,2p)$ is entirely described by a word of length p on the alphabet of three letters $\{X,Y,Z\}$. The block

$C_0 \cup C_1$ is called *initial block* and the block $C_{2p-1} \cup C_{2p}$ is called *terminal block*. These extremal blocks are not considered here as consecutive blocks.

By Lemma 15, $FS(j, 2p) \backslash M$ has no 6-cycles if and only if $FS(j, 2p) \backslash M$ is a unique even cycle. It is an easy matter to prove that two consecutive blocks do not induce a 6-cycle if and only if they are not of the same type. Then the possible configurations for two consecutive blocks are XY, XZ, YX, YZ, ZX and ZY. To eliminate a possible 6-cycle in $C_0 \cup C_{2p-1}$ we have to determine for every $j \in \{1, 2, 3\}$ the forbidden extremal configurations. An extremal configuration shall be denoted by a word on two letters in $\{X, Y, Z\}$ such that the left letter denotes the type of the initial block $C_0 \cup C_1$ and the right letter denotes the type of the terminal block $C_{2p-1} \cup C_{2p}$. We suppose that the extremal blocks are connected for $j = 1$ by the edges $x_{2p-1}z_0$, $y_{2p-1}x_0$ and $z_{2p-1}y_0$, for $j = 2$ by the edges $x_{2p-1}x_0$, $y_{2p-1}z_0$ and $z_{2p-1}y_0$ and for $j = 3$ by the edges $x_{2p-1}x_0$, $y_{2p-1}y_0$ and $z_{2p-1}z_0$. Then, it is easy to verify that we have the following result.

Lemma 21. *Let M be a perfect matching of type* 2.0 *of $G = FS(j, 2p)$ (with $p \geq 2$) such that the* 2*-factor $G \setminus M$ is a hamiltonian cycle. Then the forbidden extremal configurations are*

XY, YZ *and* ZX *for* $FS(1, 2p)$,

XX, YZ *and* ZY *for* $FS(2, 2p)$,

and XX, YY *and* ZZ *for* $FS(3, 2p)$.

Thus, any perfect matching M of type 2.0 of $FS(j, 2p)$ such that the 2-factor $G \setminus M$ is a hamiltonian cycle is totally characterized by a word of length p on the alphabet $\{X, Y, Z\}$ having no two identical consecutive letters and such that the sub-word [initial letter][terminal letter] is not a forbidden configuration. Then, we are in position to obtain the number of such perfect matchings in $FS(j, 2p)$. Let us denote by $\mu'_{2.0}(j, 2p)$ (respectively $\mu'_{2.1}(j, 2p)$, $\mu'_2(j, 2p)$) the number of perfect matchings of type 2.0 (respectively type 2.1, type 2) complementary to a hamiltonian cycle in $FS(j, 2p)$. Clearly $\mu'_2(j, 2p) = \mu'_{2.0}(j, 2p) + \mu'_{2.1}(j, 2p)$ and $\mu'_{2.0}(j, 2p) = \mu'_{2.1}(j, 2p)$.

Theorem 22. *The numbers $\mu'_2(j, 2p)$ of perfect matchings of type* 2 *complementary to hamiltonian cycles in $FS(j, 2p)$ ($j \in \{1, 2, 3\}$) are given by:*

$$\mu'_2(1, 2p) = 2^{p+1} + (-1)^{p+1}2,$$

$$\mu_2'(2,2p) = 2^{p+1},$$
$$\textit{and } \mu_2'(3,2p) = 2^{p+1} + (-1)^p 4.$$

Proof. Consider, as previously, perfect matchings of type 2.0. Let α and β be two letters in $\{X,Y,Z\}$ (not necessarily distinct). Let $A^p_{\alpha\beta}$ be the set of words of length p on $\{X,Y,Z\}$ having no two consecutive identical letters, beginning by α and ending by a letter distinct from β. Denote the number of words in $A^p_{\alpha\beta}$ by $a^p_{\alpha\beta}$. Let $B^p_{\alpha\beta}$ be the set of words of length p on $\{X,Y,Z\}$ having no two consecutive identical letters, beginning by α and ending by β. Denote by $b^p_{\alpha\beta}$ the number of words in $B^p_{\alpha\beta}$.

Clearly, the number of words of length p having no two consecutive identical letters and beginning by α is 2^{p-1}. Then $a^p_{\alpha\beta} + b^p_{\alpha\beta} = 2^{p-1}$. The deletion of the last β of a word in $B^p_{\alpha\beta}$ gives a word in $A^{p-1}_{\alpha\beta}$ and the addition of β to the right of a word in $A^{p-1}_{\alpha\beta}$ gives a word in $B^p_{\alpha\beta}$.

Thus $b^p_{\alpha\beta} = a^{p-1}_{\alpha\beta}$ and for every $p \geq 3$ $a^p_{\alpha\beta} = 2^{p-1} - a^{p-1}_{\alpha\beta}$. We note that $a^2_{\alpha\beta} = 2$ if $\alpha = \beta$, and $a^2_{\alpha\beta} = 1$ if $\alpha \neq \beta$. If $\alpha = \beta$ we have to solve the recurrent sequence: $u_2 = 2$ and $u_p = 2^{p-1} - u_{p-1}$ for $p \geq 3$. If $\alpha \neq \beta$ we have to solve the recurrent sequence: $v_2 = 1$ and $v_p = 2^{p-1} - v_{p-1}$ for $p \geq 3$. Then we obtain $u_p = \frac{2}{3}(2^{p-1} + (-1)^p)$ and $v_p = \frac{1}{3}(2^p + (-1)^{p+1})$ for $p \geq 2$.

By Lemma 21

$$\mu_{2.0}'(1,2p) = a^p_{XY} + a^p_{YZ} + a^p_{ZX} = 3v_p = 2^p + (-1)^{p+1},$$
$$\mu_{2.0}'(2,2p) = a^p_{XX} + a^p_{YZ} + a^p_{ZY} = u_p + 2v_p = 2^p,$$
$$\text{and } \mu_{2.0}'(3,2p) = a^p_{XX} + a^p_{YY} + a^p_{ZZ} = 3u_p = 2^p + (-1)^p 2.$$

Since $\mu_2'(j,2p) = \mu_{2.0}'(j,2p) + \mu_{2.1}'(j,2p)$ and $\mu_{2.0}'(j,2p) = \mu_{2.1}'(j,2p)$ we obtain the announced results. ■

Remark 23. We see that $\mu_2'(j,2p) \simeq 2^{p+1}$ and this is to compare with the number $\mu_2(j,2p) = 2 \times 3^p$ of perfect matchings of type 2 in $FS(j,2p)$ (see backward in Section 2).

4.3. Strong matchings and Jaeger's graphs

For a given graph $G = (V,E)$ a *strong matching* (or *induced matching*) is a matching S such that no two edges of S are joined by an edge of G. That is, S is the set of edges of the subgraph of G induced by the set $V(S)$. We consider cubic graphs having a perfect matching which is the union of two strong matchings that we call *Jaeger's graph* (in his thesis [9] Jaeger called these cubic graphs *equitable*). We call *Jaeger's matching* a perfect matching M of a cubic graph G which is the union of two strong matchings M_B and M_R. Set $B = V(M_B)$ (the blue vertices) and $R = V(M_R)$ (the red vertices). An edge of G is said *mixed* if its end vertices have distinct colours. Since the set of mixed edges is $E(G) \setminus M$, the 2-factor $G \setminus M$ is even and $\mid B \mid = \mid M \mid$. Thus, every Jaeger's graph G is a cubic 3-edge colourable graphand for any Jaeger's matching $M = M_B \cup M_R$, $\mid M_B \mid = \mid M_R \mid$. See, for instance, [3] and [4] for some properties of these graphs.

In this subsection we determine the values of j and k for which a graph $FS(j,k)$ is a Jaeger's graph.

Lemma 24. *If* $G = FS(j,k)$ *is a Jaeger's graph* (with $k \geq 3$) *and* $M = M_B \cup M_R$ *is a Jaeger's matching of* G, *then* M *is a perfect matching of type* 1.

Proof. Suppose that M is of type 2 and suppose without loss of generality that there are two edges of M between C_0 and C_1, for instance x_0x_1 and y_0y_1. Then $C_0 \cap M = \{t_0z_0\}$ and $C_1 \cap M = \{t_1z_1\}$. Suppose that x_0x_1 and y_0y_1 belong to M_B. Since M_B is a strong matching, t_0z_0 and t_1z_1 belong to $M \setminus M_B = M_R$. This is impossible because M_R is also a strong matching. By symmetry there are no two edges of M_R between C_0 and C_1. Then there is one edge of M_B between C_0 and C_1, x_0x_1 for instance, and one edge of M_R between C_0 and C_1, y_0y_1 for instance. Since M_B and M_R are strong matchings, there is no edge of M in $C_0 \cup C_1$, a contradiction. Thus, M is a perfect matching of type 1. ∎

Lemma 25. *If* $G = FS(j,k)$ *is a Jaeger's graph* (with $k \geq 3$), *then either* ($j = 1$ *and* $k \equiv 1$ *or* 2 (mod 3)) *or* ($j = 3$ *and* $k \equiv 0$ (mod 3)).

Proof. Let $M = M_B \cup M_R$ be a Jaeger's matching of G. By Lemma 24 M is a perfect matching of type 1. Suppose without loss of generality that $M_B \cap E(C_0) = \{x_0t_0\}$. Since M_B is a strong matching there is no edge of M_B between C_0 and C_1. Suppose, without loss of generality, that the edge in M_R joining C_0 to C_1 is y_0y_1. Consider the claws C_0, C_1 and C_2. Since M_B and M_R are strong matchings, we can see that the choices of $x_0t_0 \in M_B$ and $y_0y_1 \in M_R$ fixes the positions of the other edges of M_B and M_R. More precisely, $\{t_1z_1, y_2t_2\} \subset M_B$ and $\{x_1x_2, z_2z'_3\} \subset M_R$. This unique configuration is depicted in Figure 5.

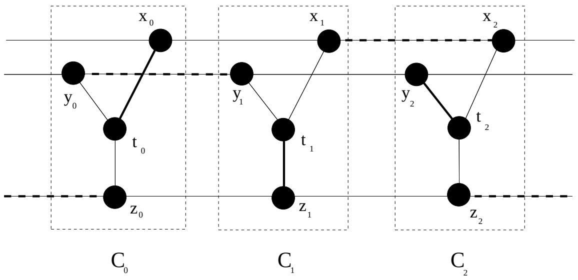

Figure 5. Strong matchings M_B (bold edges) and M_R (dashed edges).

If $k \geq 4$ then we see that $z_2z_3 \in M_R$, $x_3t_3 \in M_B$, and $y_3y'_4 \in M_R$. So, the local situation in C_3 is similar to that in C_0, and we can see that there is a unique Jaeger's matching $M = M_B \cup M_R$ such that $x_0t_0 \in M_B$ and $y_0y_1 \in M_R$ in the graph $FS(j,k)$. We have to verify the coherence of the connections between the claws C_{k-1} and C_0. We note that $M_B = M \cap (\cup_{i=0}^{i=k-1} E(C_i))$ and M_R is a strong matching included in the 2-factor induced by $\cup_{i=0}^{i=k-1}\{V(C_i) \setminus \{t_i\}\}$.

Case 1. $k = 3p$ with $p \geq 1$.
We have $x_0t_0 \in M_B$, $y_{k-1}t_{k-1} \in M_B$, $x_{k-2}x_{k-1} \in M_R$ and $z'_{k-1}z_0 = z_{k-1}z'_0 \in M_R$ (that is, $z_{k-1}z_0 \in M_R$). Thus, $z_{k-1}z_0$, $y_{k-1}y_0$ and $x_{k-1}x_0$ are edges of $FS(j,3p)$ and we must have $j = 3$.

Case 2. $k = 3p+1$ with $p \geq 1$.
We have $x_0t_0 \in M_B$, $x_{k-1}t_{k-1} \in M_B$ (that is, $x_{k-1}x_0 \notin E(G)$), $z_{k-2}z_{k-1} \in M_R$ and $z'_{k-1}z_0 = y_{k-1}y'_0 \in M_R$ (that is, $y_{k-1}z_0 \in M_R$). Thus, $y_{k-1}z_0$, $x_{k-1}y_0$ and $z_{k-1}x_0$ are edges of $FS(j,3p+1)$ and we must have $j = 1$.

Case 3. $k = 3p+2$ with $p \geq 1$.
We have $x_0t_0 \in M_B$, $z_{k-1}t_{k-1} \in M_B$, $y_{k-2}y_{k-1} \in M_R$ and $z'_{k-1}z_0 = x_{k-1}x'_0 \in M_R$ (that is $x_{k-1}z_0 \in M_R$). Thus, $x_{k-1}z_0$, $y_{k-1}x_0$ and $z_{k-1}y_0$ are edges of $FS(j,3p+2)$ and we must have $j = 1$. ■

Remark 26. It follows from Lemma 25 that for every $k \geq 3$ the graph $FS(2,k)$ is not a Jaeger's graph. This is obvious when k is odd, since the flower snarks have chromatic index 4.

Then, we obtain the following.

Theorem 27. *For $j \in \{1,2,3\}$ and $k \geq 2$, the graph $G = FS(j,k)$ is a Jaeger's graph if and only if*

either $k \equiv 1$ or $2 \pmod 3$ and $j = 1$,

or $k \equiv 0 \pmod 3$ and $j = 3$.

Moreover, $FS(1,2)$ has 3 Jaeger's matchings and for $k \geq 3$ a Jaeger's graph $G = FS(j,k)$ has exactly 6 Jaeger's matchings.

Proof. For $k = 2$ we remark that $FS(1,2)$ (that is the cube) has exactly three distinct Jaeger's matchings M_1, M_2 and M_3. Following our notations: $M_1 = \{x_0t_0, t_1z_1\} \cup \{y_0y_1, z_0x_1\}$, $M_2 = \{z_0t_0, t_1y_1\} \cup \{y_0z_1, x_0x_1\}$ and $M_3 = \{y_0t_0, t_1x_1\} \cup \{z_0z_1, x_0y_1\}$.

For $k \geq 3$, by Lemma 25, condition

$$(*) \qquad (j = 1 \text{ and } k \equiv 1 \text{ or } 2 \pmod 3) \text{ or } (j = 3 \text{ and } k \equiv 0 \pmod 3)$$

is a necessary condition for $FS(j,k)$ to be a Jaeger's graph.

Consider the function $\Phi_{X,Y} : V(G) \to V(G)$ such that for every i in $\mathbf{Z}_k$, $\Phi_{X,Y}(t_i) = t_i$, $\Phi_{X,Y}(z_i) = z_i$, $\Phi_{X,Y}(x_i) = y_i$ and $\Phi_{X,Y}(y_i) = x_i$. Define similarly $\Phi_{X,Z}$ and $\Phi_{Y,Z}$. For $j = 1$ or 3 these functions are automorphisms of $FS(j,k)$. Thus, the process described in the proof of Lemma 25 is a constructive process of all Jaeger's matchings in a graph $FS(j,k)$ (with $k \geq 3$) verifying condition (*).

We remark that for any choice of an edge e of C_0 to be in M_B there are two distinct possible choices for an edge f between C_0 and C_1 to be in M_R, and such a pair $\{e,f\}$ corresponds exactly to one Jaeger's matching. Then, a Jaeger's graph $FS(j,k)$ (with $k \geq 3$) has exactly 6 Jaeger's matchings. ■

Remark 28. The *Berge-Fulkerson Conjecture* states that if G is a bridgeless cubic graph, then there exist six perfect matchings $M_1, \ldots, M_6$ of G (not necessarily distinct) with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$ (this conjecture is attributed to Berge in [16] but appears in [5]). Using each colour of a cubic 3-edge colourable graph twice, we see that such a graph fulfils the Berge-Fulkerson Conjecture. Very few is known about this conjecture excepted that it holds for the Petersen graph and for cubic 3-edge colourable graphs. So, Berge-Fulkerson Conjecture holds for Jaeger's graphs, but generally we do not know if we can find six distinct perfect matchings. We remark that if $FS(j,k)$, with $k \geq 3$, is a Jaeger's graph then its six Jaeger's matchings are such that every edge is contained in exactly two of them.

5. 2-FACTOR HAMILTONIAN CUBIC GRAPHS

Recall that a simple graph of maximum degree $d > 1$ with edge chromatic number equal to d is said to be a *Class* 1 *graph*. For any d-regular simple graph (with $d > 1$) of even order and of Class 1, for any minimum edge-colouring of such a graph, the set of edges having a given colour is a perfect matching (or 1-factor). Such a regular graph is also called a 1-*factorable graph*. A Class 1 d-regular graph of even order is *strongly hamiltonian* or *perfectly* 1-*factorable* (or is a *Hamilton graph* in the Kotzig's terminology [10]) if it has an edge colouring such that the union of any two colours is a hamiltonian cycle. Such an edge colouring is said to be *a Hamilton decomposition* in the Kotzig's terminology. In [11] by using two operations ρ and π (described also in [10]) and starting from the θ-graph (two vertices joined by three parallel edges) he obtains all strongly hamiltonian cubic graphs, but these operations do not always preserve planarity. In his paper [10] he describes a method for constructing planar strongly hamiltonian cubic graphs and he deals with the relation between strongly hamiltonian cubic graphs and 4-regular graphs which can be decomposed into two hamiltonian cycles. See also [12] and a recent work on strongly hamiltonian cubic graphs [2] in which the authors give a new construction of strongly hamiltonian graphs.

A Class 1 regular graph such that every edge colouring is a Hamilton decomposition is called a *pure Hamilton graph* by Kotzig [10]. Note that K_4 is a pure Hamilton graph and every cubic graph obtained from K_4 by a sequence of triangular extensions is also a pure Hamilton cubic graph. In the paper [10] of Kotzig, a consequence of his Theorem 9 (p.77) concerning pure Hamilton graphs is that the family of pure Hamilton graphs that he exhibits is precisely the family obtained from K_4 by triangular extensions. Are there others pure Hamilton cubic graphs ? The answer is "yes".

We remark that 2-factor hamiltonian cubic graphs defined above (see Definition 18) are pure Hamilton graphs (in the Kotzig's sense) but the converse is false because K_4 is 2-factor hamiltonian and the pure Hamilton cubic graph on 6 vertices obtained from K_4 by a triangular extension (denoted by PR_3) is not 2-factor hamiltonian. Observe that the operation of triangular extension preserves the property "pure Hamilton", but does not preserve the property "2-factor hamiltonian". The Heawood graph H_0 (on 14 vertices) is pure Hamiltonian, more precisely it is 2-factor hamiltonian (see [7] Proposition 1.1 and Remark 2.7). Then, the graphs obtained from the Heawood graph H_0 by triangular extensions are also pure Hamilton graphs.

A *minimally 1-factorable* graph G is defined by Labbate and Funk [7] as a Class 1 regular graph of even order such that every perfect matching of G is contained in exactly one 1-factorization of G. In their article they study bipartite minimally 1-factorable graphs and prove that such a graph G has necessarily a degree $d \leq 3$. If G is a minimally 1-factorable cubic graph then the complementary 2-factor of any perfect matching has a unique decomposition into two perfect matchings, therefore this 2-factor is a hamiltonian cycle of G, that is G is 2-factor hamiltonian. Conversely it is easy to see that any 2-factor hamiltonian cubic graph is minimally 1-factorable. The complete bipartite graph $K_{3,3}$ and the Heawood graph H_0 are examples of 2-factor hamiltonian bipartite graph given by Labbate and Funk. Starting from H_0, from $K_{1,3}$ and from three copies of any tree of maximum degree 3 and using three operations called *amalgamations* the authors exhibit an infinite family of bipartite 2-factor hamiltonian cubic graphs, namely the $poly - HB - R - R^2$ graphs (see [7] for more details). Except H_0, these graphs are exactly cyclically 3-edge connected. Others structural results about 2-factor hamiltonian bipartite cubic graph are obtained in [13], [14]. These results have been completed and a simple method to generate 2-factor hamiltonian bipartite cubic graphs was given in [6].

Proposition 29 (Lemma 3.3, [6]). *Let G be a 2-factor hamiltonian bipartite cubic graph. Then G is 3-connected and* $\mid V(G) \mid \equiv 2 \pmod 4$.

Let G_1 and G_2 be disjoint cubic graphs, $x \in v(G_1)$, $y \in v(G_2)$. Let x_1, x_2, x_3 (respectively y_1, y_2, y_3) be the neighbours of x in G_1 (respectively, of y in G_2). The cubic graph G such that $V(G) = (V(G_1) \setminus \{x\}) \cup (V(G_2) \setminus \{y\})$ and $E(G) = (E(G_1) \setminus \{x_1x, x_2x, x_3x\}) \cup (E(G_2) \setminus \{y_1y, y_2y, y_3y\}) \cup \{x_1y_1, x_2y_2, x_3y_3\}$ is said to be a *star product* and G is denoted by $(G_1, x) * (G_2, y)$. Since $\{x_1y_1, x_2y_2, x_3y_3\}$ is a cyclic edge-cut of G, a star product of two 3-connected cubic graphs has cyclic edge-connectivity 3.

Proposition 30 (Proposition 3.1, [6]). *If a bipartite cubic graph G can be represented as a star product $G = (G_1, x) * (G_2, y)$, then G is 2-factor hamiltonian if and only if G_1 and G_2 are 2-factor hamiltonian.*

Then, taking iterated star products of $K_{3,3}$ and the Heawood graph H_0 an infinite family of 2-factor hamiltonian cubic graphs is obtained. These graphs (excepted $K_{3,3}$ and H_0) are exactly cyclically 3-edge connected. In [6] the authors conjecture that the process is complete.

Conjecture 31 (Funk, Jackson, Labbate, Sheehan (2003)[6]). Let G be a bipartite 2-factor hamiltonian cubic graph. Then G can be obtained from $K_{3,3}$ and the Heawood graph H_0 by repeated star products.

The authors specify that a smallest counterexample to Conjecture 31 is a cyclically 4-edge connected cubic graph of girth at least 6, and that to show this result it would suffice to prove that H_0 is the only 2-factor hamiltonian cyclically 4-edge connected bipartite cubic graph of girth at least 6. Note that some results have been generalized in [1].

To conclude, we may ask what happens for non bipartite 2-factor hamiltonian cubic graphs. Recall that K_4 and $FS(1,3)$ (the "Triplex Graph" of Robertson, Seymour and Thomas [15]) are 2-factor hamiltonian cubic graphs. By Corollary 19 the graphs $FS(j,k)$ with k odd and $j=1$ or 3 introduced in this paper form a new infinite family of non bipartite 2-factor hamiltonian cubic graphs. We remark that they are cyclically 6-edge connected. Can we generate others families of non bipartite 2-factor hamiltonian cubic graphs ? Since PR_3 (the cubic graph on 6 vertices obtained from K_4 by a triangular extension) is not 2-factor hamiltonian and $PR_3 = K_4 * K_4$, the star product operation is surely not a possible tool.

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