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k-INDEPENDENCE STABLE GRAPHS UPON EDGE REMOVAL

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Abstract

Let k be a positive integer and $G = (V(G), E(G))$ a graph. A subset S of $V(G)$ is a k-independent set of G if the subgraph induced by the vertices of S has maximum degree at most $k-1$. The maximum cardinality of a k-independent set of G is the k-independence number $\beta_k(G)$. A graph G is called β_k^-*-stable* if $\beta_k(G-e) = \beta_k(G)$ for every edge e of $E(G)$. First we give a necessary and sufficient condition for β_k^--stable graphs. Then we establish four equivalent conditions for β_k^--stable trees.

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1. Introduction

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* is $N[v] = N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v of G, denoted by $d_G(v)$, is the size of its open neighborhood. Specifically, for a vertex v in a rooted tree T, we denote by $C(v)$ and $D(v)$ the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of a rooted tree T induced by $D[v]$, and is denoted by T_v.

In [2] Fink and Jacobson generalized the concept of independent sets. Let k be a positive integer. A subset S of V is *k-independent* if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k-1$. A k-independent set S of G is maximal if for every vertex $v \in V - S$, $S \cup \{v\}$ is not k-independent. The *k-independence number* $\beta_k(G)$ is the maximum cardinality of a k-independent set of G. Notice that 1-independent sets are independent, and so $\beta_1(G) = \beta(G)$. If S is a k-independent set of G of size $\beta_k(G)$, then we call S a $\beta_k(G)$-set. A vertex in a k-independent set S is said to be *full* if it has exactly $k-1$ neighbors in S, and a vertex in $V - S$ with at least k neighbors in S is said to be *k-dominated* by S.

In [3] Gunther, Hartnell and Rall studied the graphs whose independence numbers are unaffected by addition or deletion of any edge. They gave constructive characterizations of such trees.

A graph G is called *β_k^--stable* if $\beta_k(G - e) = \beta_k(G)$ for every edge e of $E(G)$. In this paper we are interested in determining conditions under which a graph G is β_k^--stable. In Section 2, we characterize the β_k^--stable trees by proving the following:

Theorem 1. *Let T be a tree. Then for every positive integer k the following conditions are equivalent:*

(a) *T is a β_k^--stable tree.*

(b) *T has a unique $\beta_k(T)$-set.*

(c) *for every $\beta_k(T)$-set S, each vertex $x \in V - S$ is $(k+1)$-dominated by S or there are at least two full vertices in $N(x) \cap S$.*

(d) *$\Delta(T) \leq k-1$ or $T \in \mathcal{F}_k$ (The family $\mathcal{F}_k$ is defined in Section 2).*

We note the result in [3] concerning trees whose independence number is unaffected by the deletion of an edge is a special case of Theorem 1.

2. β_k^--Stable Graphs

We begin with the following observation.

Observation 2. *Let G be a graph. If $uv \in E(G)$ and $\beta_k(G - uv) > \beta_k(G)$, then u and v are in every $\beta_k(G - uv)$-set.*

Proposition 3. *For any graph G and edge $e \in E(G)$, $\beta_k(G) \leq \beta_k(G-e) \leq \beta_k(G) + 1$.*

Proof. The lower bound is immediate from the fact that every k-independent set of a graph G is also a k-independent set of any spanning subgraph of G. Suppose that $\beta_k(G - uv) > \beta_k(G)$ for some edge $uv \in E(G)$, and let S be a $\beta_k(G - uv)$-set for some $uv \in E(G)$. By Observation 2, both u and v are in S. Then $S - \{u\}$ is a k-independent set of G implying that $\beta_k(G) \geq |S| - 1 = \beta_k(G - uv) - 1$. ∎

Next we provide a necessary and sufficient condition for β_k^--stable graphs.

Theorem 4. *A graph G is β_k^--stable if and only if for every $\beta_k(G)$-set S, each vertex $x \in V - S$ is $(k+1)$-dominated by S or there are at least two full vertices in $N(x) \cap S$.*

Proof. Let G be a β_k^--stable graph and S any $\beta_k(G)$-set. Assume there is a vertex $x \in V - S$ having at most k neighbors in S and there is at most one full vertex in $N(x) \cap S$. Let y be the full vertex in $N(x) \cap S$, if one exists, and an arbitrary vertex in $N(x) \cap S$ otherwise. Then $S \cup \{x\}$ is a k-independent set of $G - xy$, and so $\beta_k(G - xy) \geq |S| + 1 > \beta_k(G)$, contradicting the assumption that G is β_k^--stable.

Conversely, let $e = uv$ be any edge of $E(G)$ and S a $\beta_k(G - e)$-set. Assume that $\beta_k(G-e) > \beta_k(G)$. By Observation 2, u and v are in S. Then $S' = S - \{u\}$ is a k-independent set of G. Thus $\beta_k(G-e) > \beta_k(G) \geq |S'| = \beta_k(G - e) - 1$, and so Proposition 3 implies that S' is a $\beta_k(G)$-set. Since $u \in S$, u has in $G - e$ at most $k-1$ neighbors in S. Thus u has in G at most k neighbors in S'. Moreover, $N(u) \cap S'$ contains at most v as a full vertex in

G for otherwise S is not a k-independent set since it would contain a vertex having more than $k-1$ neighbors in S. But then S' is a $\beta_k(G)$-set for which $u \notin S'$ and u does not satisfy the conditions of the theorem, a contradiction. Thus $\beta_k(G-e) = \beta_k(G)$ for every $e \in E(G)$, and hence G is a β_k^--stable graph. ■

The following result shows that graphs with unique $\beta_k(G)$-sets are β_k^--stable.

Theorem 5. *If G is a graph with a unique $\beta_k(G)$-set, then G is a β_k^--stable graph.*

Proof. Let S be the unique $\beta_k(G)$-set. If every vertex of $V-S$ is $(k+1)$-dominated by S, then by Theorem 4, G is β_k^--stable. Now assume that $u \in V-S$ is a vertex with at most k neighbors in S. Assume further that $N(u) \cap S$ contains at most one full vertex. Let y be the full vertex in $N(u) \cap S$ if one exists and an arbitrary vertex in $N(u) \cap S$ otherwise. Then $\{u\} \cup (S - \{y\})$ is second $\beta_k(G)$-set, a contradiction. Thus for every vertex $u \in V-S$ not $(k+1)$-dominated by S, $S \cap N(u)$ contains at least two full vertices, and so by Theorem 4, G is β_k^--stable. ■

Note that the converse of Theorem 5 is not true for arbitrary graphs. Clearly the complete graph K_n, $n \geq 4$, is a β_2^--stable graph but any two vertices of K_n form a $\beta_2(K_n)$-set. Our next result shows that the converse of Theorem 5 holds for trees.

Lemma 6. *If T is a β_k^--stable tree, then T has a unique $\beta_k(T)$-set.*

Proof. Assume that T is β_k^--stable. Clearly the result holds if $\Delta(T) \leq k-1$, since $V(T)$ is the unique $\beta_k(T)$-set. Suppose that $\Delta(T) \geq k$, and let $B(T) = \{x \in V(T) : \deg_T(x) \geq k\}$. We proceed by induction on $|B(T)|$. If $|B(T)| = 1$, then the unique vertex in $B(T)$ should have degree at least $k+1$ for otherwise removing any edge incident to such a vertex increases the k-independence number, a contradiction. It follows that $V(T) - B(T)$ is the unique $\beta_k(T)$-set. Assume that every β_k^--stable tree T' with $|B(T')| < |B(T)|$ has a unique $\beta_k(T')$-set.

We now root T at a vertex r of maximum eccentricity. Let w be a vertex of degree at least k at maximum distance from r. Such a vertex exists since $\Delta(T) \geq k$. Let u be the parent of w in the rooted tree, and v be the parent of u. Let S be a $\beta_k(T)$-set. We distinguish between two cases.

Case 1. $d_T(w) \geq k+1$. Let $T' = T - T_w$. If $w \in S$, then at least one child of w, say w', is not in S. But then $S \cup \{w'\}$ is a k-independent set of $T - ww'$, a contradiction. Thus w belongs to no $\beta_k(T)$-set. It follows that $D(w) \subseteq S$. Now it can be seen that $\beta_k(T) = \beta_k(T') + \beta_k(T_w)$. Since T is a β_k^--stable tree, $\beta_k(T - uw) = \beta_k(T) = \beta_k(T') + \beta_k(T_w)$. Moreover, if for some edge $e \in E(T')$, $\beta_k(T' - e) > \beta_k(T')$, then $\beta_k(T - e) \geq \beta_k(T' - e) + \beta_k(T_w) > \beta_k(T') + \beta_k(T_w) = \beta_k(T)$, and so T is not β_k^--stable, a contradiction. It follows that for every edge $e \in E(T')$, $\beta_k(T' - e) = \beta_k(T')$ and so T' is a β_k^--stable. By induction on T', T' has a unique $\beta_k(T')$-set, say X. Since no $\beta_k(T)$-set contains w, $S \cap V(T')$ is a $\beta_k(T')$-set. Hence $S \cap V(T') = X$. Moreover, $S \cap V(T_w) = D(w)$. Thus, S is the unique $\beta_k(T)$-set.

Case 2. $d_T(w) = k$. By our choice of w, every descendant of w has degree at most $k-1$. Hence, $w \in S$ for otherwise by Theorem 4, w is $k+1$ dominated by S or $N(w) \cap S$ contains two full vertices, which is impossible. Assume that u is in S. Since $w \in S$, it follows that at least one child of w, say w', is not in S. But then $S \cup \{w'\}$ is a k-independent set of $T - uw$ with $|S \cup \{w'\}| > \beta_k(T)$, contradicting our assumption that T is β_k^--stable. Hence $u \notin S$. We may assume that every child of u has degree at most k, otherwise Case 1 applies. It follows that $D(u) \subseteq S$. Note that $D(u)$ is a $\beta_k(T_u)$-set, and we have shown that $S \cap V(T_u) = D(u)$ for any $\beta_k(T)$-set S. Let $T' = T - T_u$, and let $S' = S \cap V(T')$. Since $u \notin S$ and S' is a k-independent set, we conclude that S' is a $\beta_k(T')$-set. Moreover, since T is a β_k^--stable tree, $\beta_k(T - uv) = \beta_k(T) = \beta_k(T') + \beta_k(T_u)$. Now if S' does not satisfy conditions of Theorem 4, then clearly $S = S' \cup D(u)$ does not satisfy these conditions in T, and so T is not β_k^- -stable, a contradiction. It follows that T' is a β_k^--stable tree, and by our inductive hypothesis on T', $S \cap V(T')$ is the unique $\beta_k(T')$-set. Since u does not belong to any $\beta_k(T)$-set, $S = D(u) \cup S'$ is the unique $\beta_k(T)$-set. ■

Lemma 7. *Let T_1 and T_2 be trees with unique β_k-sets S_1 and S_2, respectively. If T is a tree obtained from $T_1 \cup T_2$ by adding an edge uv where $u \in V(T_1)$ and $v \in V(T_2) - S_2$, then $S_1 \cup S_2$ is the unique $\beta_k(T)$-set.*

Proof. Let T_1 and T_2 be trees with unique β_k-sets S_1 and S_2, respectively, and let T be a tree obtained from $T_1 \cup T_2$ by adding an edge uv where $u \in V(T_1)$ and $v \in V(T_2) - S_2$. Clearly, since $v \notin S_2$, $S_1 \cup S_2$ is a k-independent set of T. Thus, $\beta_k(T) \geq |S_1 \cup S_2|$. Let D be a $\beta_k(T)$-set, and let $D_1 = D \cap V(T_1)$ and $D_2 = D \cap V(T_2)$. Since D_i is a k-independent set in T_i, we have $\beta_k(T_i) \geq |D_i|$ for $i \in \{1,2\}$. Hence, $\beta_k(T_1) + \beta_k(T_2) \geq |D_1| + |D_2| = |D| = \beta_k(T)$. Therefore, $\beta_k(T) = \beta_k(T_1) + \beta_k(T_2)$ and D is a $\beta_k(T)$-set. Moreover, it follows that D_i is a k-independent set of T_i having cardinality $\beta_k(T_i)$ for $i \in \{1,2\}$ and so $D_i = S_i$ implying that $D = S_1 \cup S_2$ is the unique $\beta_k(T)$-set. ■

In [1], Blidia, Chellali and Volkmann defined the following trees. For a positive integer p, a nontrivial tree T is called an $\mathcal{N}_p$-*tree* if T contains a vertex, say w, of degree at least $p-1$ and $\deg_T(x) \leqslant p-1$ for every vertex of $x \in V(T) - \{w\}$. We will call w the *special vertex* of T. The subdivided star $K_{1,p}$ $(p \geqslant 3)$ is an example of an $\mathcal{N}_p$-*tree.*

We define a related family of trees, which we call $\mathcal{N}^*_{k,j}$-trees. A tree T is an $\mathcal{N}^*_{k,j}$-tree with special vertex w if $N(w)$ contains $j \geq 0$ vertices of degree k, the remaining vertices in T except possibly w have degree at most $k-1$, and if $j \leq 1$, $d_T(w) \geq k+1$. We note that if $j \geq 2$, the only degree restriction on the special vertex w is that $d_T(w) \geq j$. An $\mathcal{N}_k$-tree with special vertex of degree at least $k+1$ is an example of an $\mathcal{N}^*_{k,j}$-tree. A tree T is a weak $\mathcal{N}^*_{k,1}$-tree with special vertex w if w has degree at most k, $N(w)$ contains one vertex of degree k, and the remaining vertices in T except possibly w have degree at most $k-1$.

Observation 8. *For an $\mathcal{N}^*_{k,j}$-tree T with special vertex w, $V(T) - \{w\}$ is the unique $\beta_k(T)$-set.*

In order to characterize trees T with a unique $\beta_k(T)$-set, we define the family $\mathcal{F}_k$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_p$ $(p \geq 1)$ of trees, where $T_1 = T^*$ is an $\mathcal{N}^*_{k,j}$-tree, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the four operations listed below.

- Operation $\mathcal{O}_1$: Attach an $\mathcal{N}_k$-tree with special vertex z of degree at least $k+1$ by adding an edge from z to any vertex of T_i.
- Operation $\mathcal{O}_2$: Attach an $\mathcal{N}_k$-tree with special vertex z of degree k by adding an edge from z to any vertex belonging to a $\beta_k(T_i)$-set.
- Operation $\mathcal{O}_3$: Attach an $\mathcal{N}^*_{k,j}$-tree with special vertex z, where $j \geq 1$, by adding an edge from z to any vertex in T_i.
- Operation $\mathcal{O}_4$: Attach a weak $\mathcal{N}^*_{k,1}$-tree T^* with special vertex z, by adding the edge zx, where x is a vertex in a $\beta_k(T_i)$-set, with the condition that if x is not full, then z has degree k in T^*.

We state two lemmas.

Lemma 9. *Let T be a tree and k a positive integer. If $\Delta(T) \leq k-1$ or $T \in \mathcal{F}_k$, then T has a unique $\beta_k(T)$-set.*

Proof. It is clear that if $\Delta(T) \leq k-1$, then $V(T)$ is the unique $\beta_k(T)$-set. Suppose now that $\Delta(T) \geqslant k$ and $T \in \mathcal{F}_k$. Then T is obtained from a sequence $T_1, T_2, \ldots, T_p$ ($p \geq 1$) of trees, where $T_1 = T^*$ with special vertex w, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the four operations defined above. Clearly the property is true if $p = 1$. This establishes the basis case.

Assume now that $p \geqslant 2$ and that the result holds for all trees $T \in \mathcal{F}_k$ that can be constructed from a sequence of length at most $p-1$, and let $T' = T_{p-1}$. By the inductive hypothesis, T' has a unique $\beta_k(T')$-set. Let T be a tree obtained from T' and S a $\beta_k(T)$-set. We consider the following four cases.

Case 1. T is obtained from T' by using Operation $\mathcal{O}_1$. Let H be the $\mathcal{N}_k$-tree with special vertex z of degree at least $k+1$ added to T'. Note that $V(H) - \{z\}$ is the unique $\beta_k(H)$-set, and since T' has a unique $\beta_k(T')$-set, say S', Lemma 7 implies that $S' \cup (V(H) - \{z\})$ is the unique $\beta_k(T)$-set.

Case 2. T is obtained from T' by using Operation $\mathcal{O}_2$. Let H be an $\mathcal{N}_k$-tree with special vertex z of degree k added to T' with edge uz, where u is a vertex of a $\beta_k(T')$-set S'. Clearly $S' \cup (V(H) - \{z\})$ is a k-independent set of T and so $\beta_k(T) \geq \beta_k(T') + |V(H)| - 1$. Moreover, if S contains z, then since $d_T(z) = k+1$ at least one of its neighbors in H is not in S, and hence z can be substituted by such a vertex in S. Therefore we may assume that $z \notin S$, and hence $V(H) - \{z\} \subseteq S$. Thus $S \cap V(T')$ is a k-independent set of T' implying that $\beta_k(T') \geq \beta_k(T) - |V(H)| + 1$, and the following equality is obtained $\beta_k(T) = \beta_k(T') + |V(H)| - 1$. Now assume that S is not the unique $\beta_k(T)$-set, and let M be a second $\beta_k(T)$-set. Note that we have seen that $z \notin S$. Since at most $|V(H)| - 1$ vertices from H are in M, it follows that $|M \cap V(T')| \geq \beta_k(T')$. Since T' has a unique $\beta_k(T')$-set, $M \cap V(T') = S \cap V(T')$ is the unique $\beta_k(T')$-set. Hence $u \in M$. If $z \in M$, then two vertices of $N_H(z)$, say $y', y'' \notin M$ but then $\{y', y''\} \cup (M - \{z\})$ is a k-independent set of T larger than M which is impossible. Thus $z \notin M$. It follows that M contains $V(H) - \{z\}$, implying that $M = S$, a contradiction. Therefore S is the unique $\beta_k(T)$-set.

Case 3. T is obtained from T' by using Operation $\mathcal{O}_3$. Then T is obtained from T' by adding an $\mathcal{N}^*_{k,j}$-tree T^* with special vertex z by adding the edge zx, where $x \in V(T')$. From Observation 8, we know that $V(T^*) - \{z\}$ is the unique $\beta_k(T^*)$-set. Since T' has the unique $\beta_k(T')$-set S', it follows from Lemma 7 that $S' \cup (\ V(T^*) - \{z\})$ is the unique $\beta_k(T)$-set.

Case 4. T is obtained from T' by using Operation $\mathcal{O}_4$. Then T is obtained from T' by adding a weak $\mathcal{N}^*_{k,1}$-tree T_0 with special vertex z by adding the edge zx, where $x \in \beta_k(T')$-set S'. Then $S' \cup (V(T_0) - \{z\})$ is a k-independent set of T and hence $\beta_k(T) \geq \beta_k(T') + |V(T_0)| - 1$. Also since $N_{T_0}(z)$ contains a vertex, say y, of degree k, S does not contain all vertices of $N[y]$. Hence we may assume that $z \notin S$. It follows that $V(T_0) - \{z\} \subseteq S$ and so $S \cap T'$ is a k-independent set implying that $\beta_k(T') \geq \beta_k(T) - |V(T_0)| + 1$. Thus we have $\beta_k(T) = \beta_k(T') + |V(T_0)| - 1$. Assume now that S is not the unique $\beta_k(T)$-set, and let M be a second $\beta_k(T)$-set. Since T_0 contains a vertex of degree k, M does not contain all vertices of $V(T_0)$. If $z \notin M$ or $x \notin M$, then $M \cap V(T')$ would be a second $\beta_k(T')$-set, a contradiction. Thus $z \in M$ and $x \in M$. The uniqueness of a $\beta_k(T')$-set implies that $M \cap V(T')$ is the unique $\beta_k(T')$-set. Clearly x is not full in $M \cap V(T')$. By our construction in that case both y and z have degree k in T_0. Then there are two vertices y' and y'' in $N_{T_0}(z)$ that do not belong to M, but then $\{y', y''\} \cup (M - \{z\})$ would be a k-independent set of T larger than M, a contradiction. Thus S is the unique $\beta_k(T)$-set. ∎

Lemma 10. *Let T be a tree and k a positive integer. If T admits a unique $\beta_k(T)$-set, then either $\Delta(T) \leq k-1$ or $T \in \mathcal{F}_k$.*

Proof. If $\Delta(T) \leq k-1$, we are finished. Suppose that $\Delta(T) \geqslant k$, and let $B(T) = \{x \in V(T) : \deg_T(x) \geqslant k\}$. Clearly $B(T) \neq \emptyset$. We use an induction on the size of $B(T)$. If $|B(T)| = 1$, then T is an $\mathcal{N}_k$-tree with special vertex, say z, of degree at least $k+1$, for otherwise $V(T) - \{z\}$ and $V(T) - \{z'\}$ are two $\beta_k(T)$-sets, where z' is any vertex adjacent to z. Hence T is an $\mathcal{N}^*_{k,j}$-tree. This establishes the basis case.

Let $|B(T)| \geqslant 2$ and assume that every tree T' with $|B(T')| < |B(T)|$ having a unique $\beta_k(T')$-set is in $\mathcal{F}_k$. Let T be a tree with a unique $\beta_k(T)$-set S.

Root T at a vertex r of maximum eccentricity, and let w be a vertex of degree at least k at maximum distance from r. Let u be the parent of w in the rooted tree. We distinguish between three cases.

Case 1. $d_T(w) \geq k+2$. Let $T' = T - T_w$. Clearly $|B(T')| < |B(T)|$. The uniqueness of S implies that w does not belong to S for otherwise it can be replaced by one of at least two vertices of $N[w] - \{u\}$ not in S. It follows that $\beta_k(T) = \beta_k(T') + |V(T_w)| - 1$ and $S \cap V(T')$ is the unique $\beta_k(T')$-set. Applying the inductive hypothesis, $T' \in \mathcal{F}_k$ and hence $T \in \mathcal{F}_k$ since it is obtained from T' by using Operation $\mathcal{O}_1$.

Case 2. $d_T(w) = k+1$. If $w \in S$, then a child w' of w is not in S. Therefore $\{w'\} \cup (S - \{w\})$ is a second $\beta_k(T)$-set, a contradiction. Thus $w \notin S$ and so $u \in S$ for otherwise $\{w\} \cup (S - \{w'\})$ is a second $\beta_k(T)$-set, a contradiction. Now let $T' = T - T_w$. It is straightforward to show that $\beta_k(T) = \beta_k(T') + |V(T_w)| - 1$. The uniqueness of S implies that $S \cap V(T')$ is the unique $\beta_k(T')$-set, where $u \in S \cap V(T')$. Since $|B(T')| < |B(T)|$ the inductive hypothesis on T' implies that $T' \in \mathcal{F}_k$. Thus $T \in \mathcal{F}_k$ because it is obtained from T' by using Operation $\mathcal{O}_2$.

Case 3. $d_T(w) = k$. Assume for a contradiction that $w \notin S$. Then S must contain u else $S \cup \{w\}$ is a k-independent set of T larger than S. Hence $\{w\} \cup (S - \{u\})$ is a second $\beta_k(T)$-set, a contradiction. Therefore $w \in S$. If $u \in S$, then $k \geq 2$ and a child w' of w is not in S and so $\{w'\} \cup (S - \{u\})$ is a second $\beta_k(T)$-set, a contradiction. Thus $u \notin S$. By our choice of w, $D[w] \subseteq S$ and hence w is a full vertex in S. Also our choice of w implies that every child of u has degree at most k and each vertex in $D(u) - N(u)$ has degree at most $k-1$. Thus, S contains all descendants of u. If w is the unique full vertex in S adjacent to u and u has at most k neighbors in S, then $\{u\} \cup (S - \{w\})$ would be a second $\beta_k(T)$-set, a contradiction. Thus either u is adjacent to at least two full vertices in S or u is adjacent to at least $k+1$ vertices in S. Let $T' = T - T_u$. If $B(T') = \emptyset$, then T is an $\mathcal{N}^*_{k,j}$-tree and hence $T \in \mathcal{F}_k$. Thus assume that $B(T') \neq \emptyset$, and let v be the parent of u. Note that $V(T_u) - \{u\}$ is a k-independent set. It can be seen that $\beta_k(T) = \beta_k(T') + |V(T_u)| - 1$ and $S \cap V(T')$ is a $\beta_k(T')$-set. Moreover, the uniqueness of S implies that $S \cap V(T')$ is the unique $\beta_k(T')$-set. Thus by induction on T', $T' \in \mathcal{F}_k$. Now if T_u is an $\mathcal{N}^*_{k,j}$-tree with special vertex u, where $j \geq 1$, then $T \in \mathcal{F}_k$ because it is obtained from T' by using Operation $\mathcal{O}_3$. Hence assume that T_u is not an $\mathcal{N}^*_{k,j}$-tree. This implies that w is the only child of u with degree k and u has degree at most k in T_u. Thus, T_u is a weak $\mathcal{N}^*_{k,1}$-tree. Recall that u is adjacent to two full vertices in S or u is adjacent to at least $k+1$ vertices in S. If u is adjacent to two full vertices in S, then since w is the only full vertex in $D(u)$, it follows that v is full in S.

Since $u \notin S$, it follows that v is full in $S \cap V(T')$. If u is adjacent to $k+1$ vertices in S, then u has degree k in T_u and v is in S. Thus, $v \in S \cap V(T')$. In both cases, T can be obtained from T' by using Operation $\mathcal{O}_4$. Hence $T \in \mathcal{F}_k$. ■

According to Theorems 4, 5, and Lemmas 6, 9 and 10, we have completed the proof of Theorem 1.

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