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# NORDHAUS-GADDUM RESULTS FOR WEAKLY CONVEX DOMINATION NUMBER OF A GRAPH

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**Abstract**

Nordhaus-Gaddum results for weakly convex domination number of a graph $G$ are studied.

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## 1. Introduction

Let $G = (V, E)$ be a connected undirected graph of order $n$. The *neighbourhood* of a vertex $v \in V$ in $G$ is the set $N_G(v)$ of all vertices adjacent to $v$ in $G$. For a set $X \subseteq V$, *the open neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and *the closed neighbourhood* $N_G[X] = N_G(X) \cup X$. The *degree* $\deg_G(v)$ of a vertex $v$ in $G$ is the number of edges incident to $v$, $\deg_G(v) = |N_G(v)|$. The minimum and maximum degree of a vertex in $G$ we denote $\delta(G)$ and $\Delta(G)$, respectively. If $deg_G(v) = n - 1$, then $v$ is called an *universal vertex* of $G$. A set $D \subseteq V$ is a *dominating set* of $G$ if $N_G[D] = V$. The *domination number* of $G$, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in $G$.

Given a graph $G$ and a set $S \subseteq V$, the *private neighbourhood* of $v \in S$ relative to $S$ is defined as $PN[v, S] = N_G[v] - N_G[S - \{v\}]$, that is, $PN[v, S]$ denotes the set of all vertices of the closed neighbourhood of $v$, which are not dominated by any other vertex of $S$. The vertices of $PN[v, S]$ are called *private neighbours* of $v$ relative to $S$.

The *distance* $d_G(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of the shortest $(u - v)$ path in $G$. A $(u - v)$ path of length $d_G(u, v)$ is called $(u - v)$-*geodesic*. A set $X \subseteq V$ is *weakly convex* in $G$ if for every two vertices $a, b \in X$ there exists an $(a - b)$- geodesic in which all vertices belong to $X$. A set $X \subseteq V$ is a *weakly convex dominating set* if $X$ is both weakly convex and dominating. The *weakly convex domination number* $\gamma_{wcon}(G)$ of a graph $G$ equals the minimum cardinality of a weakly convex dominating set. Weakly convex domination number was first introduced by Jerzy Topp, Gdańsk University of Technology, 2002.

The classical paper of Nordhaus and Gaddum [4] established the following inequalities for the chromatic numbers $\chi$ and $\bar{\chi}$ of a graph $G$ and its complement $\overline{G}$, where $n = |V|$:

$$2\sqrt{n} \leq \chi + \bar{\chi} \leq n + 1,$$
$$n \leq \chi\bar{\chi} \leq \frac{(n+1)^2}{4}.$$

There are a large number of results in the graph theory literature of the form $\alpha + \bar{\alpha} \leq n \pm \epsilon$, where $\epsilon \in Q$, for a domination parameter $\alpha$. Results of this form have previously been obtained for example for the domination number $\gamma$ [3] and the connected domination number $\gamma_c$ [2].

**Theorem 1.** *For any graph $G$ such that $G$ and $\overline{G}$ are connected,*

1. $\gamma(G) + \gamma(\overline{G}) \leq n + 1$,
2. $\gamma_c(G) + \gamma_c(\overline{G}) \leq n + 1$.

We are concerned with analogous inequalities involving weakly convex domination number. For unexplained terms and symbols see [1].

## 2. Results

Since $G$ and $\overline{G}$ must be connected, we consider graphs $G$ with $n(G) \geq 4$. We begin with the following result of Nordhaus-Gaddum type for weakly convex domination number.

**Theorem 2.** *For any graph $G$ such that $G$ and $\overline{G}$ are connected,* $4 \leq \gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) \leq n + 2$.

***Proof.*** If there is an universal vertex in $G$, then $\overline{G}$ is not connected. Thus there is no universal vertex in $G$ and no universal vertex in $\overline{G}$ and hence $\gamma_{wcon}(G) \geq 2$ and $\gamma_{wcon}(\overline{G}) \geq 2$. Thus $\gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) \geq 4$. Notice that equality $\gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) = 4$ holds if $G \cong P_4$.

Of course $\gamma_{wcon}(G) \leq n$ and $\gamma_{wcon}(G) \leq n$. We consider some cases, depending on the diameter of $G$.

*Case* 1. If $diam(G) = 1$, then there is an universal vertex in $G$ and $\overline{G}$ are not connected.

*Case* 2. If $diam(G) \geq 3$, then let $x, y$ be two vertices of $V$ such that $d_G(x, y) = diam(G)$. Then $\{x, y\}$ is a weakly convex dominating set of $\overline{G}$ and $\gamma_{con}(G) + \gamma_{con}(\overline{G}) \leq n + 2$.

*Case* 3. Let $diam(G) = 2$. If $diam(G) \geq 3$, then we can exchange $G$ and $\overline{G}$ and we have Case 2. Thus $diam(G) = 2$ and $diam(\overline{G}) = 2$. Let $x$ be any vertex of $G$. Since $diam(G) = 2$, for every $v \in V$ is $d_G(v, x) \leq 2$. Let $Y = \{y \in V : d_G(x, y) = 1\}$ and $Z = \{z \in V : d_G(x, z) = 2\}$, $|Y| = k, |Z| = l$, where $k, l \geq 1$ (if $l = 0$, then there is an universal vertex in $G$ and $\overline{G}$ are not connected). Then $n = k + l + 1$ and it is easy to observe that $D = \{x\} \cup Y$ is a connected dominating set of $G$. For every two vertices $u, v$ belonging to $D$, the distance between $u$ and $v$ is not greater than two and if $d_G(u, v) = 2$, then $x$ belonging to $D$ is on $(u, v)$-geodesic. Thus $D$ is a weakly convex dominating set of $G$ and $\gamma_{wcon}(G) \leq |D| = k + 1$.

Since $\overline{G}$ is connected and $diam(\overline{G}) = 2$, every vertex from $Y$ has a neighbour in $\{x\} \cup Z$ in $\overline{G}$ and hence $D' = \{x\} \cup Z$ is a connected dominating set of $\overline{G}$. For every two vertices $u, v$ belonging to $D'$, the distance between $u$ and $v$ is not greater than two and if $d_G(u, v) = 2$, then $x$ belonging to $D'$ is on $(u, v)$-geodesic. Thus $D'$ is a weakly convex dominating set of $\overline{G}$ and $\gamma_{wcon}(\overline{G}) \leq |D'| = l + 1$.

Thus $\gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) \leq k + 1 + l + 1 \leq n + 1 \leq n + 2$. ∎

The next theorem concerns of the graphs $G$ for which weakly convex domination number is equal to the number of vertices. Let $g(G)$ denotes the girth of the graph $G$.

**Theorem 3.** *If $G$ is a connected graph with $\delta(G) \geq 2$ and $g(G) \geq 7$, then $\gamma_{wcon}(G) = n$.*

***Proof.*** Let $G$ be a connected graph with $\delta(G) \geq 2$ and $g(G) \geq 7$. Suppose that $\gamma_{wcon}(G) < n$. Let $D$ be a minimum weakly convex dominating set of $G$. Since $\gamma_{wcon}(G) < n$, there exists a vertex $x$ in $G$ such that $x \notin D$. Denote $N_G(x) = \{x_1, \ldots, x_p\}$, where $p \geq 2$ (because $\delta(G) \geq 2$). It is easy to observe that since $g(G) \geq 7$, for every $x_i, x_j$ is $x_i x_j \notin E(G)$ for $1 \leq i, j \leq p$.

Notice that for every $x_i, x_j$, where $x_i \neq x_j$ and $1 \leq i, j \leq p$ we have $d_G(x_i, x_j) = 2$ and every shortest path between $x_i$ and $x_j$ contains $x$.

Suppose there are vertices $x_1, x_2 \in N_G(x)$ such that $x_1, x_2 \in D$. Then, since $D$ is weakly convex, there is a vertex $v \in D$ such that $v \in N_G(x_1) \cap N_G(x_2)$. But then we can find a cycle $C = (x_1, x, x_2, v, x_1)$ which length is less than seven, what gives a contradiction.

Thus $|N_G(x) \cap D| \leq 1$. Since $x$ has to be dominated, we have $|N_G(x) \cap D| = 1$. Without loss of generality assume that $x_1 \in N_G(x) \cap D$. Thus, since $\delta(G) \geq 2$, there exists at least one vertex belonging to $N_G(x)$ say $x_2$, such that $x_2 \notin D$. Since $\delta(G) \geq 2$ and $x_2$ is dominated, there exists a vertex $y \in N_G(x_2)$ such that $y \neq x$ and $y \in D$. Since $g(G) \geq 7$, we have $N_G(y) \cap N_G(x) = \emptyset$ and $N_G(y) \cap N_G(x_i) = \emptyset$, where $1 \leq i \leq p$.

Since $D$ is a weakly convex set, $d_G(y, x_1) = 3$ and there is a $(x_1 - y)$-geodesic $P_1$ such that all vertices of $P_1$ belong to $D$. Thus we have at least two $(x_1 - y)$-geodesics: $P_1$ and $P_2 = (x_1, x, x_2, y)$ what produces a cycle of length less than seven. That gives contradiction with $g(G) \geq 7$ and hence we have $\gamma_{wcon}(G) = n$. ■

The simplest example of a graph $G$ such that $\gamma_{wcon}(G) = n$ can be a graph $G = C_n$ with $n \geq 7$. For $\overline{C_n}$ we have $\gamma_{wcon}(\overline{C_n}) = 2$ and $\gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) = n + 2$.

**Corollary 4.** *If $G$ and $\overline{G}$ are connected, $\delta(G) \geq 2$ and $g(G) \geq 7$, then $\gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) = n + 2$.*

**Theorem 5.** *For any graph $G$ such that $G$ and $\overline{G}$ are connected, $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq (\lfloor \frac{n}{2} \rfloor + 1)^2$. Furthermore, $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) = (\lfloor \frac{n}{2} \rfloor + 1)^2$ if and only if $G$ or $\overline{G}$ is isomorphic to $C_5$.*

***Proof.*** Again we consider three cases, depending on the diameter of $G$.

If $diam(G) = 1$, then $\gamma_{wcon}(G) = 1$ and $\overline{G}$ is not connected.

If $diam(G) \geq 3$, then similarly like in the proof of Theorem 2, $\gamma_{wcon}(\overline{G}) = 2$ and since $n \geq 4$, $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq 2n < (\lfloor \frac{n}{2} \rfloor + 1)^2$.

Let $diam(G) = 2$. Similarly like in the proof of the previous theorem, let $x$ be any vertex of $G$, let $Y = \{y \in V : d_G(x, y) = 1\}$ and $Z = \{z \in V : d_G(x, z) = 2\}$, $|Y| = k, |Z| = l$, where $k, l \geq 1$.

If $k = 1$, then $\gamma_{wcon}(G) = 1$, there is an universal vertex in $G$ and $\overline{G}$ is not connected.

If $k = 2$, then, since $\{x\} \cup Y$ is a weakly convex dominating set of $G$, $\gamma_{wcon}(G) \leq 3$. Let $Y = \{u, v\}$. Notice that $\{x\}$ dominates itself and $Z$ in $\overline{G}$ and to dominate $Y$ in $\overline{G}$, it is enough to take two vertices $a, b$ from $Z$ such that $au \in E(\overline{G})$ and $bv \in E(\overline{G})$ (such vertices $a, b$ must exist since $\overline{G}$ is connected and $diam(\overline{G}) = 2$). Since $a, b \in Z$, $ax \in E(\overline{G})$ and $bx \in E(\overline{G})$ and thus $\{x, a, b\}$ is a weakly convex dominating set of $\overline{G}$. Hence $\gamma_{wcon}(\overline{G}) \leq 3$.

Since $G$ and $\overline{G}$ are connected and $diam(G) = 2$, we have $|Z| \geq 2$ and $n \geq 5$. It is easy to observe that $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq (\lfloor \frac{n}{2} \rfloor + 1)^2$.

If $\gamma_{wcon}(G) = 3$, $\gamma_{wcon}(\overline{G}) = 3$ and $n = 5$ we have equality $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) = (\lfloor \frac{n}{2} \rfloor + 1)^2$ and $C_5$ realizes this equality. In the other cases we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < (\lfloor \frac{n}{2} \rfloor + 1)^2$.

Now let $k \geq 3$. Since $\{x\} \cup Y$ is a weakly convex dominating set of $G$, we have $\gamma_{wcon}(G) \leq k + 1$. We consider three cases:

*Case* 1. If $l > k$, then $k < \lfloor \frac{n}{2} \rfloor$. Observe that $x$ dominates itself and $Z$ in $\overline{G}$. Since $\overline{G}$ is connected and $diam(\overline{G}) = 2$, every vertex from $Y$ has a neighbour in $Z$. Let $Y = \{y_1, \ldots, y_k\}$ and let $\{z_1, \ldots, z_k\}$ be the set of vertices from $Z$ such that $y_1z_1 \in E(\overline{G}), \ldots, y_kz_k \in E(\overline{G})$. Thus $\{x\} \cup \{z_1, \ldots, z_k\}$ is a weakly convex dominating set of $\overline{G}$ and $\gamma_{wcon}(\overline{G}) \leq k + 1$. Hence $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq (k + 1)^2$ and since $k < \lfloor \frac{n}{2} \rfloor$, we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < (\lfloor \frac{n}{2} \rfloor + 1)^2$.

*Case* 2. If $l = k$, then $k \leq \lfloor \frac{n}{2} \rfloor$ and $l \leq \lfloor \frac{n}{2} \rfloor$. Since $\{x\} \cup Z$ is a weakly convex dominating set of $\overline{G}$, we have $\gamma_{wcon}(\overline{G}) \leq l + 1$. Thus $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq (k + 1)(l + 1) \leq (\lfloor \frac{n}{2} \rfloor + 1)^2$.

*Case* 3. If $l < k$, then $l < \lfloor \frac{n}{2} \rfloor$. Similarly like in Case 2 we have $\gamma_{wcon}(\overline{G}) \leq l + 1$. Notice that $\{x\}$ dominates itself and $Y$ in $G$ and to dominate $Z$ in $G$ it is enough to take $l$ vertices from $Y$. Thus $\gamma_{wcon}(G) \leq l + 1$ and $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq (l + 1)^2 < (\lfloor \frac{n}{2} \rfloor + 1)^2$.

We have already shown that for $C_5$ equality $\gamma_{con}(G)\gamma_{con}(\overline{G}) = (\lfloor \frac{n}{2} \rfloor + 1)^2$ holds. Conversely, let $G$ be a graph for which we have equality. Then (from the earlier part of the proof) we have $diam(G) = 2$ and $l = k$.

If $k = 2$, then $l = 2$ and $n = 5$. Since $diam(G) = 2$, there is no end vertex in $Z$. Let $Z = \{z_1, z_2\}, Y = \{y_1, y_2\}$. If both $z_1, z_2$ have two neighbours in $Y$, then $\overline{G}$ is not connected. If one vertex of $Z$, without loss of generality if $z_1$ has two neighbours in $Y$, then $\gamma_{wcon}(G) = 2 = \gamma_{wcon}(\overline{G})$ and $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < (\lfloor \frac{n}{2} \rfloor + 1)^2$. Thus every of vertices $z_1$ and $z_2$ has only one neighbour in $Y$. If $z_1, z_2$ have a common neighbour in $Y$, say $y_1$, then $y_1$ is an end vertex in $\overline{G}$ and $diam(\overline{G}) > 2$. Thus every vertex from $Z$ has exactly one neighbour in $Y$ and every vertex from $Y$ has exactly one neighbour in $Z$, without loss of generality let $z_1y_1 \in E(G)$ and $z_2y_2 \in E(G)$. Since there is no end vertex in $G$, we have $z_1z_2 \in E(G)$. If $y_1y_2 \in E(G)$, then we have an end vertex in $\overline{G}$ and $diam(\overline{G}) > 2$; hence $y_1y_2 \notin E(G)$ and $G \cong C_5$.

Now let $l = k, k \geq 3$. We distinguish two cases.

1. There exists a vertex $y \in Y$ such that $PN[y, Y] = \emptyset$. Then $(\{x\} \cup Y) - \{y\}$ is a weakly convex dominating set of $G$ and $\gamma_{con}(G) \leq k$. Since $\{x\} \cup Z$ is a weakly convex dominating set of $\overline{G}$, we have $\gamma_{wcon}(\overline{G}) \leq l + 1$ and since $k \leq \lfloor \frac{n}{2} \rfloor$ and $l \leq \lfloor \frac{n}{2} \rfloor$, we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq k(l + 1) \leq \lfloor \frac{n}{2} \rfloor(\lfloor \frac{n}{2} \rfloor + 1) < (\lfloor \frac{n}{2} \rfloor + 1)^2$.
2. For every $y \in Y$ we have $PN[y, Y] \neq \emptyset$. Let us denote $Y = \{y_1, \ldots, y_k\}$, $Z = \{z_1, \ldots, z_k\}$ and $PN[y_1, Y] = \{z_1\}, \ldots, PN[y_k, Y] = \{z_k\}$. Then $\{x, z_1, z_2\}$ is a weakly convex dominating set of $\overline{G}$ and $\gamma_{wcon}(\overline{G}) \leq 3$. Thus we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq 3(k + 1) < (\lfloor \frac{n}{2} + 1)^2 \rfloor$.

Hence if $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) = (\lfloor \frac{n}{2} \rfloor + 1)^2$, then $G \cong C_5$. ∎

**Corollary 6.** *If $G$ and $\overline{G}$ are connected, $diam(G) \leq 2$ and $G \neq C_5$, then* $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor(\lfloor \frac{n}{2} \rfloor + 1)$.

**Theorem 7.** *If $G$ and $\overline{G}$ are connected, $G \neq C_7$ and $G \neq C_5$, then* $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor(\lfloor \frac{n}{2} \rfloor + 1)$.

***Proof.*** Let $G$ be a graph such that $G$ and $\overline{G}$ are connected and $G \neq C_5$ and $G \neq C_7$. From Corollary 6, if $diam(G) \leq 2$, then $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor(\lfloor \frac{n}{2} \rfloor + 1)$; so let $diam(G) \geq 3$. Then $\gamma_{wcon}(\overline{G}) = 2$ and $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq 2n \leq \lfloor \frac{n}{2} \rfloor(\lfloor \frac{n}{2} \rfloor + 1)$ for $n \geq 8$.

Since $diam(G) \geq 3$ and $G, \overline{G}$ are connected, we have $n \geq 4$.

If $n = 4$, then $G \cong \overline{G} \cong P_4$ and $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < \lfloor \frac{n}{2} \rfloor(\lfloor \frac{n}{2} \rfloor + 1)$.

If $n = 5$, then $\gamma_{wcon}(G) \leq 3$ and since $\gamma_{wcon}(\overline{G}) = 2$ we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor(\lfloor \frac{n}{2} \rfloor + 1)$.

If $n = 6$, then $\gamma_{wcon}(G) \leq 4$ and since $\gamma_{wcon}(\overline{G}) = 2$ is $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)$.

If $n = 7$, then, since $G \neq C_7$, we have $\gamma_{wcon}(G) \leq 5$ and since $\gamma_{wcon}(\overline{G}) = 2$, again we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)$. ■

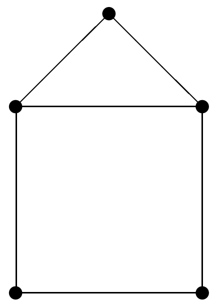

Figure 1. Graph $G_1$.

The example of the extremal graph of Theorem 7 can be the graph $G_1$ from Figure 1.

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