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NORDHAUS-GADDUM RESULTS FOR WEAKLY CONVEX DOMINATION NUMBER OF A GRAPH

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Abstract

Nordhaus-Gaddum results for weakly convex domination number of a graph G are studied.

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1. INTRODUCTION

Let G = (V, E) be a connected undirected graph of order n. The neighbourhood of a vertex $v \in V$ in G is the set $N_G(v)$ of all vertices adjacent to v in G. For a set $X \subseteq V$, the open neighbourhood $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$ and the closed neighbourhood $N_G[X] = N_G(X) \cup X$. The degree deg_G(v) of a vertex v in G is the number of edges incident to v, deg_G(v) = $|N_G(v)|$. The minimum and maximum degree of a vertex in G we denote $\delta(G)$ and $\Delta(G)$, respectively. If $deg_G(v) = n - 1$, then v is called an universal vertex of G. A set $D \subseteq V$ is a dominating set of G if $N_G[D] = V$. The domination number of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G.

Given a graph G and a set $S \subseteq V$, the private neighbourhood of $v \in S$ relative to S is defined as $PN[v, S] = N_G[v] - N_G[S - \{v\}]$, that is, PN[v, S]denotes the set of all vertices of the closed neighbourhood of v, which are not dominated by any other vertex of S. The vertices of PN[v, S] are called *private neighbours* of v relative to S.

The distance $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest (u - v) path in G. A (u - v) path of length $d_G(u, v)$ is called (u - v)-geodesic. A set $X \subseteq V$ is weakly convex in G if for every two vertices $a, b \in X$ there exists an (a - b)- geodesic in which all vertices belong to X. A set $X \subseteq V$ is a weakly convex dominating set if X is both weakly convex and dominating. The weakly convex domination number $\gamma_{wcon}(G)$ of a graph G equals the minimum cardinality of a weakly convex dominating set. Weakly convex domination number was first introduced by Jerzy Topp, Gdańsk University of Technology, 2002.

The classical paper of Nordhaus and Gaddum [4] established the following inequalities for the chromatic numbers χ and $\overline{\chi}$ of a graph G and its complement \overline{G} , where n = |V|:

$$2\sqrt{n} \le \chi + \bar{\chi} \le n+1,$$
$$n \le \chi \bar{\chi} \le \frac{(n+1)^2}{4}.$$

There are a large number of results in the graph theory literature of the form $\alpha + \bar{\alpha} \leq n \pm \epsilon$, where $\epsilon \in Q$, for a domination parameter α . Results of this form have previously been obtained for example for the domination number γ [3] and the connected domination number γ_c [2].

Theorem 1. For any graph G such that G and \overline{G} are connected,

1. $\gamma(G) + \gamma(\overline{G}) \le n+1$, 2. $\gamma_c(G) + \gamma_c(\overline{G}) \le n+1$.

We are concerned with analogous inequalities involving weakly convex domination number. For unexplained terms and symbols see [1].

2. Results

Since G and \overline{G} must be connected, we consider graphs G with $n(G) \ge 4$. We begin with the following result of Nordhaus-Gaddum type for weakly convex domination number.

Theorem 2. For any graph G such that G and \overline{G} are connected, $4 \leq \gamma_{wcon}(\overline{G}) + \gamma_{wcon}(\overline{G}) \leq n+2$.

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Proof. If there is an universal vertex in G, then \overline{G} is not connected. Thus there is no universal vertex in \overline{G} and no universal vertex in \overline{G} and hence $\gamma_{wcon}(G) \ge 2$ and $\gamma_{wcon}(\overline{G}) \ge 2$. Thus $\gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) \ge 4$. Notice that equality $\gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) = 4$ holds if $G \cong P_4$.

Of course $\gamma_{wcon}(G) \leq n$ and $\gamma_{wcon}(G) \leq n$. We consider some cases, depending on the diameter of G.

Case 1. If diam(G) = 1, then there is an universal vertex in G and \overline{G} are not connected.

Case 2. If $diam(G) \geq 3$, then let x, y be two vertices of V such that $d_G(x, y) = diam(G)$. Then $\{x, y\}$ is a weakly convex dominating set of \overline{G} and $\gamma_{con}(G) + \gamma_{con}(\overline{G}) \leq n+2$.

Case 3. Let diam(G) = 2. If $diam(G) \ge 3$, then we can exchange Gand \overline{G} and we have Case 2. Thus diam(G) = 2 and $diam(\overline{G}) = 2$. Let xbe any vertex of G. Since diam(G) = 2, for every $v \in V$ is $d_G(v, x) \le 2$. Let $Y = \{y \in V : d_G(x, y) = 1\}$ and $Z = \{z \in V : d_G(x, z) = 2\}$, |Y| = k, |Z| = l, where $k, l \ge 1$ (if l = 0, then there is an universal vertex in G and \overline{G} are not connected). Then n = k + l + 1 and it is easy to observe that $D = \{x\} \cup Y$ is a connected dominating set of G. For every two vertices u, v belonging to D, the distance between u and v is not greater than two and if $d_G(u, v) = 2$, then x belonging to D is on (u, v)-geodesic. Thus D is a weakly convex dominating set of G and $\gamma_{wcon}(G) \le |D| = k + 1$.

Since \overline{G} is connected and $diam(\overline{G}) = 2$, every vertex from Y has a neighbour in $\{x\} \cup Z$ in \overline{G} and hence $D' = \{x\} \cup Z$ is a connected dominating set of \overline{G} . For every two vertices u, v belonging to D', the distance between u and v is not greater than two and if $d_G(u, v) = 2$, then x belonging to D'is on (u, v)-geodesic. Thus D' is a weakly convex dominating set of \overline{G} and $\gamma_{wcon}(\overline{G}) \leq |D'| = l + 1$.

Thus
$$\gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) \leq k+1+l+1 \leq n+1 \leq n+2.$$

The next theorem concerns of the graphs G for which weakly convex domination number is equal to the number of vertices. Let g(G) denotes the girth of the graph G.

Theorem 3. If G is a connected graph with $\delta(G) \ge 2$ and $g(G) \ge 7$, then $\gamma_{wcon}(G) = n$.

Proof. Let G be a connected graph with $\delta(G) \geq 2$ and $g(G) \geq 7$. Suppose that $\gamma_{wcon}(G) < n$. Let D be a minimum weakly convex dominating set of G. Since $\gamma_{wcon}(G) < n$, there exists a vertex x in G such that $x \notin D$. Denote $N_G(x) = \{x_1, \ldots, x_p\}$, where $p \geq 2$ (because $\delta(G) \geq 2$). It is easy to observe that since $g(G) \geq 7$, for every x_i, x_j is $x_i x_j \notin E(G)$ for $1 \leq i, j \leq p$.

Notice that for every x_i, x_j , where $x_i \neq x_j$ and $1 \leq i, j \leq p$ we have $d_G(x_i, x_j) = 2$ and every shortest path between x_i and x_j contains x.

Suppose there are vertices $x_1, x_2 \in N_G(x)$ such that $x_1, x_2 \in D$. Then, since D is weakly convex, there is a vertex $v \in D$ such that $v \in N_G(x_1) \cap$ $N_G(x_2)$. But then we can find a cycle $C = (x_1, x, x_2, v, x_1)$ which length is less than seven, what gives a contradiction.

Thus $|N_G(x) \cap D| \leq 1$. Since x has to be dominated, we have $|N_G(x) \cap D| = 1$. Without loss of generality assume that $x_1 \in N_G(x) \cap D$. Thus, since $\delta(G) \geq 2$, there exists at least one vertex belonging to $N_G(x)$ say x_2 , such that $x_2 \notin D$. Since $\delta(G) \geq 2$ and x_2 is dominated, there exists a vertex $y \in N_G(x_2)$ such that $y \neq x$ and $y \in D$. Since $g(G) \geq 7$, we have $N_G(y) \cap N_G(x) = \emptyset$ and $N_G(y) \cap N_G(x_i) = \emptyset$, where $1 \leq i \leq p$.

Since D is a weakly convex set, $d_G(y, x_1) = 3$ and there is a $(x_1 - y)$ -geodesic P_1 such that all vertices of P_1 belong to D. Thus we have at least two $(x_1 - y)$ -geodesics: P_1 and $P_2 = (x_1, x, x_2, y)$ what produces a cycle of length less than seven. That gives contradiction with $g(G) \ge 7$ and hence we have $\gamma_{wcon}(G) = n$.

The simplest example of a graph G such that $\gamma_{wcon}(G) = n$ can be a graph $G = C_n$ with $n \geq 7$. For $\overline{C_n}$ we have $\gamma_{wcon}(\overline{C_n}) = 2$ and $\gamma_{wcon}(G) + \gamma_{wcon}(\overline{G}) = n + 2$.

Corollary 4. If G and \overline{G} are connected, $\delta(G) \geq 2$ and $g(G) \geq 7$, then $\gamma_{wcon}(\overline{G}) + \gamma_{wcon}(\overline{G}) = n + 2$.

Theorem 5. For any graph G such that G and \overline{G} are connected, $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq (\lfloor \frac{n}{2} \rfloor + 1)^2$. Furthermore, $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) = (\lfloor \frac{n}{2} \rfloor + 1)^2$ if and only if G or \overline{G} is isomorphic to C_5 .

Proof. Again we consider three cases, depending on the diameter of G. If diam(G) = 1, then $\gamma_{wcon}(G) = 1$ and \overline{G} is not connected.

If $diam(G) \ge 3$, then similarly like in the proof of Theorem 2, $\gamma_{wcon}(\overline{G}) = 2$ and since $n \ge 4$, $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \le 2n < (\lfloor \frac{n}{2} \rfloor + 1)^2$.

Let diam(G) = 2. Similarly like in the proof of the previous theorem, let x be any vertex of G, let $Y = \{y \in V : d_G(x, y) = 1\}$ and $Z = \{z \in V : d_G(x, z) = 2\}, |Y| = k, |Z| = l$, where $k, l \ge 1$.

If k = 1, then $\gamma_{wcon}(G) = 1$, there is an universal vertex in G and \overline{G} is not connected.

If k = 2, then, since $\{x\} \cup Y$ is a weakly convex dominating set of G, $\gamma_{wcon}(G) \leq 3$. Let $Y = \{u, v\}$. Notice that $\{x\}$ dominates itself and Z in \overline{G} and to dominate Y in \overline{G} , it is enough to take two vertices a, b from Z such that $au \in E(\overline{G})$ and $bv \in E(\overline{G})$ (such vertices a, b must exist since \overline{G} is connected and $diam(\overline{G}) = 2$). Since $a, b \in Z$, $ax \in E(\overline{G})$ and $bx \in E(\overline{G})$ and thus $\{x, a, b\}$ is a weakly convex dominating set of \overline{G} . Hence $\gamma_{wcon}(\overline{G}) \leq 3$.

Since G and \overline{G} are connected and diam(G) = 2, we have $|Z| \ge 2$ and $n \ge 5$. It is easy to observe that $\gamma_{wcon}(\overline{G}) \le (\lfloor \frac{n}{2} \rfloor + 1)^2$.

If $\gamma_{wcon}(G) = 3$, $\gamma_{wcon}(\overline{G}) = 3$ and n = 5 we have equality $\gamma_{wcon}(\overline{G}) = (\lfloor \frac{n}{2} \rfloor + 1)^2$ and C_5 realizes this equality. In the other cases we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < (\lfloor \frac{n}{2} \rfloor + 1)^2$.

Now let $k \ge 3$. Since $\{x\} \cup Y$ is a weakly convex dominating set of G, we have $\gamma_{wcon}(G) \le k + 1$. We consider three cases:

Case 1. If l > k, then $k < \lfloor \frac{n}{2} \rfloor$. Observe that x dominates itself and Z in \overline{G} . Since \overline{G} is connected and $diam(\overline{G}) = 2$, every vertex from Y has a neighbour in Z. Let $Y = \{y_1, \ldots, y_k\}$ and let $\{z_1, \ldots, z_k\}$ be the set of vertices from Z such that $y_1z_1 \in E(\overline{G}), \ldots, y_kz_k \in E(\overline{G})$. Thus $\{x\} \cup \{z_1, \ldots, z_k\}$ is a weakly convex dominating set of \overline{G} and $\gamma_{wcon}(\overline{G}) \leq$ k + 1. Hence $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq (k + 1)^2$ and since $k < \lfloor \frac{n}{2} \rfloor$, we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < (\lfloor \frac{n}{2} \rfloor + 1)^2$.

Case 2. If l = k, then $k \leq \lfloor \frac{n}{2} \rfloor$ and $l \leq \lfloor \frac{n}{2} \rfloor$. Since $\{x\} \cup Z$ is a weakly convex dominating set of \overline{G} , we have $\gamma_{wcon}(\overline{G}) \leq l+1$. Thus $\gamma_{wcon}(\overline{G}) \leq (k+1)(l+1) \leq (\lfloor \frac{n}{2} \rfloor + 1)^2$.

Case 3. If l < k, then $l < \lfloor \frac{n}{2} \rfloor$. Similarly like in Case 2 we have $\gamma_{wcon}(\overline{G}) \leq l+1$. Notice that $\{x\}$ dominates itself and Y in G and to dominate Z in G it is enough to take l vertices from Y. Thus $\gamma_{wcon}(G) \leq l+1$ and $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq (l+1)^2 < (\lfloor \frac{n}{2} \rfloor + 1)^2$.

We have already shown that for C_5 equality $\gamma_{con}(G)\gamma_{con}(\overline{G}) = (\lfloor \frac{n}{2} \rfloor + 1)^2$ holds. Conversely, let G be a graph for which we have equality. Then (from the earlier part of the proof) we have diam(G) = 2 and l = k.

If k = 2, then l = 2 and n = 5. Since diam(G) = 2, there is no end vertex in Z. Let $Z = \{z_1, z_2\}, Y = \{y_1, y_2\}$. If both z_1, z_2 have two neighbours in Y, then \overline{G} is not connected. If one vertex of Z, without loss of generality if z_1 has two neighbours in Y, then $\gamma_{wcon}(G) = 2 = \gamma_{wcon}(\overline{G})$ and $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < (\lfloor \frac{n}{2} \rfloor + 1)^2$. Thus every of vertices z_1 and z_2 has only one neighbour in Y. If z_1, z_2 have a common neighbour in Y, say y_1 , then y_1 is an end vertex in \overline{G} and $diam(\overline{G}) > 2$. Thus every vertex from Z has exactly one neighbour in Y and every vertex from Y has exactly one neighbour in Z, without loss of generality let $z_1y_1 \in E(G)$ and $z_2y_2 \in E(G)$. Since there is no end vertex in \overline{G} and $diam(\overline{G}) > 2$; hence $y_1y_2 \notin E(G)$, then we have an end vertex in \overline{G} and $diam(\overline{G}) > 2$; hence $y_1y_2 \notin E(G)$ and $G \cong C_5$.

Now let $l = k, k \ge 3$. We distinguish two cases.

- 1. There exists a vertex $y \in Y$ such that $PN[y, Y] = \emptyset$. Then $(\{x\} \cup Y) \{y\}$ is a weakly convex dominating set of G and $\gamma_{con}(G) \leq k$. Since $\{x\} \cup Z$ is a weakly convex dominating set of \overline{G} , we have $\gamma_{wcon}(\overline{G}) \leq l+1$ and since $k \leq \lfloor \frac{n}{2} \rfloor$ and $l \leq \lfloor \frac{n}{2} \rfloor$, we have $\gamma_{wcon}(\overline{G}) \leq k(l+1) \leq \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1) < (\lfloor \frac{n}{2} \rfloor + 1)^2$.
- 2. For every $y \in Y$ we have $PN[y, Y] \neq \emptyset$. Let us denote $Y = \{y_1, \ldots, y_k\}$, $Z = \{z_1, \ldots, z_k\}$ and $PN[y_1, Y] = \{z_1\}, \ldots, PN[y_k, Y] = \{z_k\}$. Then $\{x, z_1, z_2\}$ is a weakly convex dominating set of \overline{G} and $\gamma_{wcon}(\overline{G}) \leq 3$. Thus we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq 3(k+1) < (\lfloor \frac{n}{2} + 1)^2 \rfloor$.

Hence if $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) = (\lfloor \frac{n}{2} \rfloor + 1)^2$, then $G \cong C_5$.

Corollary 6. If G and \overline{G} are connected, $diam(G) \leq 2$ and $G \neq C_5$, then $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1).$

Theorem 7. If G and \overline{G} are connected, $G \neq C_7$ and $G \neq C_5$, then $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1).$

Proof. Let G be a graph such that G and \overline{G} are connected and $G \neq C_5$ and $G \neq C_7$. From Corollary 6, if $diam(G) \leq 2$, then $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)$; so let $diam(G) \geq 3$. Then $\gamma_{wcon}(\overline{G}) = 2$ and $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) \leq 2n \leq \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)$ for $n \geq 8$.

Since $diam(G) \ge 3$ and G, \overline{G} are connected, we have $n \ge 4$. If n = 4, then $G \cong \overline{G} \cong P_4$ and $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < |\frac{n}{2}|(|\frac{n}{2}|+1)$.

If n = 5, then $\gamma_{wcon}(G) \leq 3$ and since $\gamma_{wcon}(\overline{G}) = 2$ we have $\gamma_{wcon}(\overline{G}) \gamma_{wcon}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1).$

If n = 6, then $\gamma_{wcon}(G) \leq 4$ and since $\gamma_{wcon}(\overline{G}) = 2$ is $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1).$

If n = 7, then, since $G \neq C_7$, we have $\gamma_{wcon}(G) \leq 5$ and since $\gamma_{wcon}(\overline{G}) = 2$, again we have $\gamma_{wcon}(G)\gamma_{wcon}(\overline{G}) < \lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)$.

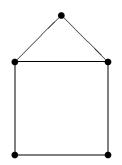


Figure 1. Graph G_1 .

The example of the extremal graph of Theorem 7 can be the graph G_1 from Figure 1.

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