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CHVÁTAL-ERDÖS TYPE THEOREMS

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Abstract

The Chvátal-Erdös theorems imply that if G is a graph of order $n \geq 3$ with $\kappa(G) \geq \alpha(G)$, then G is hamiltonian, and if $\kappa(G) > \alpha(G)$, then G is hamiltonian-connected. We generalize these results by replacing the connectivity and independence number conditions with a weaker minimum degree and independence number condition in the presence of sufficient connectivity. More specifically, it is noted that if G is a graph of order n and $k \geq 2$ is a positive integer such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - k)/(k + 1)$, and $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian. It is shown that if G is a graph of order n and $k \geq 3$ is a positive integer such that $\kappa(G) \geq 4k^2+1$, $\delta(G) > (n+k^2-2k)/k$, and $\delta(G) \geq \alpha(G)+k-2$, then G is hamiltonian-connected. This result supports the conjecture that if G is a graph of order n and $k \geq 3$ is a positive integer such that $\kappa(G) \geq k$, $\delta(G) > (n+k^2-2k)/k$, and $\delta(G) \geq \alpha(G)+k-2$, then G is hamiltonian-connected, and the conjecture is verified for $k = 3$ and 4.

Keywords: Hamiltonian, Hamiltonian-connected, Chvátal-Erdös condition, independence number.

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1. Introduction

We deal only with finite simple graphs and our notation generally follows the notation of Chartrand and Lesniak in [1]. Given a subset of vertices (or subgraph) H of a graph G and a vertex v, let $d_H(v)$ denote the degree of v relative to H, and $N_H(v)$ the neighborhood of v in H. The minimum degree, independence number, and connectivity of G will be denoted by $\delta(G)$, $\alpha(G)$, and $\kappa(G)$ respectively.

Two classical results of Chvátal and Erdös [2] are the following:

Theorem 1. *If G is a graph of order $n \geq 3$ such that $\kappa(G) \geq \alpha(G)$, then G is hamiltonian.*

Theorem 2. *If G is a graph of order $n \geq 3$ such that $\kappa(G) > \alpha(G)$, then G is hamiltonian-connected.*

The following result on the existence of hamiltonian cycles, which is an analogue of Theorem 1, will be proved. Actually, we note that Proposition 1 is an easy consequence of a result of Fraisse [5] and follows from a result of Ota [6] with the appropriate interpretation of the condition on α.

Proposition 1. *Let G be a graph of order n and $k \geq 2$ a positive integer such that $\kappa(G) \geq k$, $\delta(G) > (n+k^2-k-1)/(k+1)$. If $\delta(G) \geq \alpha(G)+k-2$, then G is hamiltonian.*

Corresponding to the hamiltonian result of Theorem 1 and an analogue to Theorem 2, we make the following conjecture.

Conjecture 1. Let G be a graph of order n and $k \geq 3$ a positive integer such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.

We prove the following two results supporting Conjecture 1. The first result has the same degree and independence number conditions and conclusion as the conjecture, but requires a higher connectivity assumption on the graph. The second result verifies the conjecture for the cases $k = 3$ and 4.

Theorem 3. *Let G be a graph of sufficiently large order n and $k \geq 3$ a positive integer such that $\kappa(G) \geq 4k^2 + 1$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.*

Theorem 4. *Let G be a graph of sufficiently large order n and $k = 3$ or 4 such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.*

In Section 2 we will give a short proof of Proposition 1 and present a family of graphs which show that none of the conditions in Proposition 1 and Theorem 4 can be weakened. In Section 3 we prove the main results.

2. Preliminary Results and Sharpness Examples

We begin by describing graphs H_i for $1 \leq i \leq 5$ which demonstrate the sharpness of the conditions in Propositon 1 and Theorem 4.

For $k \geq 2$, let $H_1(k) = K_k + (k+1)K_{(n-k)/(k+1)}$, where $n \equiv k \bmod (k+1)$. Since there are $k+1$ components in the graph $H_1(k) - K_k$, the graph $H_1(k)$ is not hamiltonian. Also, $\kappa(H_1(k)) = k$, $\delta(H_1(k)) = (n+k^2-k-1)/(k+1)$, and $\alpha(H_1(k)) = k+1 \leq \delta(H_1(k))$ for n sufficiently large.

For $k \geq 2$, let $H_2(k) = K_{(n-k)/(k+1)} + ((n+1)/(k+1))K_k$ where $n \equiv k \bmod (k+1)$. Since there are strictly more than $(n-k)/(k+1)$ components in $H_2(k) - K_{(n-k)/(k+1)}$, the graph $H_2(k)$ is not hamiltonian. Also, $\delta(H_2(k)) = (n+k^2-k-1)/(k+1)$, and $\alpha(H_2(k)) = (n+1)/(k+1) = \delta(H_2(k)) - (k-2)$.

For $k \geq 3$, let $H_3(k) = K_k + kK_{(n-k)/k}$, when $n \equiv 0 \bmod k$. Since there are as many components in $H_3(k) - K_k$ as there are vertices in K_k, the graph $H_3(k)$ is not hamiltonian-connected. Also, $\kappa(H_3(k)) = k$, $\delta(H_3(k)) = (n + k^2 - 2k)/k$, and $\alpha(H_3(k)) = k \leq \delta(H_1(k))$ for n sufficiently large.

For $k \geq 3$, let $H_4(k) = K_{n/k} + (n/k)K_{k-1}$ where $n \equiv 0 \bmod k$. The graph $H_4(k)$ is not hamiltonian-connected, since there are as many components in $H_4(k) - K_{n/k}$ as there are vertices in $K_{n/k}$. Also, $\delta(H_4(k)) = (n+k^2-2k)/k$, and $\alpha(H_4(k)) = n/k = \delta(H_4(k)) - (k-2)$.

For $(n+1)/3 < \delta(G) < n/2$, the graph $H_5(\delta) = K_\delta + (\overline{K}_\delta \cup K_{n-2\delta})$ is δ-connected, not hamiltonian and not hamiltonian-connected, and $\alpha(H_5(\delta)) = \delta(H_5(\delta)) + 1$.

The graph $H_1(k)$ implies the minimum degree condition of Theorem 1 cannot be decreased for $k \geq 2$. Note that the graph $H_2(k-1)$ satisfies the relationship $\delta(H_2(k-1)) = \alpha(H_2(k-1)) + k - 3$, and all of the other conditions of Theorem 1, so the conditions cannot be decreased in Theorem 1 for $k \geq 3$. The graph $H_5(\delta)$ verifies the sharpness of Theorem 1 when $k = 2$.

The graph $H_3(k)$ implies that the minimum degree conditions of Conjecture 1 and Theorem 4 cannot be decreased. Note that the graph $H_4(k-1)$ satisfies the relationship $\delta(H_4(k-1)) \geq \alpha(H_4(k-1)) + k - 3$ and all of the other conditions of Conjecture 1, so the conditions cannot be decreased in Conjecture 1 and also Theorem 4 for $k \geq 4$. The graph $H_5(\delta)$ verifies the sharpness of Conjecture 1 and Theorem 4 when $k = 3$.

The above examples verify the sharpness of Theorem 1, Conjecture 1 and Theorem 4 when n satisfies the appropriate congruence relative to k. Analogous examples, which are less symmetric, can be described for general n.

Before starting our main proofs, for convenience we describe additional notation. Given a positive integer p, $\sigma_p(G) = \min\{d(v_1)+d(v_2)+\cdots+d(v_p) : \{v_1, v_2, \ldots, v_p\}$ is an independent set of $G\}$. Given a positive integer λ, a cycle C in a graph G is a *D_λ-cycle* if each component of $G - C$ has fewer than λ vertices. With this notation, we can state the following result of Fraisse in [5].

Theorem A. *If G is a k-connected graph of order $n \geq 3$ with $\sigma_{k+1}(G) \geq n + k(k-1)$, then G contains a D_k-cycle.*

***Proof* (*of Proposition* 1).** Select a maximal length cycle C of G that is a D_k-cycle. Theorem A implies that such a cycle exists. If C is hamiltonian, then the proof is complete, so assume not. Select a vertex v in one of the components, say H, of $G - C$. Since $|H| \leq k-1$, we know $d_H(v) \leq k-2$, and so $d_C(v) \geq \delta(G) - k + 2$. Let $S = N_C(v)$, and let S^+ denote the successors of S on C for some orientation of C. Since C is a maximal length D_k-cycle, the set $S^* = S^+ \cup \{v\}$ is an independent set with at least $\delta(G) - k + 3$ vertices. This is a contradiction, since $\alpha(G) \leq \delta(G) - k + 2$. This completes the proof of Proposition 1. ■

Proposition 1 is also a direct consequence of the main result of Ota [6], and is related to the following corollary of the main result of Ota [6].

Theorem B. *Let G be a k-connected graph of order $n (n \geq 3)$ with $\alpha(G) \leq (n+1)/(k+1) + 1$. If $\sigma_{k+1}(G) \geq n + k^2 - k$, then G is hamiltonian.*

3. MAIN RESULTS

Theorem 3. *Let G be a graph of sufficiently large order n and $k \geq 3$ a positive integer such that $\kappa(G) \geq 4k^2 + 1$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.*

Proof. Suppose that G is not hamiltonian-connected, and select distinct vertices $x, y \in V(G)$ and let P be a path in G from x to y with the maximum number of vertices, say $m < n$. Let $t = 4k^2 + 1$, so $\kappa(G) \geq t$.

We would like to show that each vertex of $H = G - P$ may be adjacent to at most $\alpha(G)$ vertices of P. Suppose a vertex $v \in H$ has a set S of at least $\alpha(G) + 1$ adjacencies in P. Since, for either orientation of P, at most one vertex of S will not have a successor, we know $|S^+| \geq |S| - 1$. Because P is a path of maximum length, $S^+ \cup \{v\}$ must be an independent set of order at least $\alpha(G)+1$, which is a contradiction. Thus, $\delta(H) \geq \delta(G) - \alpha(G) \geq k-2$.

Let s be the cardinality of a maximum length cycle $C \subseteq H$, if a cycle exists. If H has no cycle, let $s = 1$.

Claim 1. $s < 2(m-1)/(t-1)$.

Proof. Since the conditions of Theorem 3 imply that G is hamiltonian, so there is a path between x and y that contains at least $(n+1)/2$ vertices. Thus, we have $m \geq \delta(G)$, since $\delta(G) \geq (n+1)/2$ would imply hamiltonian-connected by the result of Dirac ([3]). Assume $s < t = 4k^2 + 1$. Then, since $m \geq \delta(G) \geq (n + k^2 + k - 1)/(k+1)$, for n sufficiently large, $s < 2(m-1)/(t-1)$. Thus, assume $s \geq t$. There exist t vertex-disjoint paths between C and P. Two of these paths, say Q_1 and Q_2, have end vertices in P with at most $(m-1)/(t-1) - 1$ vertices of P between them. The end vertices of Q_1 and Q_2 on C have at least $(s-2)/2$ vertices between them in one direction around the cycle C. The maximality of P implies that $(s-2)/2+2 \leq (m-1)/(t-1)-1$. Therefore, $s < 2(m-1)/(t-1)$. This completes the proof of the claim. ■

Claim 2. The order of the path P is given by $m \geq kn/(k+1)$.

Proof. Our proof is by contradiction. Since the longest cycle in H has length s, the endvertices of a longest path in H have degree less than s. Therefore, there exist two vertices, say u and v, such that $d_H(u), d_H(v) \leq s$. Furthermore, since $\delta(H) \geq k-2$, this path, denoted by Q, from u to v in H has at least $k-1$ vertices.

Since P is a maximum length path, no vertex of H can be adjacent to two consecutive vertices of P. Let $U = N_P(u), V = N_P(v)$, $W = U \cap V$ and, without loss of generality, assume that $|U| \leq |V|$. Thus, there are $|U| + |V| - |W|$ vertices of P adjacent to either u or v, and there are the same number, or possibly one more or one less of "open" intervals of P with no adjacencies to either u or v. Since the path P is of maximum length, each "open" interval contains at least one vertex, and those intervals between a vertex in U and a vertex in V, which will we call "long" intervals, contain at least $k-1$ vertices. There are at least $|W|-1$ such "long" intervals. Thus,

$$\begin{aligned} m &\geq |U|+|V|-|W|+(k-1)(|W|-1)+(|U|+|V|-2|W|) \\ &= 2|U|+2|V|+(k-4)|W|-k+1. \end{aligned} \tag{1}$$

All of the vertices in $(U \cup V)^+ \cup \{u\}$ are independent, since any edge between vertices in this set would contradict the fact that P was chosen of maximum length. This implies that $|U|+|V|-|W| \leq \alpha(G) \leq \delta(G)-k+2$. Therefore $(\delta(G)-d_H(u)) + (\delta(G)-d_H(v)) - \delta(G)+k-2 \leq |W|$, and so

$$|W| \geq \delta(G) - 4(m-1)/(t-1)+k.$$

Hence, by Equation 1, we get

$$\begin{aligned} m &\geq 4(\delta(G)-s+1)+(k-4)|W|-k+1 \\ &\geq 4(\delta(G)-2(m-1)/(t-1)+1) \\ &\quad + (k-4)(\delta(G)-4(m-1)/(t-1)+k)-k+1. \end{aligned}$$

Using the bounds $m < kn/(k+1)$ and $\delta(G) \geq (n+k^2-2k)/k$ in the previous equation gives

$$(t+4k-9)kn/(k+1) > (t-1)(n+k^2-2k) + (t-1)(k^2-5k+5) + 4k-8.$$

However, this implies $t \leq 4k^2 - 8k + 1$ which is a contradiction. Finally we may conclude that $|P| = m \geq kn/(k+1)$ completing the proof of the claim. ∎

Assume that P is not a hamiltonian path. Select a longest path Q in H, say with q vertices and with end vertices u and v. Note that since $\delta(H) \geq k-2$, we get $q \geq k-1$. Recall that each vertex of H has at most $\alpha(G)$ adjacencies in P. In fact, $|N_P(Q)| \leq \alpha(G)$ for the same reason. Assume that $s = d_H(u) \geq d_H(v)$, thus u has $s \geq k-2$ adjacencies on Q. Denote the predecessors of these s vertices by $\{u = u_1, u_2, \dots u_s\}$. Between v and any of the vertices u_i for $1 \leq i \leq s$, there is a path with q vertices, and between u_i and u_j for $i \neq j$ there is a path with at least $(s+1)/2$ vertices. There is no loss of generality in assuming that $d_H(u_i) \leq s$ for all i, and so each vertex of $S = \{u_1, u_2, \dots, u_s, v\}$ has at least $\delta(G) - s$ adjacencies in P.

As before, let $U = N_P(u), V = N_P(v)$, $W = U \cap V$, and assume that $|U| \leq |V|$. Since $d_P(u), d_P(v) \geq (n/k) + k - 2 - s$ and $\alpha(G) \leq n/k$, we have $|W| = |U| + |V| - |U \cup V| \geq 2((n/k) + k - 2 - s) - n/k = (n/k) + 2k - 4 - 2s$. This implies

$$\begin{aligned} n - s - 1 \geq n - q &\geq (s+2)|W| + (|U| - |W|) + (|V| - |W|) - 1 \\ &\geq m \geq (s+2)((n/k) + 2k - 4 - 2s) + 2(s + 2 - k) - 1. \end{aligned}$$

Therefore, we know

$$2s^2 - ((n/k) + 2k - 5)s + n - (2n)/k - 2k + 4 \geq 0.$$

However, for $k - 2 \leq s \leq n/(2k) - 1$ the previous inequality is contradicted, so we can assume that $s > n/(2k) - 1$. We have previously shown in Claim 1 that $s \leq n/(2k^2)$. Therefore, the assumption that the path P was not a hamiltonian path between x and y is contradicted. This completes the proof of Theorem 3. ∎

The conditions of Theorem 3 are sharp except for the condition on the connectivity $\kappa(G)$, but for small values of k, we prove Theorem 4 which uses the sharp condition for $\kappa(G)$.

Theorem 4. *Let G be a graph of sufficiently large order n and $k = 3$ or 4 such that $\kappa(G) \geq k$, $\delta(G) > (n + k^2 - 2k)/k$. If $\delta(G) \geq \alpha(G) + k - 2$, then G is hamiltonian-connected.*

Proof. Suppose that G is not hamiltonian-connected, and select distinct vertices x and y and let P be a path of G from x to y with a maximum number of vertices, say $m < n$. Let $H = G - P$. If $\kappa(G) \geq 4k^2 + 1$, then the proof is complete by Theorem 3, so we can assume that $k \leq \kappa(G) \leq 4k^2$.

We would like to show each vertex of H can be adjacent to at most $\alpha(G)$ vertices of P. Otherwise, a vertex $v \in H$ has a set S of at least $\alpha(G) + 1$ adjacencies in P. Note that $|S^+| \geq |S| - 1$, since at most one vertex of S will not have a successor. Since P is a path of maximum length, this implies that $S^+ \cup \{v\}$ is an independent set of order at least $\alpha(G) + 1$, a contradiction. Thus, $\delta(H) \geq \delta(G) - \alpha(G) \geq k - 2$.

We next show that $|P| \geq (k-1)n/k + k - 1$, so assume not. Select a minimal cutset S of G, and let $C_1, C_2, \ldots, C_t$ be the components of $G - S$. Thus, $k \leq |S| = s \leq 4k^2$, and we can assume that $|C_1| \geq |C_2| \geq \cdots \geq |C_t|$.

First consider the case $k = 3$, and so $\delta(G) \geq n/3 + 2$, $\alpha(G) \leq \delta(G) - 1$, and $\kappa(G) \leq 36$. If $t \geq 4$, then $|C_t| \leq (n - s)/t$, and for any vertex $v \in C_t$, $d(v) \leq (n - s)/t + s - 1 < n/3$, a contradiction. Therefore, $t \leq 3$. If $t = 3$, then $n/3 + s - 6 \geq |C_1| \geq |C_2| \geq |C_3| \geq n/3 + 3 - s$, and $\delta(C_i) \geq n/3 - s + 2$ for each i. If $s = 3$, then for any vertex v in C_3, $d(v) \leq (n-3)/3 + 2$, a contradiction. Thus, $s \geq 4$. Each of the graphs C_i are nearly complete, and are hamiltonian-connected even after the deletion of any small number of vertices. Also, there is an s-matching between S and each of the components C_i, since S is a minimal cut set. Since $s \geq 4$, independent of the location of the vertices x and y, it is an easy and straightforward case analysis to show that there is a path P from x to y containing all of the vertices of $G - S$ and either $2, 3$ or 4 vertices of S. Thus, in this case, $|P| \geq n - s + 2 > 2n/3 + 2$.

We now consider the case when $t = 2$. Hence, $n/3 + 3 - s \leq |C_2| \leq |C_1| \leq 2n/3 - 3$, and $\delta(C_i) \geq n/3 + 2 - s$ for each i. The component C_2 is hamiltonian-connected, but if $|C_1| \leq 2n/3 + 3 - 2s$, the component C_1 is also hamiltonian-connected. Consider the case when C_1 is hamiltonian-connected. If one of x or y is not in C_1, then it is straightforward to form a path from x to y using all of the vertices of C_1 and C_2 along with 2 or 3 vertices of S. This implies $|P| \geq n - s + 2 > 2n/3 + 2$. If x and y are both in C_1 then there is a hamiltonian path Q in C_1 from x to y. There is also a matching with s edges between S and Q. Using two of these s edges, whose end vertices in Q are of minimum distance apart on Q, along with a hamiltonian path of C_2 gives a path P from x to y of length at least $n - (|C_1| - s)/(s-1) - (s-2) \geq n - (2n/3 - 3 - 3s)/(s-1) - s + 2 > n/3 + 2$. Thus, we can assume that C_1 is not hamiltonian-connected, and so $|C_1| \geq 2n/3 + 4 - 2s$.

If $\kappa(C_1) \leq 2$, then there is a cut set, say S', with $|S'| = 1$ or 2, such that $C_1 - S'$ has two components, say C_1' and C_1''. The minimum degree in each component is at least $n/3 - s$, so each of these components, and also C_2 is nearly complete. Also, there are $s - 2$ independent edges between S and each of the components C_1' and C_1'' and s independent edges between S and C_2. Hence, just as in the case when there were 3 components of $G - S$, it is an easy and straightforward case analysis to find a path P with at least $n - s + 2$ vertices from x to y, independent of the location of x and y. Therefore, we can assume that $\kappa(C_1) \geq 3$.

Since C_1 is 3-connected, by a result of Dirac [3], there is a cycle C in C_1 of length at least $2n/3 + 4 - 2s$, and there are s independent paths from S to C. Select two end vertices of these s paths that have a minimum distance between them on C. If x and y are not in C_1, then a path from x to y can be formed using all of the vertices of C_2 (since C_2 is hamiltonian-connected), at least two vertices of S, and all of the vertices of C except for possibly $(|C| - s)/s$. Thus the path P will have at least $(n/3 + 3 - s) + 2 + (s - 1)(2n/3 + 4 - 2s)/s \geq 7n/9 + 7 - 3s > 2n/3 + 2$ vertices. If x and y are in C_1, then by a result of Enomoto in [4] there is a path between x and y with at least $2n/3 + 4 - 2s$ vertices. Thus, just as in the case of the cycle, a path with at least $7n/9 + 8 - 5s/2 > 2n/3 + 2$ vertices can be formed. If $x \in C_1$ and $y \notin C_1$, then using a path Q from x to some vertex z in C_1 with a least $2n/3 + 4 - 2s$ vertices, a path between x and y can be formed using all of the vertices of Q and C_2, thereby using more than $2n/3 + 2$ vertices. This completes the proof of the claim that there is a path from x to y with at least $2n/3 + 2$ vertices.

Let P be a path between x and y of maximum length m, and let $H = G - P$. Select a path Q with a maximum number of vertices, say q, in H with end vertices u and v. Without loss of generality, let $s = d_H(u) \geq d_H(v)$, which means u has s adjacencies in Q and $q \geq s+1$. Denote the predecessors of these s vertices by $S' = \{u = u_1, u_2, \ldots, u_s\}$, and let $S = S' \cup \{v\}$. Observe that no vertex of H can be adjacent to two consecutive vertices of P, since P is a maximum length path. Let $U = N_P(u), V = N_P(v)$, $W = U \cap V$, and so $|U|, |V| \geq \delta(G) - s + 1$. There are $|U| + |V| - |W|$ vertices of P adjacent to either u or v, and there are the same number or possibly one more or one less "open" interval of P with no adjacencies to either u or v. Since the path P is of maximal length, each of the "open" intervals will have at least one vertex, and those intervals between a vertex in U and a vertex in V, which will we call "long" intervals, will have at least q vertices.

If $U = V = W$, then $|W| \geq n/3 + 3 - q$, and there will be at least $|W| - 1$ "long" intervals. Hence,

$$n - q \geq m \geq (q + 1)(n/3 + 2 - q) + 1.$$

This implies the inequality $q^2 - (n/3 + 2)q + 2n/3 - 3 \geq 0$. However, for $2 \leq q \leq n/3$, this gives a contradiction. If $q = 1$, then u has at least $\delta(G)$ adjacencies in P, which implies the existence of an independent set of order $\delta(G) > \alpha(G)$, a contradiction.

In general, if s is small, then u and v will have nearly identical neighborhoods in P. More specifically, $|U \cup V| \leq \alpha(G) \leq \delta(G) - 1$ to avoid an independent set with more than $\alpha(G)$ vertices. Since $|U|, |V| \geq \delta(G) - s$, this implies that $|U \cap V| \geq \delta(G) - 2s + 1$. An immediate consequence of this is that there are $\delta(G) - 2s$ vertex disjoint intervals of P (between the common adjacencies of u and v on P) each with at least $s+2$ vertices. This implies

$$n \geq (s + 1) + (\delta(G) - 2s)(s + 2) + 1 \geq (s + 1) + (n/3 - 2s + 2)(s + 2) + 1,$$

which is a contradiction for $s \leq 3$. Thus, we assume $q \geq 5$ and $s \geq 4$. When $U \neq V$, we get

$$\begin{aligned} n - q \geq m &\geq |U| + |V| - |W| + q|W| + (|U| + |V| - 2|W|) - 1 \\ &= 2|U| + 2|V| + (q - 3)|W| - 1. \end{aligned}$$

Since $n > 2|U| + 2|V|$, we know $|U| < n/4$ and $s \geq n/3 + 2 - n/4 > n/12$, for otherwise this gives a contradiction.

Let R be the set of r vertices of P with at least two adjacencies in S. The interval between two adjacencies of distinct vertices of S will have at least $(s+1)/2$ vertices. If $r \geq \delta(G)-s$, then there will be at least $\delta(G)-s-1$ distinct intervals of P with at least $(s+1)/2$ vertices with no adjacencies in Q, and one of the intervals will have at least q such vertices. This implies that

$$n - q \geq m \geq (\delta(G) - s - 2)(s + 1)/2 + q + \delta(G) - s.$$

The previous equation implies that $s^2-(\delta(G)-5)s-3\delta(G)+2+2n-4q \geq 0$. However, this is a contradiction for $4 \leq s < n/3-3$, so we may assume that $r < \delta(G) - s$.

There are at least $r-1$ distinct intervals of P with at least $(s+1)/2$ vertices with no adjacencies in S and also an additional s intervals with this same property because the predecessor and the successor of the interval are from distinct vertices of S. There are at least $s(\delta(G)-s-r-1)$ additional intervals with at least one vertex with no adjacencies in S. This implies that

$$n-s-1 \geq m \geq (r-1)(s+3)/2 + s(s+3)/2 + s(\delta(G)-s-r-1)2.$$

Since the lower bound on m in the previous equation is a decreasing function of r, this implies that the extreme value when $r = \delta(G)-s-1$ is also a lower bound, and so $n \geq (\delta(G)-2)(s+3)/2+s+1 > (n/3)(7/2)+4$, a contradiction. This completes the proof of the case $k=3$. The proof of the case $k=4$ is identical, except the analysis to show that there is a path between x and y with at least $3n/4+3$ vertices is much more tedious. This completes the proof of Theorem 4. ∎

4. Questions

The obvious problem is to extend Theorem 4 to all values of $k \geq 5$ and verify Conjecture 1 when the order n of the graph is sufficiently large. It is also desirable to be able to drop the n sufficiently large condition.

Many degree conditions that imply a graph is hamiltonian have analogues that imply much more, such as panconnected, hamiltonian ordered, etc. Are there similar analogues for the Chvátal-Erdös type conditions?

References

[1] G. Chartrand and L. Lesniak, Graphs and Digraphs (Chapman and Hall, London, 1996).

[2] V. Chvátal and P. Erdös, *A note on Hamiltonian circuits*, Discrete Math **2** (1972) 111–113.

[3] G.A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. **2** (1952) 69–81.

[4] H. Enomoto, *Long paths and large cycles in finite graphs*, J. Graph Theory **8** (1984) 287–301.

[5] P. Fraisse, *D_λ-cycles and their applications for hamiltonian cycles*, Thése de Doctorat d'état (Université de Paris-Sud, 1986).

[6] K. Ota, *Cycles through prescribed vertices with large degree sum*, Discrete Math. **145** (1995) 201–210.

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