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ON LOCATING-DOMINATION IN GRAPHS

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Abstract

A set D of vertices in a graph $G = (V, E)$ is a locating-dominating set (LDS) if for every two vertices u, v of $V - D$ the sets $N(u) \cap D$ and $N(v) \cap D$ are non-empty and different. The locating-domination number $\gamma_L(G)$ is the minimum cardinality of a LDS of G, and the upper locating-domination number, $\Gamma_L(G)$ is the maximum cardinality of a minimal LDS of G. We present different bounds on $\Gamma_L(G)$ and $\gamma_L(G)$.

Keywords: upper locating-domination number, locating-domination number.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The *open neighborhood* $N(v)$ of a vertex v consists of the vertices adjacent to v, the *closed neighborhood* of v is defined by $N[v] = N(v) \cup \{v\}$ and $d_G(v) = |N(v)|$ is the *degree* of v. We denote by $\delta(G)$ the *minimum degree* of a graph G.

A set $D \subseteq V$ is a *dominating set* if every vertex of $V - D$ has a neighbor in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A set $D \subseteq V$ is a *locating-dominating set* **(LDS)** if it is dominating and every two vertices x, y of $V - D$ satisfy $N(x) \cap D \neq N(y) \cap D$. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of an LDS of G, and the *upper locating-domination number*, $\Gamma_L(G)$ is the maximum cardinality of a minimal LDS of G. An LDS of minimum cardinality is called a $\gamma_L(G)$-set, and we define a $\Gamma_L(G)$-set likewise. Locating-domination was introduced by Slater [12, 13]. For recent studies on locating-domination we cite [3, 4] and [5]. The *independence number* $\beta_0(G)$ and the *independent domination number* $i(G)$ are the maximum and the minimum cardinality of a set that is both independent and dominating in G, respectively.

So far as we know, no work has been done on the upper locating-dominating number. In this paper we present some bounds on the upper locating-dominating number as well as for the locating-domination number of a graph G. Before presenting the main results, we introduce some definitions and notation.

A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. We denote the set of leaves of a graph G by $L(G)$, the set of support vertices by $S(G)$, and let $\ell(G) = |L(G)|$ and $s(G) = |S(G)|$. Denote by T_x the subtree induced by a vertex x and its descendants in a rooted tree T. The *diameter* $\mathrm{diam}(G)$ of a graph G is the maximum distance over all pairs of vertices of G. The *girth* of a graph G, denoted $g(G)$, is the length of a shortest cycle (if any) in G. Note that if the graph does not contain any cycles, then its girth is defined to be infinity. The *corona* of a graph G is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. We denote by P_n and C_n the path and the cycle on n vertices, respectively.

2. Upper Locating-Domination Number

It is known that every connected graph G of order $n \geq 2$ verifies $\gamma_L(G) \leq n - 1$. We will see that this bound remains valid for the upper locating-domination number and we characterize the graphs achieving equality.

Theorem 1. *Every connected nontrivial graph G of order n satisfies $\Gamma_L(G) \leq n-1$, with equality if and only if G is a complete graph or a star.*

Proof. Clearly the upper bound follows from the fact that the entire vertex set of G is a locating-dominating set but is not minimal.

Let S be a $\Gamma_L(G)$-set of size $n-1$ and assume that $V - S = \{u\}$. First assume that $A = S - N(u) \neq \emptyset$. The minimality of S implies that A is independent for otherwise for some vertex $z \in A$ having a neighbor in $A, S - \{z\}$ is an LDS of G, a contradiction. Since G is connected, every vertex $v \in A$ has at least one neighbor in S. Also $N(v) = N(u)$ for otherwise $S - \{v\}$ is an LDS of G, a contradiction. Now if $|N(u)| \geq 2$, then since every vertex $u' \in N(u)$ is adjacent to all A, $S - \{u'\}$ is an LDS of G, contradicting the minimality of S. Hence $|N(u)| = 1$, and so G is a star. Now assume that $A = \emptyset$. If all edges exist between the vertices of S, then G is a complete graph. So let x, y be any two non-adjacent vertices of S. If x has a neighbor in S, then $S - \{x\}$ is an LDS of G, a contradiction. Thus x is isolated in S and likewise for y. It follows that S is independent and hence G is a star.

The converse is obvious. ■

As an immediate consequence of Theorem 1 we obtain the following corollary to disconnected graphs.

Corollary 2. *If G is a graph of order n with $\delta(G) \geq 1$ and p components, then $\Gamma_L(G) \leq n-p$, with equality if and only if every component is a complete graph or a star.*

Next we give the exact value of the upper locating-domination number for paths.

Theorem 3. *For every path P_n,*

$$\Gamma_L(P_n) = \begin{cases} 4k & \textit{if} \quad n = 7k, \\ 4k+1 & \textit{if} \quad n = 7k+1 \ \textit{or} \ n = 7k+2, \\ 4k+2 & \textit{if} \quad n = 7k+3 \ \textit{or} \ n = 7k+4, \\ 4k+3 & \textit{if} \quad n = 7k+5, \\ 4k+4 & \textit{if} \quad n = 7k+6. \end{cases}$$

Proof. We use an induction on the order n. It is a easy to check the result for $n \leq 7$. Let $n \geq 8$ and assume that every path $P_{n'}$ of order n'

with $1 \leq n' < n$ satisfies the theorem. Let P_n be a path with $V(P_n) = \{u_1, u_2, \ldots, u_n\}$, and let D be any $\Gamma_L(P_n)$-set. We claim that there is such a D with $|D \cap \{u_1, u_2, \ldots, u_7\}| = 4$. First we note that no three consecutive vertices are in D. It follows that $|D \cap \{u_1, u_2, \ldots, u_7\}| \leq 4$. Suppose now that $|D \cap \{u_1, u_2, \ldots, u_7\}| \leq 3$. If $u_7 \in D$, then u_2, u_4 should be in D but then $\{u_1, u_3\} \cup D - \{ u_2\}$ is a minimal LDS of P_n larger than D, a contradiction. Thus $u_7 \notin D$. Clearly $D' = D \cap \{ u_8, \ldots, u_n\}$ is an LDS of the path $P_{n'} = P_n - \{u_1, u_2, \ldots, u_7\}$. If D' is minimal, then $\{u_1, u_2, u_5, u_6\} \cup D'$ is a minimal LDS larger than D. Hence we assume that D' is not minimal and so there is a vertex $v \in D'$ such that $D' - \{v\}$ is a minimal LDS for $P_{n'}$. But then $\{u_1, u_2, u_5, u_6\} \cup (D' - \{v\})$ would be a $\Gamma_L(P_n)$-set since we have supposed that $|D \cap \{u_1, u_2, \ldots, u_7\}| \leq 3$. Therefore we have an LDS D with $|D \cap \{u_1, u_2, \ldots, u_7\}| = 4$. Now to complete the proof we consider the following two cases.

Case 1. $u_7 \notin D$. By the previous observations we can assume that $u_1, u_2, u_5, u_6 \in D$ and $u_3, u_4 \notin D$. Let $P_{n'}$ be the path resulting from P_n by removing the vertices $u_1, u_2, \ldots, u_7$. Thus $D - \{u_1, u_2, u_5, u_6\}$ is a minimal locating-dominating set of $P_{n'}$ and so $\Gamma_L(P_{n'}) \geq \Gamma_L(P_n) - 4$. Also every $\Gamma_L(P_{n'})$-set can be extended to a minimal locating-dominating set of P_n by adding the set $\{u_1, u_2, u_5, u_6\}$ and so $\Gamma_L(P_{n'}) = \Gamma_L(P_n) - 4$. Applying the inductive hypothesis on $P_{n'}$ and by examining case by case the values of n' we obtain the desired result.

Case 2. $u_7 \in D$. We distinguish three subcases.

Subcase 2.1. $u_6 \notin D$ and $u_8 \in D$. Then by our claim $|D \cap \{u_1, u_2, \ldots, u_5\}| = 3$, say u_1, u_3, u_4. Thus $D - \{u_1, u_2, \ldots, u_7\}$ is a minimal LDS of the path induced by $P_n - \{u_1, u_2, \ldots, u_7\} = P_{n'}$ and so $\Gamma_L(P_{n'}) \geq \Gamma_L(P_n) - 4$. The equality is obtained since every $\Gamma_L(P_{n'})$-set can be extended to a minimal LDS for P_n by adding the set $\{u_1, u_2, u_5, u_6\}$. Using induction on $P_{n'}$ by considering as above all possible values of n' we obtain the result.

Subcase 2.2. $u_6, u_8 \notin D$. If $n = 8$, then $\{u_1, u_2, u_5, u_6, u_8\}$ is a minimal LDS of P_8 larger than D, a contradiction. Thus $n \geq 9$. First assume that $u_9 \in D$. By our claim, and without loss of generality, we can assume that $\{u_2, u_3, u_5\} \subset D$. Let $D' = D - \{u_2, u_3, u_5, u_7\}$. Note that $\{u_2, u_3, u_5, u_7\}$ is a minimal LDS of the path induced by $\{u_1, u_2, \ldots, u_8\}$. Now if D' is not a minimal LDS of $P_n - \{u_1, u_2, \ldots, u_7\}$, then u_8 and u_{10} have u_9 as a unique neighbor in D' but then $\{u_1, u_6, u_8\} \cup D - \{u_3, u_7\}$ is a minimal LDS of P_n larger than D, a contradiction. Thus D' is a minimal LDS of $P_{n'}$, where $n' = n - 7$. It can be seen easily that $\Gamma_L(P_n) = \Gamma_L(P_{n'}) + 4$. Using induction on $P_{n'}$ we obtain the result. Now assume that $u_9 \notin D$. Clearly $u_{10} \in D$. By our claim we assume that $\{u_2, u_3, u_5\} \subset D$. Let $D'' = D - \{u_2, u_3, u_5, u_7\}$ and P' be the path resulting from P_n by removing the vertices $u_1, \ldots, u_9$. Note that D'' is an LDS of P'. If D'' is minimal, then $D'' \cup \{u_1, u_2, u_5, u_6, u_8\}$ is a minimal LDS of P_n larger than D, a contradiction. Thus we assume that D'' is not minimal. Then there is a vertex $w \in D''$ such that $D'' - \{w\}$ is a minimal LDS of P' but then $\{u_1, u_2, u_5, u_6, u_8\} \cup (D'' - \{w\})$ is a $\Gamma_L(P_n)$-set that does not contain u_7 and such a case has been considered in Case 1.

Subcase 2.3. $u_6 \in D$. Then $u_8 \notin D$ for otherwise $D - \{u_7\}$ would be an LDS, contradicting the minimality of D. Likewise $u_5 \notin D$ for otherwise $D - \{u_6\}$ is LDS of G. It follows, without loss of generality, that $u_2, u_3 \in D$. Then $\{u_5\} \cup D - \{u_6\}$ is a $\Gamma_L(P_n)$-set that does not contain either u_6 or u_8, and such a case has been considered in Subcase 2.2. ■

Theorem 4. *If G is a graph with girth $g(G) \geq 5$, then every maximum independent set S is a minimal locating-dominating set. Furthermore, if $\delta(G) \geq 2$, then $V - S$ is a locating-dominating set.*

Proof. Let S be a $\beta_0(G)$-set. We first prove that S is a locating-dominating set of G. If $\beta_0(G) = 1$, then $G = K_1$ or K_2, and the result is valid. So assume that $\beta_0(G) \geq 2$. Suppose that S is not an LDS of G. Then there exists at least two vertices, say $u, v \in V - S$ with $N(u) \cap S = N(v) \cap S$. If u and v have two common neighbors in S, then the subgraph induced by u, v and their neighbors in S contains a cycle C_4, contradicting the fact that $g(G) \geq 5$. Thus u and v have a unique common neighbor in S, say w. If $uv \in E$, then $\{u, v, w\}$ induces a cycle C_3. Thus u and v are not adjacent but then $\{u, v\} \cup S - \{w\}$ is an independent set larger than S, a contradiction. Thus S is a locating-dominating set of G. Now since S is a minimal dominating set, it follows that S is a minimal locating-dominating set.

Now if $\delta(G) \geq 2$, then since $g(G) \geq 5$ no two vertices of S can have the same neighborhood intersection. Thus $V - S$ is a locating-dominating set. ■

As an immediate consequence we have the following corollaries.

Corollary 5. *If G is a graph with girth $g(G) \geq 5$, then $\Gamma_L(G) \geq \beta_0(G) \geq \gamma_L(G)$.*

Corollary 6 (Blidia *et al.* [5] 2008)**.** *If T is a tree, then $\beta_0(T) \geq \gamma_L(T)$.*

Corollary 7. *If G is a graph of order n with $\delta(G) \geq 2$ and girth $g(G) \geq 5$, then $\gamma_L(G) \leq n/2$.*

Note that the difference between $\Gamma_L(G) - \beta_0(G)$ can be arbitrarily large even for trees. To see this consider the tree T_t obtained by $t \geq 1$ copies of a path P_6 and connecting the third vertex in each P_6 to the third vertex in the next P_6. Clearly the set of leaves and support vertices form a $\Gamma_L(T_t)$-set of size $4t$ while $\beta_0(T_t) = 3t$.

Theorem 8. *If T is a nontrivial tree of order n with ℓ leaves, then, $\Gamma_L(T) \leq \frac{2n+\ell-2}{3}$, and this bound is sharp.*

Proof. We use an induction on the order n of T. The result can easily be checked for every tree T with $\mathrm{diam}(T) \in \{1,2,3\}$. Assume that every tree of order $n' < n$ with ℓ' leaves satisfies $\Gamma_L(T') \leq \frac{2n'+\ell'-2}{3}$. Let T be a tree of order n with $\mathrm{diam}(T) \geq 4$, and let D be any $\Gamma_L(T)$-set.

If there is a support vertex u adjacent to at least two leaves, then let $T' = T - \{u'\}$, where u' is a leaf neighbor of u belonging to D. Such a leaf exists since D contains either all leaves of u or all except one. Then $D - \{u'\}$ is a minimal LDS of T' and so $\Gamma_L(T') \geq |D| - 1$. Since $n' = n-1$, $\ell(T') = \ell - 1$, by induction on T', we obtain $\Gamma_L(T) \leq \frac{2n+\ell-2}{3}$. From now on we assume that $\ell - s(T) = 0$.

Assume now that T contains two adjacent support vertices x, y. Let T_x and T_y be the subtrees obtained from T by removing the edge xy. Then $D_x = D \cap V(T_x)$ is a minimal LDS of T_x and likewise $D_y = D \cap V(T_y)$ for T_y. Thus $\Gamma_L(T_x) + \Gamma_L(T_y) \geq |D_x| + |D_y| = \Gamma_L(T)$. Since $\mathrm{diam}(T) \geq 4$, one of T_x or T_y, say T_y has diameter at least two. Using the induction, $\Gamma_L(T_y) \leq \frac{2|V(T_y)|+|L(T_y)|-2}{3}$. If $\mathrm{diam}(T_x) = 1$, then $\Gamma_L(T_x) = 1 = \frac{2|V(T_x)|+1-2}{3}$ and $\ell = |L(T_y)| + 1$. If $\mathrm{diam}(T_x) \geq 2$, then by induction $\Gamma_L(T_x) \leq \frac{2|V(T_x)|+|L(T_x)|-2}{3}$, and, since $|V(T_x)| + |V(T_y)| = n$ and $\ell = |L(T_y)| + |L(T_x)|$, for both cases we obtain the desired result. Hence we may assume that no two support vertices of T are adjacent.

We now root the tree at a leaf r of maximum eccentricity $\mathrm{diam}(T) \geq 4$. Let u be a support vertex at distance $\mathrm{diam}(T) - 1$ from r on a longest path starting at r. Let v, w be the parents of u and v, respectively, and let u' be the unique leaf-neighbor of u. Since v cannot be a support vertex, the subtree T_v rooted at v is a subdivided star.

We first suppose that $v \in D$. Let $T' = T - \{u, u'\}$. Note that D contains either u or u', and hence $D \cap V(T')$ is a minimal LDS of T'. Therefore $\Gamma_L(T') \geq |D| - 1$ and $\ell(T') \leq \ell$. By induction on T', we obtain $\Gamma_L(T) < \frac{2n+\ell-2}{3}$.

Now assume that $v \notin D$. We consider the following two cases.

Case 1. $d_T(v) = k \geq 3$. If some child of v, say z, does not belong to D, then let $T' = T - \{z, z'\}$, where z' is the unique leaf adjacent to z. That case is similar to the above situation and so $\Gamma_L(T) < \frac{2n+\ell-2}{3}$. Thus we assume that D contains all children of v. Let $T' = T - T_v$. Clearly D contains no leaf of T_v and $D \cap V(T')$ is a minimal LDS of T'. Thus $\Gamma_L(T') \geq |D| - k + 1$ and $\ell(T') \leq \ell - k + 1$. The result immediately follows by using induction on T'.

Case 2. $d_T(v) = k = 2$. Since $\mathrm{diam}(T) \geq 4$, let z be the parent of w. If $z = r$, then $T = P_5$, and $\Gamma_L(T) < \frac{2n+\ell-2}{3}$. Thus $r \neq z$. To complete our proof we need to consider the following three situations for D. Recall that $v \notin D$. So we have either $u', w \in D$ and $u \notin D$, or $u, w \in D$ and $u' \notin D$ or $u, u' \in D$ and $w \notin D$.

If the first situation occurs, then let $T' = T - \{u, u'\}$. In the second situation, $D - \{u\}$ is a minimal LDS either for $T' = T - \{u, u'\}$ or $T' = T - \{v, u, u'\}$, and the subtree T' for which $D - \{u\}$ is a minimal LDS will be considered. We leave it to the reader check that the above two situations provide $\Gamma_L(T) \leq \frac{2n+\ell-2}{3}$.

Consider now the last situation $u, u' \in D$ and $w \notin D$. Let $T' = T - \{u', u, v\}$. Then $D - \{u', u\}$ is a minimal LDS of T' and hence $\Gamma_L(T') \geq |D| - 2$. By using induction and the facts $n' = n - 3$ and $\ell(T') = \ell$ we obtain $\Gamma_L(T) \leq \frac{2n+\ell-2}{3}$.

The bound is sharp for nontrivial stars. ■

The following lower bound on the independence number for bipartite graphs is given in [2].

Proposition 9 (Blidia, Chellali, Favaron, Meddah [2] 2007)**.** *If G is a bipartite graph, then*

$$\beta_0(G) \geq \frac{n + \ell(G) - s(G)}{2}.$$

By using Theorem 8 and Proposition 9 we obtain an upper bound on $\Gamma_L(T)$ for trees in terms of $\beta_0(T)$, number of leaves and support vertices.

Corollary 10. *If T is a nontrivial tree, then*

$$\Gamma_L(T) \leq \frac{4}{3}\beta_0(T) - \frac{1}{3}(\ell(T) - 2s(T) + 2).$$

We also obtain

Corollary 11. *If T is a nontrivial tree, then*

$$\Gamma_L(T) - \beta_0(T) \leq \frac{1}{6}\left(n - \ell(T) + 3s(T) - 4\right).$$

3. Locating-Domination Number

We begin by recalling the following two results given in [4] on the locating-domination number for trees.

Theorem 12 (Blidia *et al.* [4] 2007)**.** *If T is a tree of order $n \geq 2$, then $\gamma_L(T) \leq (n + \ell(T) - s(T))/2$.*

Theorem 13 (Blidia *et al.* [4] 2007)**.** *If T is a tree of order $n \geq 3$, then $\gamma_L(T) \geq (n + \ell(T) - s(T) + 1)/3$.*

A *block graph* is a graph in which every block (maximal 2-connected graph) is a clique. It is well known that block graphs are C_4 and $(K_4 - e)$-free graphs.

Proposition 14. *If G is a block graph with $\delta(G) \geq 2$, then $\gamma_L(G) + \beta_0(G) \leq n$.*

Proof. Let S be any $\beta_0(G)$-set. Then every vertex of S has at least two neighbors in $V - S$, and since G is C_4 and $(K_4 - e)$-free each pair of vertices $x, y \in S$ satisfy $N(x) \cap (V - S) \neq N(y) \cap (V - S)$. It follows that $V - S$ is an LDS of G implying that $\gamma_L(G) \leq |V - S| = n - \beta_0(G)$. ■

The bound in Proposition 14 is attained for the graph G_k formed by k triangles sharing the same vertex of a path P_2.

Next we show that the locating-domination number is bounded below by the independent domination number for the class of trees.

Theorem 15. *If T is a tree, then $\gamma_L(T) \geq i(T)$.*

Proof. We use an induction on the order of T. It is a routine matter to check the result if $\mathrm{diam}(T) \in \{0, 1, 2, 3\}$. Thus assume that every tree T' of order n' less than n satisfies $\gamma_L(T') \geq i(T')$. Let T be a tree of order n and diameter at least four. Let D be a $\gamma_L(T)$-set containing all support vertices. If any support vertex, say x, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf x' adjacent to x and belonging to D. Then $D - \{x'\}$ is an LDS of T' and so $\gamma_L(T') \leq \gamma_L(T) - 1$. If S is any $i(T')$-set, then either S or $S \cup \{x'\}$ is a maximal independent set of T, and so $i(T) \leq i(T') + 1$. Using induction on T' we obtained the desired result. Thus we assume that every support vertex of T is adjacent to exactly one leaf.

We now root the tree at a leaf r of maximum eccentricity $\mathrm{diam}(T)$. Let u be the vertex at distance $\mathrm{diam}(T) - 1$ from r on a longest path starting at r. Let v be the parent of u and let u' the unique leaf adjacent to u.

If v is a support vertex, then let $T' = T - \{u, u'\}$. Then $\gamma_L(T') \leq \gamma_L(T) - 1$ and $i(T) \leq i(T') + 1$. By induction on T' we have $\gamma_L(T) \geq i(T)$. Thus v is not a support vertex and so T_v is a subdivided star. Then since D contains the support vertices of T_v, $v \notin D$ (else we replace it by its parent). Let $T' = T - T_v$. Then $D \cap V(T')$ is an LDS of T' and so $\gamma_L(T') \leq \gamma_L(T) - d_T(v) + 1$. Moreover, every $i(T')$-set union the set of support vertices in T_v is a maximal independent set of T implying that $i(T) \leq i(T') + d_T(v) - 1$. Using induction on T' we obtain $\gamma_L(T) \geq i(T)$. ■

By Corollary 5 and Theorem 15 we obtain the following inequality chain relating locating-domination and independence parameters for every tree T:

$$i(T) \leq \gamma_L(T) \leq \beta_0(T) \leq \Gamma_L(T). \tag{1}$$

In [11], Ravindra showed that a tree T satisfies $i(T) = \beta_0(T)$ if and only if T is a single vertex or T is a corona of a tree. By Ravindra's result equality throughout (1) holds if and only if T is a single vertex or T is a corona of some tree T'. However for the class of trees T_t defined in Section 2, we have $\Gamma_L(T_t) - i(T_t) = 4t - 2t = 2t$.

The graphs G of even order and without isolated vertices with $\gamma(G) = n/2$ have been characterized independently by Payan and Xuong [10] and Fink, Jacobson, Kinch and Roberts [7].

Theorem 16 (Payan, Xuong [10] 1982 and Fink *et al.* [7] 1985)**.** *Let G be a graph of even order n without isolated vertices. Then $\gamma(G) = n/2$ if and only if each component of G is either a cycle C_4 or the corona of a connected graph.*

Observation 17. *Let T be a tree of order at least three. Then $\gamma(T) \leq \frac{n-\ell(T)+s(T)}{2}$ with equality if and only if T is a tree of order $\ell(T) + s(T)$.*

Proof. Clearly the result holds if T is a star. Assume T is not a star and let T^* be the tree obtained from T by removing for every support vertex of T all its leaves except one. Since there is a minimum dominating set containing all support vertices we have $\gamma(T) = \gamma(T^*)$. Also T^* has order $n - \ell(T) + s(T)$ and by the well known Ore's theorem $\gamma(T^*) \leq \frac{n-\ell(T)+s(T)}{2}$. Now by Theorem 16 $\gamma(T^*) = \frac{n-\ell(T)+s(T)}{2}$ if and only if T^* is a corona of some tree T' and so T is a tree where every vertex is either a leaf or a support vertex, that is T has order $\ell(T) + s(T)$. ∎

Recall that a set $R \subseteq V(G)$ is a *packing set* of G if $N[x] \cap N[y] = \emptyset$ holds for any two distinct vertices $x, y \in R$. The *packing number* $\rho(G)$ is the maximum cardinality of a packing in G.

Proposition 18. *For every connected nontrivial graph G, $\gamma_L(G) \leq n - \rho(G)$.*

Proof. Let R be a maximum packing set of G. Then since $N[x] \cap N[y] = \emptyset$ for any two distinct vertices $x, y \in R$, $V - R$ is a locating-dominating set of G and so $\gamma_L(G) \leq |V - R| = n - \rho(G)$. ∎

Farber [6] proved that the domination number and packing number are equal for any strongly chordal graph including the class of trees. Thus we have the following corollary to Proposition 18.

Corollary 19. *For every nontrivial tree T, $\gamma_L(T) + \gamma(T) \leq n$, with equality if and only if T is a tree of order $\ell(T) + s(T)$.*

Proof. Assume that $\gamma_L(T) + \gamma(T) = n$. If T is a star, then it has order $\ell(T) + s(T)$. Thus we assume that T is not a star. Then by Theorem 12 and Observation 17 we have $\gamma_L(T) = \frac{n+\ell(T)-s(T)}{2}$ and $\gamma(T) = \frac{n-\ell(T)+s(T)}{2}$. It follows that T is a tree of order $\ell(T) + s(T)$.

The converse is obvious. ∎

Next we extend the upper bound in Theorem 12 to bipartite graphs with no cycle C_4. We assume that $\ell(P_2) = s(P_2) = 2$.

Theorem 20. *If G is a connected nontrivial bipartite graph with no cycle C_4, then $\gamma_L(G) \le (n + \ell(G) - s(G))/2 \le \Gamma_L(G)$.*

Proof. If G is a tree, then by Corollary 5, Proposition 9 and Theorem 12 the result holds. Thus assume that G is not a tree. Let D be a set of leaves of G chosen as follows: for every support vertex u adjacent in G to two or more leaves, put in D all the leaves adjacent to u except one. Then $|D| = \ell(G) - s(G)$. Now consider the subgraph G' obtained from G by removing all its leaves. Since G is not a tree and is C_4-free, G' is nontrivial and then has a unique bipartition A, B into non-empty independent sets. Clearly every leaf of G' is a support vertex in G and every vertex of G' different from a leaf has degree at least two neighbors in G'. Let $A' = A - S(G)$ and $B' = B - S(G)$, and assume that $|A'| \le |B'|$. Since G is a bipartite graph containing no cycle C_4, no two vertices of B (resp., A) have common neighbors in $D \cup S(G) \cup A'$ (resp., $D \cup S(G) \cup B'$). Thus each of the sets $D \cup S(G) \cup A'$ and $D \cup S(G) \cup B'$ is a minimal locating-dominating set of G. Therefore $\gamma_L(T) \le |D \cup S(G) \cup A'|$ and $\Gamma_L(G) \ge |D \cup S(G) \cup B'|$. By using the facts that $|A'| \le (n - \ell(G) - s(G))/2 \le |B'|$ the results are proved. ■

Note that the upper bound on $\gamma_L(G)$ is not valid if G is a bipartite graph containing a cycle C_4. To see this consider the cycle C_4 by attaching one new vertex at a vertex of the cycle. Clearly $\gamma_L(G) = 3 > (n + \ell(G) - s(G))/2$.

Theorem 21. *If G is a connected graph of order $n \ge 2$ with at most one cycle, then $\gamma_L(G) \le (n + \ell(G) - s(G) + 1)/2$.*

Proof. If G is a tree, then by Theorem 12 the result is true. Thus we assume that G contains a cycle C. Clearly if $G = C$, then $\gamma_L(G) \le (n+1)/2$. So we assume that $G \ne C$ and hence G contains a vertex of degree at least three. Assume that the result does not hold and let G be the smallest connected unicycle graph such that $\gamma_L(G) > (n + \ell(G) - s(G) + 1)/2$.

We first assume that all support vertices are on the cycle C. If C contains only one support vertex b, then let A be a maximum independent set of $G[V(C)]$ that contains b. Then $|A| = \lfloor |C|/2 \rfloor$, and A union the set of leaves L_b adjacent to b is an LDS of G of size at most $(n + |L_b|)/2 = (n + \ell(G) - s(G) + 1)/2$, a contradiction. Thus C contains at least two support vertices.

Assume that C contains two consecutive support vertices x and y joined by a nontrivial path in which every vertex has degree two in G. Let H be the set of vertices on such a path between x and y. Thus $|H| \geq 2$. Let $G' = G - H$. Then $n' = n - |H|, \ell(G') = \ell(G)$, $s(G') = s(G)$. Each of G' and $P_{|H|}$ is a tree and so by Theorem 12, $\gamma_L(G') \leq (n' + \ell(G') - s(G'))/2$ and $\gamma_L(P_{|H|}) \leq (|H|+1)/2$. Then $\gamma_L(G) \leq (n' + \ell(G') - s(G'))/2 + (|H|+1)/2 = (n + \ell(G) - s(G) + 1)/2$, a contradiction. Thus the distance between every two consecutive support vertices on C is one or two. Hence $n = s(G) + \ell(G) + k$, where k is the number of vertices of degree two. Clearly $k \leq s(G)$. Let $L'(G)$ be a set of leaves so that for each support vertex we put in $L'(G)$ all its leaves except one. Then $|L'(G)| = \ell(G) - s(G)$ and clearly $S(G) \cup L'(G)$ is an LDS of G. Thus $\gamma_L(G) \leq s(G) + \ell(G) - s(G) = \ell(G) < (n + \ell(G) - s(G) + 1)/2$, contradicting our assumption. It follows that G contains at least one support vertex that does not belong to C.

Let u be a support vertex of G at maximum distance from C. Let v be the neighbor of u in the unique path from u to C. First assume $d(v) \geq 3$, and let $G' = G - (L_u \cup \{u\})$, where L_u is the set of leaves adjacent to u. If S' is any $\gamma_L(G')$-set, then $S' \cup L_u$ is an LDS of G and so $\gamma_L(G) \leq \gamma_L(G') + |L_u|$. Now since G' has order n' less than n and $n' = n - (|L_u|+1), \ell(G') = \ell(G) - |L_u|$ and $s(G') = s(G) - 1$ it follows that

$$\gamma_L(G) \leq (n' + \ell(G') - s(G') + 1)/2 + |L_u| = (n + \ell(G) - s(G) + 1)/2,$$

a contradiction. Thus suppose that $d(v) = 2$ and let w be the second neighbor of v. If $d(w) = 2$ or $d(w) \geq 3$ and w is not a support vertex, then let $G' = G - (L_u \cup \{u\})$. Then G' satisfies the theorem and we have $n' = n - (|L_u| + 1), \ell(G') = \ell(G) - |L_u| + 1$, $s(G') = s(G)$. Since every $\gamma_L(G')$-set can be extended to a locating-dominating set of G by adding L_u, we obtain

$$\gamma_L(G) \leq (n' + \ell(G') - s(G') + 1)/2 + |L_u| = (n + \ell(G) - s(G) + 1)/2,$$

a contradiction. Thus we finally assume that $d(w) \geq 3$ and w is a support vertex and let $G' = G - (L_u \cup \{u, v\})$. Then G' satisfies the theorem and we have $n' = n - (|L_u| + 2), \ell(G') = \ell(G) - |L_u|$, $s(G') = s(G) - 1$. Since there is a $\gamma_L(G')$-set that contains w such a set can be extended to an LDS of G by adding $\{u\} \cup L_u - \{u'\}$, where u' is any leaf adjacent to u. It follows that $\gamma_L(G) \leq (n' + \ell(G') - s(G') + 1)/2 + |L_u| < (n + \ell(G) - s(G) + 1)/2$, a contradiction. This completes the proof. ■

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