## **RADIO NUMBER FOR SOME THORN GRAPHS**

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#### Abstract

For a graph G and any two vertices u and v in G, let d(u, v) denote the distance between u and v and let diam(G) be the diameter of G. A multilevel distance labeling (or radio labeling) for G is a function fthat assigns to each vertex of G a positive integer such that for any two distinct vertices u and v,  $d(u, v) + | f(u) - f(v) | \ge \text{diam}(G) + 1$ . The largest integer in the range of f is called the span of f and is denoted span(f). The radio number of G, denoted rn(G), is the minimum span of any radio labeling for G. A thorn graph is a graph obtained from a given graph by attaching new terminal vertices to the vertices of the initial graph. In this paper the radio numbers for two classes of thorn graphs are determined: the caterpillar obtained from the path  $P_n$  by attaching a new terminal vertex to each non-terminal vertex and the thorn star  $S_{n,k}$  obtained from the star  $S_n$  by attaching k new terminal vertices to each terminal vertex of the star.

**Keywords:** multilevel distance labeling, radio number, caterpillar, diameter.

2010 Mathematics Subject Classification: 05C15, 05C78.

### 1. INTRODUCTION

Radio labeling (or multilevel distance labeling) of graphs is motivated by restrictions in assigning channel frequencies for radio transmitters [6]. More precisely, for a set of given stations, it is required to assign to each station a channel such that interference is avoided and the span of assigned channels

is minimized. Channels are positive integers, and the level of interference is related to the distances between stations. For small distances the interference is stronger, so the stations that are geographically close must be assigned channels with large frequency difference, while for stations that are further apart this difference can be small. This type of problem can be modeled by a graph, where vertices represent stations and every vertex has a positive number assigned, representing a channel. Every pair of close stations is connected by an edge.

Let G be a connected graph with vertex set V(G) and diameter diam(G). For any two vertices u and v of G, d(u, v) represents the distance between them. A vertex u for which there exists a vertex v such as d(u, v) = diam(G)is called a *peripheral vertex*. A radio labeling (or multilevel distance labeling) of G is a one-to-one mapping  $f : V(G) \to \mathbb{Z}^+$  which assigns to each vertex a positive integer, satisfying the condition

$$d(u,v) + |f(u) - f(v)| \ge \operatorname{diam}(G) + 1$$

for every two distinct vertices u, v. This condition is referred to as *radio* condition (or multilevel distance labeling condition). The span of f, denoted by span(f), is the maximum integer in the range of f. The radio number of G, denoted rn(G), is the smallest span in all radio labelings of G. Since the radio condition contains only the difference of the labels, a radio labeling realizing rn(G) must have the minimum label equal to 1.

For many classes of graphs is not easy to determine their radio number. For radio numbers of paths and cycles in [2] and [3] only upper bounds were obtained. Later, in [8], Liu and Zhu determined the exact values of these radio numbers. In [9] Rahim and Tomescu considered radio labelings for helm graphs (a helm graph  $H_n$  is obtained from the wheel  $W_n$  by attaching a vertex of degree one to each of the *n* vertices of the cycle of the wheel).

Liu [7] determined a lower bound for the radio number of trees and characterized the trees achieving this bound. To be able to discuss these results, we introduce the following notions.

Let T be a tree rooted at a vertex w. For any two vertices u and v, if u is on the path connecting w and v, then u is an *ancestor* of v and v is a *descendent* of u. The *level function* on V(T), for a fixed root w, is defined by

$$L_w(u) = d(w, u), \ \forall u \in V(T).$$

For any  $u, v \in V(T)$ , define

 $\Phi_w(u, v) = \max\{L_w(t) \mid t \text{ is a common ancestor of } u \text{ and } v\}.$ 

Let w' be a neighbor of w. The subtree induced by w' together with all the descendents of w' is called a *branch*.

**Remark 1.1** ([7]). Let T be a tree rooted at w. For any vertices u and v we have:

- (1)  $\Phi_w(u, v) = 0$  if and only if u and v belong to different branches, unless one of them is w;
- (2)  $d(u,v) = L_w(u) + L_w(v) 2\Phi_w(u,v).$

For any vertex w in T, the status of w in T is defined by

$$s_T(w) = \sum_{u \in V(T)} L_w(u) = \sum_{u \in V(T)} d(u, w).$$

The status of T is the minimum status among all vertices of T:

$$s(T) = \min\{s_T(w) \mid w \in V(T)\}.$$

A vertex  $w^*$  of T is called a *weight center* of T if  $s_T(w^*) = s(T)$ .

**Remark 1.2.** The set of all weight centers of a tree T is known as the median of T ([1]).

Because in [7] a radio labeling is also allowed to take value 0, the radio numbers and limits determined in [7] are one less than the radio numbers previously defined in this article. We will recall the results from [7], making the necessary adjustments by adding one to the bounds and radio numbers arising from these results.

**Theorem 1.3** [7]. Let T be a tree with n vertices and diameter d. Then

$$rn(T) \ge (n-1)(d+1) + 2 - 2s(T).$$

Moreover, the equality holds if and only if for every weight center  $w^*$  there exists a radio labeling f with  $f(u_1) = 1 < f(u_2) < \cdots < f(u_n)$  for which all the following properties hold, for every i with  $1 \le i \le n-1$ :

- (1)  $u_i$  and  $u_{i+1}$  belong to different branches, unless one of them is  $w^*$ ;
- (2)  $\{u_1, u_n\} = \{w^*, v\}$ , where v is some vertex with  $L_{w^*}(v) = 1$ ;
- (3)  $f(u_{i+1}) = f(u_i) + d + 1 L_{w^*}(u_i) L_{w^*}(u_{i+1}).$

Thorn graphs were introduced by Gutman in [5]. For a graph G with  $V(G) = \{v_1, v_2, \ldots, v_n\}$ , a thorn graph of G with nonnegative parameters  $p_1, p_2, \ldots, p_n$  is obtained by attaching  $p_i$  new vertices of degree one to the vertex  $v_i$ , for each  $1 \le i \le n$ . A thorn path is called *caterpillar*. In the following sections we will determine the radio number for two classes of thorn graphs: a particular class of caterpillars and the thorn star.

# 2. Radio Labeling and Radio Number for a Class of Caterpillars

For  $n \ge 2$  we denote by  $CP_n$  the caterpillar obtained from the path with n vertices  $P_n$  by attaching a new terminal vertex to each non-terminal vertex of  $P_n$ .  $CP_n$  has m = 2n - 2 vertices and diameter d = n - 1.

In this section we will determine the radio number for this type of caterpillar, more precisely we will show that:  $rn(CP_3) = 5$ ,  $rn(CP_n) = 4k^2 - 6k + 4$  for n = 2k and  $rn(CP_n) = 4k^2 - 2k + 4$  for n = 2k + 1,  $k \ge 2$ . We will consider two cases, in accordance with the parity of n.

Case 1. n is even.

Let  $n = 2k, k \ge 1$ . In this case we denote by  $v_1, \ldots, v_{2k}$  the vertices of  $P_n$  from which the caterpillar  $CP_n$  is obtained, by  $v'_{i-1}$  the terminal vertex attached to  $v_i$ , for  $2 \le i \le k$ , and by  $v'_{i+1}$  the terminal vertex attached to  $v_i$ , for  $k+1 \le i \le 2k-1$  (Figure 1).



Figure 1.  $CP_{2k}$ 

We have m = 4k - 2 and d = 2k - 1.

**Theorem 2.1.** For n = 2k,  $k \ge 1$ , the radio number for  $CP_n$  is  $rn(CP_n) = 4k^2 - 6k + 4$ .

#### **Proof.** We use Theorem 1.3.

Let  $n = 2k, k \ge 1$ . In this case  $CP_n$  has two weight centers,  $v_k$  and  $v_{k+1}$ . We have

$$s(CP_n) = s_{CP_n}(v_k) = \sum_{u \in V(CP_n)} d(u, v_k)$$
  
= 3 \cdot 1 + 4(2 + \dots + k - 1) + 2 \cdot k  
= 2k^2 - 1.

By Theorem 1.3,

$$rn(CP_n) \ge (m-1)(d+1) + 2 - 2s(T) = (4k-3)(2k) + 2 - 2(2k^2 - 1)$$
  
=  $4k^2 - 6k + 4$ .

Moreover, in order to prove equality, it suffices to find a radio labeling f for  $CP_n$  with  $span(f) = 4k^2 - 6k + 4$  (or, equivalently, a radio labeling that satisfies the properties (1)–(3) in Theorem 1.3 for every weight center of  $CP_n$ ; furthermore, since  $CP_n$  is symmetrical, it suffices to give a radio labeling for  $CP_n$  with these properties only for weight center  $v_k$ ).

For that, we order the vertices of  $CP_n$  as follows: alternate  $v_j$  and  $v'_{k+j}$  for  $j = k, k - 1, \ldots, 2$ , then  $v_1, v_{2k}$ , then alternate  $v'_j$  and  $v_{k+j}$  for  $j = k - 1, k - 2, \ldots, 1$ . We rename the vertices of  $CP_n$  in the above ordering by  $u_1, u_2, \ldots, u_m$ .

We define a labeling f for  $CP_n$  using the rules given by (2) and (3) from Theorem 1.3 as follows:  $f(u_1) = 1$ ,  $f(u_{i+1}) = f(u_i) + d + 1 - d(u_{i+1}, u_i)$  for  $1 \le i \le m - 1$ .

For example, if k = 4, the order in which the vertices are labeled and their labels are shown in Figure 2.

Since we have the following distances:  $d(v_j, v'_{k+j}) = k$ , for  $2 \le j \le k$ ;  $d(v'_{k+j}, v_{j-1}) = k+1$ , for  $2 \le j \le k$ ;  $d(v_1, v_{2k}) = 2k-1$ ,  $d(v_{2k}, v'_{k-1}) = k+1$ ;  $d(v'_j, v_{k+j}) = k$ , for  $1 \le j \le k-1$  and  $d(v_{k+j}, v'_{j-1}) = k+1$ , for  $2 \le j \le k-1$ , we obtain:

$$span(f) = f(u_m) = f(v_{k+1}) = f(u_1) + (m-1)(d+1) - \sum_{i=1}^{m-1} d(u_{i+1}, u_i)$$
$$= 4k^2 - 6k + 4.$$



Figure 2.  $CP_{2k}$ 

The following relations also hold:

 $\begin{aligned} f(v_j) &= f(v_{j+1}) + 2k - 1 \text{ for } 1 \leq j \leq k - 1; \\ f(v_j) &= f(v_{j+1}) + 2k - 1 \text{ for } k + 1 \leq j \leq 2k - 1; \\ \text{similary,} \\ f(v'_j) &= f(v'_{j+1}) + 2k - 1 \text{ for } k + 2 \leq j \leq 2k - 1; \\ f(v'_j) &= f(v'_{j+1}) + 2k - 1 \text{ for } 1 \leq j \leq k - 2; \\ f(v'_{k-1}) &= f(v'_{k+2}) + 2k - 1; \\ f(v_{2k}, v'_{k+2}) &= k; \\ &|f(v'_i) - f(v_j)| \geq 2k - 1 \text{ if } v'_i \text{ and } v_j \text{ are not consecutive in the order previously} \\ \text{established.} \end{aligned}$ 

Consecutive vertices in the ordering verify the radio constraint by construction. Then it is easy to check that for every two distinct vertices u and v the radio condition is verified, considering each particular case of pairs of vertices (both vertices are from  $P_n$ , both are terminal or they are of different type), so f is a radio labeling for  $CP_n$ . Moreover, from the way fwas defined, it satisfies the properties (1)–(3) in Theorem 1.3 for the weight center  $v_k$ , since the vertices  $u_i$  and  $u_{i+1}$  belong to different branches for  $2 \le i \le m-1, u_1 = v_k$  and  $u_m = v_{k+1}$ , with  $L_{v_k}(v_{k+1}) = d(v_k, v_{k+1}) = 1$ .

### Case 2. n is odd.

Let n = 2k + 1. For k = 1, it is easy to see that  $rn(CP_3) = 5$ ,  $CP_3$  being the star  $S_3$ . If  $k \ge 2$ , in order to label the vertices of  $CP_n$ , we denote by  $v_1, \ldots, v_{2k+1}$  the vertices of  $P_n$  from which the caterpillar is obtained and by  $v'_i$  the terminal vertex attached to  $v_i$ , for  $2 \le i \le 2k$  (Figure 3). We have m = 4k and d = 2k.



Figure 3.  $CP_{2k+1}$ 

**Theorem 2.2.** For n = 2k + 1,  $k \ge 2$  the radio number for  $CP_n$  is  $rn(CP_n) = 4k^2 - 2k + 4$ .

**Proof.** We shall see that it is not sufficient to use only Theorem 1.3 to prove the equality. Let n = 2k + 1,  $k \ge 2$ . In this case  $CP_n$  has a single weight center,  $v_{k+1}$ . We have:

$$s(CP_n) = s_{CP_n}(v_{k+1}) = \sum_{u \in V(CP_n)} d(u, v_{k+1})$$
  
= 3 \cdot 1 + 4(2 + \cdots + k) = 2k^2 + 2k - 1.

By Theorem 1.3,

$$rn(CP_n) \ge (m-1)(d+1) + 2 - 2s(T)$$
  
=  $(4k-1)(2k+1) + 2 - 2(2k^2 + 2k - 1)$   
=  $4k^2 - 2k + 3$ .

We will prove that there is no radio labeling for  $CP_n$  that satisfies the properties (1)–(3) for the weight center  $v_{k+1}$  of  $CP_n$ , so the inequality is strict.

Suppose that there exists a radio labeling f for  $CP_n$  with these properties. We order the vertices of  $CP_n$  by their labels and rename the vertices in this ordering by  $u_1, u_2, \ldots, u_m$ :  $1 = f(u_1) < f(u_2) < \cdots < f(u_m)$ . Let  $u_i, u_{i+1}, u_{i+2}$  be three consecutive vertices in this ordering,  $1 \le i \le m - 2$ . We can assume, without loss of generality, that  $d(u_i, u_{i+1}) \ge d(u_{i+1}, u_{i+2})$ . We shall prove the following claims:

- (a) If one of the vertices  $u_i$ ,  $u_{i+1}$ ,  $u_{i+2}$  belongs to the path that connects the other two, then  $\min\{d(u_i, u_{i+1}), d(u_{i+1}, u_{i+2})\} \le k$ ;
- (b)  $\min\{d(u_i, u_{i+1}), d(u_{i+1}, u_{i+2})\} \le k+1;$
- (c) If v is a peripheral vertex in  $CP_{2k+1}$  and  $v = u_i$ , then i is different from 1 and m. Moreover, if its neighboring vertices  $u_{i-1}$  and  $u_{i+1}$  are different from  $v_{k+1}$ , then one of the vertices  $u_{i-1}$  or  $u_{i+1}$  is  $v'_{k+1}$ .

**Claim** (a). We assume first that one of the vertices  $u_i$ ,  $u_{i+1}$ ,  $u_{i+2}$  belongs to the path that connects the other two.

Suppose that  $\min\{d(u_i, u_{i+1}), d(u_{i+1}, u_{i+2})\} > k$ . Then  $d(u_i, u_{i+1}) > \frac{n}{2}$ and  $d(u_{i+1}, u_{i+2}) > \frac{n}{2}$ . Because we assumed that  $d(u_i, u_{i+1}) \ge d(u_{i+1}, u_{i+2})$ ,  $u_{i+2}$  must lie on the path connecting  $u_i$  and  $u_{i+1}$ , hence  $d(u_i, u_{i+2}) = d(u_i, u_{i+1}) - d(u_{i+1}, u_{i+2})$ . By property (3) from Theorem 1.3,  $f(u_{i+1}) - f(u_i) = d + 1 - d(u_i, u_{i+1}) = n - d(u_i, u_{i+1})$ , hence we have:

$$f(u_{i+2}) - f(u_i) = f(u_{i+2}) - f(u_{i+1}) + f(u_{i+1}) - f(u_i)$$
  
=  $n - d(u_i, u_{i+1}) + n - d(u_{i+1}, u_{i+2})$   
=  $2n - (d(u_i, u_{i+1}) + d(u_{i+1}, u_{i+2}))$   
=  $2n - (d(u_i, u_{i+1}) - d(u_{i+1}, u_{i+2}) + 2d(u_{i+1}, u_{i+2}))$   
=  $2n - d(u_i, u_{i+2}) - 2d(u_{i+1}, u_{i+2})$   
<  $2n - d(u_i, u_{i+2}) - 2\frac{n}{2} = n - d(u_i, u_{i+2}).$ 

This contradicts that f is a radio labeling. It follows that Claim (a) is true.

**Claim** (b). In order to prove Claim (b) it suffices to consider the case when no vertex belongs to the path connecting the other two.

Suppose that  $\min\{d(u_i, u_{i+1}), d(u_{i+1}, u_{i+2})\} > k+1$ . It results that  $d(u_i, u_{i+1}) \ge k+2$  and  $d(u_{i+1}, u_{i+2}) \ge k+2$ . Since  $d(u_i, u_{i+1}) \ge d(u_{i+1}, u_{i+2})$ , by Theorem 1.3 (1), we can only have the following situation:  $u_{i+2}$  does not belong to the path connecting  $u_i$  and  $u_{i+1}$ , but there exists a vertex  $u'_{i+2}$  adjacent to  $u_{i+2}$  that belongs to this path. Then

$$d(u_i, u_{i+2}) = d(u_i, u_{i+1}) - d(u_{i+1}, u'_{i+2}) + 1$$
  
=  $d(u_i, u_{i+1}) - (d(u_{i+1}, u_{i+2}) - 1) + 1$   
=  $d(u_i, u_{i+1}) - d(u_{i+1}, u_{i+2}) + 2.$ 

Hence

$$f(u_{i+2}) - f(u_i) = 2n - (d(u_i, u_{i+1}) + d(u_{i+1}, u_{i+2}))$$
  
= 2n - (d(u\_i, u\_{i+2}) + 2d(u\_{i+1}, u\_{i+2}) - 2)  
= 2n - d(u\_i, u\_{i+2}) - 2d(u\_{i+1}, u\_{i+2}) + 2  
 $\leq 2n - d(u_i, u_{i+2}) - 2(k+2) + 2$ 

$$= 2n - d(u_i, u_{i+2}) - (n+3) + 2$$
  
=  $n - 1 - d(u_i, u_{i+2})$   
<  $n - d(u_i, u_{i+2})$ ,

which is a contradiction since

$$f(u_{i+2}) - f(u_i) \ge d + 1 - d(u_i, u_{i+2}) = n - d(u_i, u_{i+2}),$$

so Claim (b) follows.

Claim (c). Let v be a peripheral vertex in  $CP_{2k+1}$  (v is one of vertices  $v_1$ ,  $v_{2k+1}, v'_2, v'_{2k}$ ). For any vertex u not belonging to the same branch as v we have  $d(v, u) \ge k + 1$ . Also, d(v, u) = k + 1 holds only for those two vertices u which are also adjacent to the center  $v_{k+1}$ . Let i be an index between 1 and m such that  $v = u_i$ . By property (1) from Theorem 1.3,  $\{u_1, u_m\} = \{v_{k+1}, v^*\}$ , with  $d(v_{k+1}, v^*) = 1$ , hence i is different from 1 and m and we have  $f(u_{i-1}) < f(v) < f(u_{i+1})$ . Moreover, if  $u_{i-1}$  and  $u_{i+1}$  are different from center  $v_{k+1}$ , since min $\{d(u_{i-1}, v), d(v, u_{i+1})\} \ge k + 1$ , the assumptions from Claim (a) are not verified, so none of the vertices  $u_{i-1}, v, u_{i+1}$  belongs to the path connecting the other two, and min $\{d(u_{i-1}, v), d(v, u_{i+1})\} = k+1$ . It follows that one of the vertices  $u_{i-1}$  or  $u_{i+1}$  is  $v'_{k+1}$ .

Hence we proved Claim (c).

Since f satisfies theorem 1.3 (2), for at least three peripheral vertices their neighboring vertices  $u_{i-1}$  and  $u_{i+1}$  are different from the center  $v_{k+1}$ . It follows that at least three peripheral vertices have the property that one of the vertices  $u_{i-1}$  or  $u_{i+1}$  is  $v'_{k+1}$ , which is impossible.

Therefore there is no radio labeling f for  $CP_n$  that verifies the properties (1)–(3) in Theorem 1.3 for the weight center  $v_{k+1}$  of  $CP_n$ . Hence  $rn(CP_n) > 4k^2 - 2k + 3$ .

To prove that  $rn(CP_n) = 4k^2 - 2k + 4$  it suffices to find a radio labeling f for  $CP_n$  with  $span(f) = 4k^2 - 2k + 4$ . For that, we order the vertices of  $CP_n$  as follows:  $v_{k+1}$ ,  $v_1$ ,  $v'_{k+1}$ , then alternate  $v'_{2k-j}$  and  $v'_{k-j}$  for  $j = 0, 1, \ldots, k-2$ , then  $v_{2k+1}$ , then alternate  $v_{k-j}$  and  $v_{2k-j}$  for  $j = 0, 1, \ldots, k-2$ . We rename the vertices of  $CP_n$  in the above ordering by  $u_1, u_2, \ldots, u_m$ .

We define a labeling f for  $CP_n$  using the rules given by (3) from Theorem 1.3 as follows:  $f(u_1) = 1$ ,  $f(u_{i+1}) = f(u_i) + d + 1 - d(u_{i+1}, u_i)$  for  $1 \le i \le m - 1$ , with one exception:  $f(v_{2k+1}) = f(v'_2) + 2$ .

For example, if k = 4, the order in which the vertices are labeled and their labels are shown in Figure 4.



Figure 4.  $CP_{2k+1}$ 

As in Case 1, from the definition of f and the distances between consecutive vertices in the above ordering, we obtain:

$$span(f) = 4k + 1 + (2k - 1)(k - 2) + 2 + (2k + 1)(k - 1) = 4k^2 - 2k + 4.$$

Also, it is easy to verify that for every two distinct vertices u and v the radio condition is verified, considering each particular pair of vertices as in Case 1 and taking in consideration the following facts: every consecutive vertex in the ordering considered above verify the radio constraint by construction,  $|f(v_i) - f(v_j)| \ge 2k + 1$  if  $v_i$  and  $v_j$  are not consecutive; similary  $|f(v'_i) - f(v'_j)| \ge 2k - 1$  if  $v'_i$  and  $v'_j$  are not consecutive;  $f(v_k) - f(v'_2) = k + 2 \ge 2k + 1 - d(v_k, v'_2), f(v'_i) - f(v_1) \ge f(v'_{2k}) - f(v_1) = 2k$  for  $i \ne k + 1, f(v_{2k+1}) - f(v'_{k+2}) = k + 1 \ge 2k + 1 - d(v_{2k+1}, v'_{k+2})$  and the remaining differences for non-consecutive vertices  $v'_i$  and  $v_j$  are  $|f(v'_i) - f(v_j)| \ge 2k + 1$ .

So f is a radio labeling for  $CP_n$ , hence  $rn(CP_{2k+1}) = 4k^2 - 2k + 4$ .

## 3. RADIO LABELING AND RADIO NUMBER FOR A THORN STAR

The thorn star  $S_{n,k}$  is the graph obtained from the star graph  $S_n$  by attaching k new terminal vertices to each terminal vertex of the star. We denote by z the center of the star, with  $v_1, v_2, \ldots, v_n$  the terminal vertices from the initial star  $S_n$  and with  $u_{i1}, u_{i2}, \ldots, u_{ik}, 1 \leq i \leq n$  the new terminal vertices attached to the vertex  $v_i$ , for  $1 \leq i \leq n$ . We have  $|V(S_{n,k})| = 1 + n + nk = (k+1)n + 1$  and  $diam(S_{n,k}) = 4$ . We will show that  $rn(S_{n,k}) = (k+3)n + 2$  for  $n \ge 3$  and  $rn(S_{2,k}) = 3k + 8$ .

**Theorem 3.1.** For  $n \ge 3$  and  $k \ge 1$ ,  $rn(S_{n,k}) = (k+3)n + 2$ .

**Proof.** We will first show that  $rn(S_{n,k}) \ge (k+3)n+2$ . For that we use Theorem 1.3. The weight center of  $S_{n,k}$  is z, hence we have

$$s(S_{n,k}) = s_{S_{n,k}}(z) = \sum_{i=1..n} d(z, v_i) + \sum_{\substack{i=1..n\\j=1..k}} d(z, u_{ij}) = n + 2nk.$$

It follows that  $rn(S_{n,k}) \ge 5(k+1)n + 2 - 2(n+2nk) = (k+3)n + 2.$ 

To prove equality, it suffices to find a radio labeling f for  $S_{n,k}$  with span(f) = (k+3)n+2.

We define a label f for  $S_{n,k}$  as follows:

 $f(z) = 1, f(v_n) = 5, f(v_j) = (k+3)n + 2 - 3(n-j-1)$  for  $1 \le j \le n-1$ (vertices  $v_j$  have as labels numbers starting with kn + 8, the maximum label of these vertices being (k+3)n + 2), and terminal vertices are labeled with values from 7 to kn + 6 as follows:  $f(u_{jt}) = 7 + (j-1) + (t-1)n$ , for  $1 \le j \le n, 1 \le t \le k$ . For n = 4 and k = 3 the labeling is shown in Figure 5.



Figure 5. A radio labeling for  $S_{4,3}$ .

Hence  $span(f) = f(v_{n-1}) = (k+3)n + 2$ . It remains to verify the radio condition for each pair of vertices. We have the following cases:

•  $d(u_{jt}, v_j) = 1, 1 \le j \le n, 1 \le t \le k$ . It suffices to show that in this case we have  $|f(v_j) - f(u_{jt})| \ge 4$ .

For 
$$1 \le j \le n - 1$$
  
 $|f(v_j) - f(u_{jt})| = f(v_j) - f(u_{jt})$   
 $= (k+3)n + 2 - 3(n-j-1) - 7 - (j-1) - (t-1)n$   
 $= (k+1-t)n + 2j - 1 \ge n + 2 - 1 \ge 4.$ 

For j = n

$$|f(v_n) - f(u_{nt})| = f(u_{nt}) - f(v_n)$$
  
= 7 + (n - 1) + (t - 1)n - 5 = tn + 1 \ge n + 1 \ge 4.

•  $d(u_{jt}, u_{js}) = 2, 1 \le j \le n, 1 \le t \ne s \le k$ . In this case we have

$$|f(u_{jt}) - f(u_{js})| = |(t-1)n - (s-1)n| = |(t-s)n| \ge n \ge 3.$$

- $d(u_{jt}, u_{ls}) = 4, 1 \le j \ne l \le n, 1 \le t, s \le k$ . From the way f was defined we have  $f(u_{jt}) \ne f(u_{ls})$ .
- $d(z, u_{jt}) = 2, 1 \le j \le n, 1 \le t \le k$ . We have

$$f(u_{jt}) - f(z) \ge 7 - 1 = 6.$$

•  $d(z, v_j) = 1, 1 \le j \le n$ . We then deduce

$$f(v_j) - f(z) \ge f(v_n) - f(z) = 5 - 1 = 4.$$

•  $d(v_k, v_j) = 2, 1 \le j \ne k \le n$ . In this case the following relations hold:

$$|f(v_k) - f(v_j)| \ge 3 \text{ for } k, j < n,$$

$$f(v_j) - f(v_n) \ge f(v_1) - f(v_n) = (k+3)n + 2 - 3(n-2) - 5 = kn + 3 \ge 3.$$

•  $d(v_j, u_{it}) = 3$ , for  $1 \le j \ne i \le n, 1 \le t \le k$ . For j = n we obtain

$$|f(u_{it}) - f(v_n)| \ge 7 - 5 = 2.$$

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For 
$$j \neq n$$
  
 $|f(u_{it}) - f(v_j)| \ge (kn + 8) - (kn + 6) = 2.$ 

In all cases the radio condition is satisfied.

As in case of caterpillars  $CP_n$  with n odd, in order to prove that  $rn(S_{2,k}) = 3k + 8$  it is not sufficient to find a suitable labeling of the vertices of  $S_{2,k}$  and then apply Theorem 1.3 for the reverse inequality; we also need some additional results.

Forwards we say that the vertex  $v_1$  and the terminal vertices attached to it are vertices of type 1, vertex  $v_2$  and the terminal vertices attached to it are vertices of type 2, and the center z is of type 3.

For a radio labeling f of  $S_{2,k}$ , we order the vertices ascending by their labels and rename the terminal vertices  $u_{jt}$  in this order by  $y_1, y_2, \ldots, y_{2k}$ ; we have

$$f(y_1) < f(y_2) < \dots < f(y_{2k}).$$

We denote by Y the sequence  $y_1, y_2, \ldots, y_{2k}$ , by  $f_Y$  the sequence of the labels attached to vertices of Y:  $f(y_1), f(y_2), \ldots, f(y_{2k})$ , and with  $d_{f_Y}$  the sequence of differences between consecutive labels from  $f_Y$ , where the *i*-th element of the sequence is denoted by  $d_{f_Y}^i = f(y_{i+1}) - f(y_i)$ , for  $1 \le i \le 2k - 1$ .

On the class of radio labeling of  $S_{2,k}$  we define the function  $\Delta_Y$  as follows:

$$\Delta_Y(f) = \sum_{i=1}^{2k-1} d_{f_Y}^i = f(y_{2k}) - f(y_1).$$

### Remark 3.2.

- 1. In the sequence  $d_{f_Y}$  it is not possible to have two consecutive elements with value 1.
- 2.  $\Delta_Y$  attains a minimum only for radio labelings  $f^*$  of  $S_{2,k}$  with the sequence of differences

$$d_{f_{\mathbf{v}}^*} = \{1, 2, 1, 2, \dots, 1, 2, 1\}.$$

For those labelings  $\Delta_Y(f^*) = 3k - 2$ .

3.  $f(y_{2k}) = f(y_1) + \Delta_Y(f)$ .

**Proof.** 1. Suppose that there exists an index *i* such that  $d_{f_Y}^i = 1$  and  $d_{f_Y}^{i+1} = 1$ . It follows that

$$f(y_{i+1}) - f(y_i) = f(y_{i+2}) - f(y_{i+1}) = 1.$$

Since the pairs of vertices  $y_i$  and  $y_{i+1}$ , respectively  $y_{i+1}$  and  $y_{i+2}$  must satisfy the radio condition, it follows that  $d(y_i, y_{i+1}) = d(y_{i+1}, y_{i+2}) = 4$ , hence  $y_i$  and  $y_{i+2}$  are of the same type. We obtain  $d(y_i, y_{i+2}) = 2$ . Since the radio condition must be satisfied for the vertices  $y_i$  and  $y_{i+2}$ , it follows that  $f(y_{i+2}) - f(y_i) \ge 5 - d(y_i, y_{i+2}) = 3$ . But

$$f(y_{i+2}) - f(y_i) = f(y_{i+2}) - f(y_{i+1}) + f(y_{i+1}) - f(y_i) = 1 + 1 = 2,$$

a contradiction.

2. Using the first remark, it is obvious that the minimum can be obtained only in the conditions stated in this remark. In this conditions we have

$$\Delta_Y(f^*) = 1 \cdot k + 2 \cdot (k-1) = 3k - 2.$$

We denote  $\Delta_Y^* = \Delta_Y(f^*) = 3k - 2.$ 

**Lemma 3.3.** Let f be a radio labeling for  $S_{2,k}$ . If for a type t, with  $t \in \{1, 2\}$  there exists an index i between 1 and 2k such that  $f(y_i) < f(v_t) < f(y_{i+1})$ , then the following properties hold:

- 1.  $d_{f_V}^i \ge 4;$
- 2. If  $d_{f_Y}^i \leq 5$ , then  $y_i$  and  $y_{i+1}$  are of type 3-t;
- 3. If  $i + 2 \le 2k$ , then  $d_{f_Y}^i + d_{f_Y}^{i+1} \ge 6$ ;
- 4. If  $i-1 \ge 1$ , then  $d_{f_Y}^{i-1} + d_{f_Y}^i \ge 6$ .

**Proof.** From the radio condition we have:

$$f(v_t) - f(y_i) \ge 5 - d(v_t, y_i),$$
  
$$f(y_{i+1}) - f(v_t) \ge 5 - d(v_t, y_{i+1}).$$

It follows that

$$f(y_{i+1}) - f(y_i) \ge 10 - [d(v_t, y_i) + d(v_t, y_{i+1})].$$

But  $d(v_t, y_i)$  has value 1 if  $y_i$  is of type t, and 3 otherwise. We then obtain  $d_{f_Y}^i = f(y_{i+1}) - f(y_i) \ge 10 - (3+3) = 4$ . Moreover, if  $d_{f_Y}^i \le 5$ , then  $d(v_t, y_i) + d(v_t, y_{i+1}) \ge 5$ , from which it follows that  $d(v_t, y_i) = d(v_t, y_{i+1}) = 3$ , hence  $y_i$  and  $y_{i+1}$  are of type 3 - t.

In order to prove properties 3 and 4 of the lemma it suffices to consider the case  $d_{f_Y}^i = 4$ , since for greater values of  $d_{f_Y}^i$  the inequalities are obvious. In this case from the property 2 of the lemma it follows that  $y_i$  and  $y_{i+1}$  are of type 3 - t and  $d(v_t, y_i) = d(v_t, y_{i+1}) = 3$ .

If  $y_{i+2}$  has the same type as  $y_i$  and  $y_{i+1}$ , then  $d(y_{i+1}, y_{i+2}) = 2$  and from the radio condition we have

$$f(y_{i+2}) \ge f(y_{i+1}) + 5 - d(y_{i+1}, y_{i+2}) \ge f(y_{i+1}) + 3.$$

Then

$$d_{f_Y}^i + d_{f_Y}^{i+1} = 4 + f(y_{i+2}) - f(y_{i+1}) \ge 4 + 3 = 7.$$

Otherwise, if  $y_{i+2}$  has type t,  $d(v_t, y_{i+2}) = 1$  and from the radio condition we obtain

$$f(y_{i+2}) \ge f(v_t) + 5 - d(v_t, y_{i+2}) = f(v_t) + 4 = f(y_{i+1}) + 2,$$

hence

$$d_{f_Y}^i + d_{f_Y}^{i+1} \ge 4 + 2 = 6.$$

Property 4 can be proved analogously.

**Remark 3.4.** Let f be a radio labeling for  $S_{2,k}$ . If there exists an index i between 1 and 2k - 1 such that  $f(y_i) < f(z) < f(y_{i+1})$ , then  $d^i_{f_Y} \ge 6$ .

**Proof.** From the radio condition we have:

$$f(y_{i+1}) - f(y_i) \ge 10 - [d(z, y_i) + d(z, y_{i+1})] = 10 - (2+2) = 6.$$

Using these results we can determine a lower bound for  $rn(S_{2,k})$ .

**Theorem 3.5.** For  $k \ge 1$ ,  $rn(S_{2,k}) \ge 3k + 8$ .

**Proof.** Let f be a radio labeling for  $S_{2,k}$ . We prove that  $span(f) \ge 3k+8$ . We consider the following cases, by comparing the labels f(z),  $f(v_1)$ ,  $f(v_2)$  with the labels from  $f_Y$ .

Case 1. None of the labels f(z),  $f(v_1)$ ,  $f(v_2)$  are between  $f(y_1)$  and  $f(y_{2k})$ .

In this case the sequence of all vertices ordered by their labels is obtained starting from the sequence Y by adding, in turn, each of the vertices  $z, v_1, v_2$  at the beginning or at the end of the current sequence. We denote by z',  $v'_1$ , respectively  $v'_2$  the vertex near which  $z, v_1$ , respectively  $v_2$  are added in the sequence. Then, using the radio condition, we obtain:

$$span(f) \ge 1 + \Delta_Y(f) + |f(z) - f(z')| + |f(v_1) - f(v'_1)| + |f(v_2) - f(v'_2)|$$
  

$$\ge 1 + \Delta_Y(f) + 5 - d(z, z') + 5 - d(v_1, v'_1) + 5 - d(v_2, v'_2)$$
  

$$= \Delta_Y(f) + 16 - [d(z, z') + d(v_1, v'_1) + d(v_2, v'_2)].$$

Let  $S = d(z, z') + d(v_1, v'_1) + d(v_2, v'_2)$ . For  $t \in \{1, 2\}$  and  $1 \le i \le 2k$  we have:  $d(z, y_i) = 2$ ,  $d(v_t, y_i) = 1$  if  $y_i$  is of type t,  $d(v_t, y_i) = 3$  if  $y_i$  is of type t,  $d(v_1, v_2) = 2$  and  $d(v_t, z) = 1$ . Moreover, at most two of the vertices z',  $v'_1, v'_2$  are in Y. It follows that  $S \le 7$ .

If  $S \leq 6$ , then

$$span(f) \ge \Delta_Y(f) + 16 - S \ge \Delta_Y^* + 16 - S \ge 3k - 2 + 16 - 6 = 3k + 8.$$

If S = 7, then at least one of the vertices  $v'_t$  with  $t \in \{1, 2\}$  is  $y_1$  or  $y_{2k}$  and  $d(v_t, v'_t) = 3$ . We can assume, without loss of generality, that  $v'_1 = y_1$ . We have  $f(v_1) < f(y_1) < f(y_2)$ . We will prove that  $f(y_{2k}) \ge f(v_1) + 3k + 1$ .

From the radio condition for  $v_1$  and  $y_1$  we obtain

$$f(y_1) \ge f(v_1) + 5 - d(v_1, y_1) = f(v_1) + 2$$

and then  $f(y_2) \ge f(v_1) + 4$ .

If  $f(y_1) \ge f(v_1) + 3$ , then

$$f(y_{2k}) = f(y_1) + \Delta_Y(f) \ge f(y_1) + \Delta_Y^* \ge f(v_1) + 3 + 3k - 2 = f(v_1) + 3k + 1.$$

Otherwise we have  $f(y_1) = f(v_1) + 2$  and it follows that  $d(v_1, y_1) = 3$  and  $y_1$  is of type 2.

Moreover, if  $y_2$  is of type 1, from the radio condition we have

$$f(y_2) \ge f(v_1) + 5 - d(v_1, y_2) = f(v_1) + 4 = f(y_1) + 2.$$

Otherwise

$$f(y_2) \ge f(y_1) + 5 - d(y_1, y_2) = f(y_1) + 3(\ge f(v_1) + 4)$$

In both situations we obtain  $d_{f_Y}^1 = f(y_2) - f(y_1) \ge 2$ , hence  $\Delta_Y(f) > \Delta_Y^*$ , and the following relation holds:

$$f(y_{2k}) = f(y_1) + \Delta_Y(f) \ge f(y_1) + \Delta_Y^* + 1$$
  
$$\ge f(v_1) + 2 + 3k - 2 + 1 = f(v_1) + 3k + 1.$$

Then

$$span(f) \ge f(y_{2k}) + |f(z) - f(z')| + |f(v_2) - f(v'_2)|$$
  

$$\ge f(v_1) + 3k + 1 + 10 - [d(z, z') + d(v_2, v'_2)]$$
  

$$\ge 1 + 3k + 1 + 10 - [S - d(v_1, v'_1)]$$
  

$$\ge 3k + 12 - (7 - 1) = 3k + 8.$$

Case 2. Only one of the values  $f(v_1)$  and  $f(v_2)$  is between  $f(y_1)$  and  $f(y_{2k})$  (f(z) is not between  $f(y_1)$  and  $f(y_{2k})$ .

We can assume, without loss of generality, that  $f(v_1) \in \{f(y_1), \ldots, f(y_{2k})\}$ . Then there exists an index p between 1 and 2k - 1 such that  $f(y_p) < f(v_1) < f(y_{p+1})$ . From lemma 3.3 we have  $d_{f_Y}^p \ge 4$ .

If  $d_{f_V}^p \ge 6$ , using remark 3.2 we obtain:

$$\Delta_Y(f) \ge 6 + 1 \cdot k + 2 \cdot (k - 2) = 3k + 2.$$

Otherwise we have  $4 \leq d_{f_Y}^p \leq 5$ , and, from Lemma 3.3, it follows that  $y_p$  and  $y_{p+1}$  are of type 2 and  $k \geq 2$ . Then  $p-1 \geq 1$  or  $p+2 \geq 2k$ . We assume  $p+2 \geq 2k$ , since the case  $p-1 \geq 1$  can be treated analogously. Using lemma 3.3 it follows that  $d_{f_Y}^p + d_{f_Y}^{p+1} \geq 6$ . Moreover, since  $y_p$  and  $y_{p+1}$  are of type 2, there exists an index q between 1 and 2k-1 such that  $y_q$  and  $y_{q+1}$  are of type 1, and then

$$d_{f_Y}^q = f(y_{q+1}) - f(y_q) \ge 5 - d(y_{q+1}, y_q) = 5 - 2 = 3.$$

It follows that  $\Delta_Y(f) \ge 6 + 3 + 1 \cdot (k-1) + 2 \cdot (k-3) = 3k + 2$ . In all cases we obtain  $\Delta_Y(f) \ge 3k + 2$ , and it follows that

$$span(f) \ge f(y_{2k}) + |f(z) - f(z')| + |f(v_2) - f(v'_2)|$$
  

$$\ge 1 + \Delta_Y(f) + |f(z) - f(z')| + |f(v_2) - f(v'_2)|$$
  

$$\ge 1 + 3k + 2 + 10 - [d(z, z') + d(v_2, v'_2)]$$
  

$$\ge 3k + 3 + 10 - (2 + 3) = 3k + 8.$$

Case 3.  $f(v_1)$  and  $f(v_2)$  are between  $f(y_1)$  and  $f(y_{2k})$ , but f(z) is not. Then there exist two indices p and q between 1 and 2k - 1 such that

$$f(y_p) < f(v_1) < f(y_{p+1})$$
 and  $f(y_q) < f(v_2) < f(y_{q+1})$ .

By Lemma 3.3 we have  $d_{f_Y}^p \ge 4$  and  $d_{f_Y}^q \ge 4$ . We prove that  $\Delta_Y(f) \ge 3k+4$ . If  $d_{f_Y}^p \ge 5$  and  $d_{f_Y}^q \ge 5$ , then, from Remark 3.2, it follows that

$$\Delta_Y(f) \ge 5 + 5 + 1 \cdot k + 2 \cdot (k - 3) = 3k + 4.$$

If  $d_{f_Y}^p = 4$  and  $d_{f_Y}^q \ge 5$ , then, using the same lemma, for  $p+1 \le 2k$  we have  $d_{f_Y}^p + d_{f_Y}^{p+1} \ge 6$  and for  $p-1 \ge 1$  we have  $d_{f_Y}^{p-1} + d_{f_Y}^p \ge 6$ . Hence, if there exists,  $d_{f_Y}^{p+1} \ge 2$  and  $d_{f_Y}^{p-1} \ge 2$  we obtain

$$\Delta_Y(f) \ge 4 + 5 + 1 \cdot (k - 1) + 2 \cdot (k - 2) = 3k + 4$$

since in the sequence  $d_{f_Y}$  it is not possible to have two consecutive elements with value 1. Analogously we can prove that, if  $d_{f_Y}^p \ge 5$  and  $d_{f_Y}^q = 4$ , then  $\Delta_Y(f) \ge 3k + 4$ .

It remains to consider the situation when  $d_{f_Y}^p = d_{f_Y}^q = 4$ . Using an argument similar to the previous one, it can be proved that in the sequence  $d_{f_Y}$  the value 1 cannot be on one of the positions p-1, p+1, q-1, q+1, if such a position exist. Then  $\Delta_Y(f) \ge 4+4+1 \cdot (k-2)+2 \cdot (k-1)=3k+4$ . In all situations we have  $\Delta_Y(f) \ge 2k+4$  hence

In all situations we have  $\Delta_Y(f) \ge 3k + 4$ , hence

$$span(f) \ge 1 + \Delta_Y(f) + |f(z) - f(z')|$$
  
 
$$\ge 1 + 3k + 4 + 5 - d(z, z') \ge 3k + 10 - 2 = 3k + 8.$$

Case 4. f(z) is between  $f(y_1)$  and  $f(y_{2k})$ , but  $f(v_1)$  and  $f(v_2)$  are not. Then there exists an index p between 1 and 2k - 1 such that  $f(y_p) < f(z) < f(y_{p+1})$ . By remark 3.4 we have  $d_{f_Y}^p \ge 6$ . We assume, without loss of generality, that  $f(v_1) < f(y_1)$  and  $f(v_2)$  satisfies one of the relations:  $f(v_2) < f(v_1)$  or  $f(v_2) > f(y_{2k})$ . If p = 1, then the smallest labels are  $f(v_1) < f(y_1) < f(z) < f(y_2)$  and we obtain  $\Delta_Y(f) \ge 6 + 1 \cdot (k-1) + 2 \cdot (k-1) = 3k+3$ , hence it follows

$$span(f) \ge 1 + \Delta_Y(f) + |f(v_1) - f(y_1)| + |f(v_2) - f(v'_2)|$$
$$\ge 1 + 3k + 3 + 5 - d(v_1, y_1) + 5 - d(v_2, v'_2)$$
$$\ge 1 + 3k + 3 + 5 - 3 + 5 - 3 = 3k + 8.$$

If p > 1, then, using same type of arguments as in case 1, we will prove that  $f(y_2) \ge f(v_1) + 4$ . Since in  $d_{f_Y}$  is not possible to have two consecutive elements with value 1, it will follow that

$$f(y_{2k}) \ge f(y_2) + 6 + 1 \cdot (k-1) + 2 \cdot (k-2)$$
$$\ge f(v_1) + 4 + 3k + 1 \ge 3k + 6$$

and so

$$span(f) \ge f(y_{2k}) + |f(v_2) - f(v'_2)| \ge 3k + 6 + 5 - d(v_2, v'_2) \ge 3k + 8.$$

From the radio condition,  $f(y_1) \ge f(v_1) + 2$ . If the inequality is strict, then it is obvious that  $f(y_2) \ge f(y_1) + 1 \ge f(v_1) + 4$ . Otherwise we have  $f(y_1) = f(v_1) + 2$  and, using the radio condition, we obtain  $d(v_1, y_1) = 3$ , which implies that  $y_1$  is of type 2. As in case 1, it follows that  $f(y_2) \ge f(v_1) + 4$ .

Case 5. Only f(z) and one of the labels  $f(v_1)$  or  $f(v_2)$  are between  $f(y_1)$  and  $f(y_{2k})$ ; assume  $f(v_2)$  is between  $f(y_1)$  and  $f(y_{2k})$ .

Then there exist two indices p and q between 1 and 2k - 1 such that  $f(y_p) < f(z) < f(y_{p+1})$  and  $f(y_q) < f(v_2) < f(y_{q+1})$ . By Lemma 3.3 and Remark 3.4 we have  $d_{f_Y}^p \ge 6$  and  $d_{f_Y}^q \ge 4$ . We will prove that  $\Delta_Y(f) \ge 3k + 5$ . It will follow that

$$span(f) \ge 1 + \Delta_Y(f) + |f(v_2) - f(v'_2)| \ge 1 + 3k + 5 + 5 - 3 = 3k + 8.$$

Thus, if  $d_{f_Y}^q \ge 5$ , then

$$\Delta_Y(f) \ge 6 + 5 + 1 \cdot k + 2 \cdot (k - 3) = 3k + 5.$$

Otherwise, if  $d_{f_Y}^q = 4$ , using arguments similar to the previous cases, it follows that in the sequence  $d_{f_Y}$  value 1 cannot be on positions q-1, q+1, if these positions exist. Then  $\Delta_Y(f) \ge 4+6+1\cdot(k-1)+2\cdot(k-2)=3k+5$ .

Case 6. All of the labels f(z),  $f(v_1)$ ,  $f(v_2)$  are between  $f(y_1)$  and  $f(y_{2k})$ . Then there exist three indices p, q and r between 1 and 2k - 1 such that  $f(y_p) < f(z) < f(y_{p+1})$ ,  $f(y_q) < f(v_1) < f(y_{q+1})$  and  $f(y_r) < f(v_2) < f(y_{r+1})$  and we have  $d_{f_Y}^p \ge 6$ ,  $d_{f_Y}^q \ge 4$  and  $d_{f_Y}^r \ge 4$ .

If one of the values  $d_{f_Y}^q$  or  $d_{f_Y}^r$  is strictly greater than 4, then  $\Delta_Y(f) \ge 6 + 5 + 4 + 1 \cdot k + 2 \cdot (k - 4) = 3k + 7$ . It follows that in the sequence  $d_{f_Y}$  value 1 cannot be on positions q - 1, q + 1, r - 1, r + 1, if these positions exist, hence  $\Delta_Y(f) \ge 4 + 4 + 6 + 1 \cdot (k - 2) + 2 \cdot (k - 2) = 3k + 8$ .

In both situations we have  $span(f) \ge 1 + \Delta_Y(f) \ge 3k + 8$ .

**Theorem 3.6.** For  $k \ge 1$ ,  $rn(S_{2,k}) = 3k + 8$ .

**Proof.** By Theorem 3.5, it suffices to build a radio labeling f for  $S_{2,k}$  with span(f) = 3k + 8. Let f be a labeling defined as follows: f(z) = 1,  $f(u_{11}) = 4$ ,  $f(u_{1j}) = 4 + 3(j-1)$ , for  $2 \le j \le k$ ,  $f(u_{21}) = 5$ ,  $f(u_{2j}) = 5 + 3(j-1)$ , for  $2 \le j \le k$ ,  $f(v_1) = 1 + f(u_{2k}) + 5 - d(v_1, u_{2k}) = f(u_{2k}) + 3$ ,  $f(v_2) = f(v_1) + 5 - d(v_1, v_2) = f(u_{2k}) + 3 + 3 = f(u_{2k}) + 6$ . Then  $f(u_{2k}) = 5 + 3(k-1) = 3k + 2$  and  $span(f) = f(v_2) = 3k + 8$ .

For n = 4 and k = 3 the labeling is shown in Figure 6.



Figure 6. A radio labeling for  $S_{2,3}$ .

We prove that f is a radio labeling for  $S_{2,k}$ , by considering each possible type of pairs of vertices and verifying the radio condition for it.

• For any  $1 \le t \le 2, 1 \le j \le k$  we have

$$|f(u_{tj}) - f(z)| \ge f(u_{11}) - f(z) = 3 = 5 - d(z, u_{tj}).$$

• For any  $1 \le t \le 2$ ,  $1 \le i < j \le k$  we have

$$|f(u_{tj}) - f(u_{ti})| = 3(j-i) \ge 3 = 5 - d(u_{tj}, u_{ti}).$$

Moreover,  $d(u_{tj}, u_{t'i}) = 4$  for t' = 3 - t and  $f(u_{tj}) \neq f(u_{t'i})$  from the way f was defined.

• For any  $1 \le i < j \le k$  the following relations hold

$$\begin{split} f(v_1) - f(u_{1j}) &\geq f(v_1) - f(u_{1k}) = 3k + 5 - (3k + 1) = 4 = 5 - d(v_1, u_{1j}). \\ f(v_1) - f(u_{2j}) &\geq f(v_1) - f(u_{2k}) = 3k + 5 - (3k + 2) = 3 \geq 5 - d(v_1, u_{2j}). \\ f(v_2) - f(u_{1j}) &\geq f(v_2) - f(u_{1k}) = 3k + 8 - (3k + 1) = 7 > 5 - d(v_2, u_{1j}). \\ f(v_2) - f(u_{2j}) &\geq f(v_2) - f(u_{2k}) = 3k + 8 - (3k + 2) = 6 \geq 5 - d(v_2, u_{2j}). \end{split}$$

• We have the relations:

$$f(v_2) - f(z) \ge f(v_1) - f(z) = f(u_{2k}) + 3 - 1$$
  
=  $3k + 4 \ge 5 - d(z, v_1) = 5 - d(z, v_2),$   
 $f(v_2) - f(v_1) = 3.$ 

For related problems see the survey paper [4].

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Received 12 November 2008 Revised 8 June 2009 Accepted 8 June 2009