Discussiones Mathematicae
Graph Theory 30 (2010) 185–199

ON THE $(2, 2)$-DOMINATION NUMBER OF TREES *

You Lu, Xinmin Hou and Jun-Ming Xu

Department of Mathematics
University of Science and Technology of China
Hefei, Anhui, 230026, China

e-mail: xmhou@ustc.edu.cn

Abstract

Let $\gamma(G)$ and $\gamma_{2,2}(G)$ denote the domination number and $(2, 2)$-domination number of a graph G, respectively. In this paper, for any nontrivial tree T, we show that $\frac{2(\gamma(T)+1)}{3} \leq \gamma_{2,2}(T) \leq 2\gamma(T)$. Moreover, we characterize all the trees achieving the equalities.

Keywords: domination number, total domination number, $(2, 2)$-domination number.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

For notation and graph theory terminology we follow [2, 5, 6]. Let $G = (V(G), E(G))$ be a simple graph. For $u, v \in V(G)$, the distance $d_G(u, v)$ between u and v is the length of the shortest uv-paths in G. The diameter of G is $d(G) = \max\{d_G(u, v) : u, v \in V(G)\}$. For an integer $k \geq 1$ and $v \in V(G)$, the open k-neighborhood of v is $N_k(v, G) = \{u \in V(G) : 0 < d_G(u, v) \leq k\}$, and the closed k-neighborhood of v is $N_k[v, G] = N_k(v) \cup \{v\}$. If the graph G is clear from the context, we will simply use $N_k(v)$ and $N_k[v]$ instead of $N_k(v, G)$ and $N_k[v, G]$, respectively. The degree $deg(v)$ of v is the number of vertices in $N_1(v)$. The minimum k-degree $\delta_k(G)$ is defined by $\delta_k(G) = \min\{|N_k(v)| : v \in V(G)\}$. For $S \subseteq V(G)$, $N_k(S) = \cup_{v \in S} N_k(v)$, $N_k[S] = N_k(S) \cup S$. For convenience, we also denote $N_1(S)$

*The work was supported by NNSF of China (No.10701068 and No.10671191).

and $N_1[S]$ by $N(S)$ and $N[S]$, respectively. Let $G[S]$ be the subgraph of G induced by S.

For $S \subseteq V(G)$, S is a dominating set if $N[S] = V(G)$ and a total dominating set if $N(S) = V(G)$. The domination number $\gamma(G)$ (resp. total domination number $\gamma_t(G)$) is the minimum cardinality among all dominating sets (resp. total dominating sets) of G. Any minimum dominating set of G will be called a γ-set of G. For all graphs G without isolated vertices, $\gamma_t(G) \leq 2\gamma(G)$. If $S, T \subseteq V(G)$, we say that S dominates T in G if $T \subseteq N[S]$.

Let k and p be positive integers. A subset S of $V(G)$ is defined to be a (k, p)-dominating set of G if, for any vertex $v \in V(G) \setminus S$, $|N_k(v) \cap S| \geq p$. The (k, p)-domination number of G, denoted by $\gamma_{k,p}(G)$, is the minimum cardinality among all (k, p)-dominating sets of G. Any minimum (k, p)-dominating set of G will be called a $\gamma_{k,p}$-set of G. Clearly, for a graph G, a $(1, 1)$-dominating set is a classic dominating set, that is, $\gamma_{1,1}(G) = \gamma(G)$. For $S, T \subseteq V(G)$, we say that S (k, p)-dominates T in G if $|N_k(v) \cap S| \geq p$, for any $v \in T - S$.

The concept of (k, p)-domination in a graph G is a generalized domination which combined k-distance domination and p-domination in G. So the investigation of (k, p)-domination of G is more interesting and has received the attention of many researchers. In [1], Bean, Henning and Swart investigated the relationship between $\gamma_{k,p}(G)$ and the order of G and posed a conjecture: $\gamma_{k,p}(G) \leq \frac{p}{k+p}|V(G)|$ if G is a graph with $\delta_k(G) \geq k + p - 1$. In 2005, Fischermann and Volkmann [3] confirmed that the conjecture is valid for all positive integers k and p, where p is a multiple of k. In [7], Korneffel, Meierling, and Volkmann not only showed that $\gamma_{2,2}(G) \leq (|V(G)| + 1)/2$ without the condition $\delta_2(G) \geq 3$, but characterized all graphs achieving the equality.

In this paper, we concentrate our attention on $(2, 2)$-domination of trees and give upper and lower bounds of $\gamma_{2,2}(T)$ in terms of the domination number $\gamma(T)$. The main result is:

$$\frac{2(\gamma(T) + 1)}{3} \leq \gamma_{2,2}(T) \leq 2\gamma(T)$$

for any nontrivial tree T. Moreover, we characterize all the trees achieving the equalities.

2. The Lower Bound

For a vertex v in a rooted tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively. And we define $D[v] = D(v) \cup \{v\}$. Let $L(T)$ and $S(T)$ denote the set of the leaves and the set of the support vertices of T, respectively. We use $P_l = u_1 u_2 \cdots u_l$ to represent a path with l vertices. As an immediate consequence from the definition of a $(2,2)$-dominating set, we have

Lemma 1. *Let S be a $(2,2)$-dominating set of G. If v is a support vertex with at least two leaves in G, then $|N[v] \cap S| \geq 2$.*

Lemma 2. *Let G be a graph obtained from a graph G' by joining u_3 of a path $P_4 = u_1u_2u_3u_4$ to a vertex v of G'.*

(1) *If S is a $\gamma_{2,2}$-set of G, then $|S \cap V(P_4)| = 2$;*

(2) *If S is a $\gamma_{2,2}$-set of G containing vertices of degree one as few as possible, then $S \cap V(P_4) = \{u_2, u_3\}$.*

We introduce the family $\mathcal{T}$ of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees such that $T_1 = P_4$, $T = T_k$, and, for $k \geq 2$, T_{i+1} $(1 \leq i \leq k-1)$ is obtained recursively from T_i by one of the operations defined below.

We recall that the corona cor(G) of a graph G is a graph obtained from G by adding a pendant edge to each vertex of G. Let $H = cor(P_3)$ with vertex set $V(H) = \{u, v, w, u', v', w'\}$ and edge set $E(H) = \{uv, vw, uu', vv', ww'\}$. Let $A(T_1) = S(T_1)$.

- **Operation $\mathcal{O}_1$:** Attach a vertex by joining it to a support vertex of T_i. Let $A(T_{i+1}) = A(T_i)$.
- **Operation $\mathcal{O}_2$:** Attach a copy of H by joining w to a vertex of $A(T_i)$. Let $A(T_{i+1}) = A(T_i) \cup \{u, v\}$.
- **Operation $\mathcal{O}_3$:** Attach a copy of H by joining w' to a leaf of T_i such that the leaf is adjacent to a vertex in $A(T_i)$ which has at least two leaves in T_i.
  Let $A(T_{i+1}) = A(T_i) \cup \{u, v\}$.

By induction on the length k of the sequence of the construction of $T \in \mathcal{T}$, the following lemma is clearly true from the construction.

Lemma 3. *Let $T \in \mathcal{T}$. Then*

(1) *every vertex of $A(T)$ is a support vertex of T;*

(2) *$A(T)$ is a $(2,2)$-dominating set of T;*

(3) *$T[A(T)] = \cup_{i=1}^{t} K_2$, where t is the number of the operations $\mathcal{O}_2$ and $\mathcal{O}_3$ used by the construction of T.*

For a dominating set of a tree T, we can derive the following observation from the definition.

Lemma 4. *Let T be a tree of order at least three. Then T has a γ-set containing all the support vertices.*

From the definition of Operation $\mathcal{O}_i$ $(i = 1, 2, 3)$ and Lemma 4, we can easily prove

Lemma 5. *Let $T' \in \mathcal{T}$ and T is obtained from T' by Operation $\mathcal{O}_i$ $(i = 1, 2, 3)$.*

(1) *If $i = 1$, then $\gamma(T) = \gamma(T')$;*

(2) *If $i = 2$, then $\gamma(T) = \gamma(T') + 3$;*

(3) *If $i = 3$, then $\gamma(T) = \gamma(T') + 3$.*

The following lemma characterizes the minimum $(2,2)$-dominating set of $T \in \mathcal{T}$.

Lemma 6. *Let $T \in \mathcal{T}$ and $T \neq P_4$. Then $\gamma_{2,2}(T) = 2(\gamma(T) + 1)/3$ and $A(T)$ is the unique $\gamma_{2,2}$-set of T.*

Proof. Suppose T is obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 2)$ of trees, where $T_1 = P_4$, $T = T_k$, and, T_{i+1} $(1 \leq i \leq k-1)$ can be obtained from T_i by Operation $\mathcal{O}_j$ $(j = 1, 2$ or $3)$. We prove by induction on the length k of the sequence $T_1, T_2, \ldots, T_k$.

If $k = 2$, then $T = T_2$. It can be checked directly that the results are true for $T = T_2$. Now assume $k > 2$ and the results hold for all the trees in $\mathcal{T}$ that can be constructed from a sequence of length at most $k-1$. Let $T' = T_{k-1}$ and S be a $\gamma_{2,2}$-set of T.

If T is obtained from T' by Operation $\mathcal{O}_1$ by attaching a vertex x to a support vertex y of T', then, by Lemma 3 (2), $A(T') = A(T)$ is a $(2,2)$-dominating set of T. Hence $|S| = \gamma_{2,2}(T) \leq |A(T')|$. Let y' be a leaf of y

in T'. By the induction hypothesis on T', $\gamma_{2,2}(T') = \frac{2(\gamma(T')+1)}{3}$ and $A(T')$ is the unique $\gamma_{2,2}$-set of T'. We claim that $x \notin S$, then S is a $(2,2)$-dominating set of T' with $|S| = |A(T')|$. And, by Lemma 5, $\gamma_{2,2}(T) = |S| = |A(T')| = \gamma_{2,2}(T') = \frac{2(\gamma(T')+1)}{3} = \frac{2(\gamma(T)+1)}{3}$. Suppose to the contrary that $x \in S$, let $S' = (S \setminus \{x\}) \cup \{y'\}$ if $y' \notin S$; otherwise $(S \setminus \{x\}) \cup \{y\}$. Then S' is a $(2,2)$-dominating set of T' with $|S'| \leq |S| \leq |A(T')|$. Hence S' is a $\gamma_{2,2}$-set of T' containing a leaf y'. By the induction hypothesis on T', $S' = A(T')$, which contradicts that every vertex of $A(T')$ is a support vertex of T'.

If T is obtained from T' by Operation $\mathcal{O}_2$ by attaching H to a vertex y of $A(T')$, then, by Lemma 3 (2), $A(T) = A(T') \cup \{u,v\}$ is a $(2,2)$-dominating set of T. And so $|S| = \gamma_{2,2}(T) \leq |A(T')| + 2$. Since $y \in A(T') \subseteq S(T')$, let y' be a leaf of y in T'. By the induction hypothesis on T', $\gamma_{2,2}(T') = \frac{2(\gamma(T')+1)}{3}$ and $A(T')$ is the unique $\gamma_{2,2}$-set of T'. Now we prove $S = A(T)$. Note that $N_2[y',T'] = N_2[y',T] \setminus \{w\}$ and $N_2[y',T'] = N[y,T']$. Since S is a $\gamma_{2,2}$-set of T, $|N[y,T'] \cap S| = |N_2[y',T'] \cap S| \geq 1$. We claim that $|N[y,T'] \cap S| \geq 2$. Otherwise, we have $|N_2[y',T'] \cap S| = |N[y,T'] \cap S| = 1$. Then $|S \cap \{w,w'\}| \geq 1$ (Suppose that $S \cap \{w,w'\} = \emptyset$, then, to $(2,2)$-dominate w', $y \in S$. By $|N_2[y',T'] \cap S| = 1$, we have $y' \notin S$, and so y' can't be $(2,2)$-dominated by S, a contradiction). By Lemma 2 (1), $|S \cap \{u,v,u',v'\}| = 2$. So $|S \cap V(H)| \geq 3$. Let y'' be any vertex in $N[y,T']$ which is not contained in S. Then $(S \cap V(T')) \cup \{y''\}$ is a $(2,2)$-dominating set of T'. Since

$$|(S \cap V(T')) \cup \{y''\}| = |S \cap V(T')| + 1 \leq |S| - 3 + 1 = |S| - 2 \leq |A(T')| = \gamma_{2,2}(T'),$$

$(S \cap V(T')) \cup \{y''\}$ is a $\gamma_{2,2}$-set of T'. Since $|N[y,T'] \cap S| = 1$, $N[y,T']$ contains at least two vertices which are not in S, that is, we have at least two choices of y''. So T' has at least two distinct $\gamma_{2,2}$-sets, a contradiction with T' has a unique $\gamma_{2,2}$-set. The claim holds. Hence $S \cap V(T')$ is a $(2,2)$-dominating set of T'. By Lemma 2 (1), we have $|S \cap \{u,v,u',v'\}| = 2$, and so $|S \cap V(T')| \leq |S| - 2 = \gamma_{2,2}(T) - 2 \leq |A(T')| = \gamma_{2,2}(T')$. So $S \cap V(T')$ is the unique $\gamma_{2,2}$-set $A(T')$ of T' and $|S \cap V(H)| = 2$. It is easy to check that $S \cap V(H) = \{u,v\}$. Hence $S = (S \cap V(T')) \cup (S \cap V(H)) = A(T') \cup \{u,v\} = A(T)$. By Lemma 5, $\gamma_{2,2}(T) = |S| = \gamma_{2,2}(T') + 2 = \frac{2(\gamma(T')+1)}{3} + 2 = \frac{2(\gamma(T)+1)}{3}$.

If T is obtained from T' by Operation $\mathcal{O}_3$ by attaching H to a leaf x of T', then, by Lemma 3 (2), $A(T) = A(T') \cup \{u,v\}$ is a $(2,2)$-dominating set of T and so $|S| = \gamma_{2,2}(T) \leq |A(T')| + 2$. Let y be the support vertex of x in T' and y' another leaf of y. By the induction hypothesis on T', $\gamma_{2,2}(T') = \frac{2(\gamma(T')+1)}{3}$ and $A(T')$ is the unique $\gamma_{2,2}$-set of T'. Now we prove that $S = A(T)$. By Lemma 2 (1), $|S \cap \{u, v, u', v'\}| = 2$, and so $|S \cap (V(T') \cup \{w, w'\})| = |S| - 2 \leq |A(T')| = \gamma_{2,2}(T')$. Note that $S \cap (V(T') \cup \{w, w'\})$ $(2,2)$-dominates T' in T. We claim that $S \cap \{w, w'\} = \emptyset$. Otherwise, we have $|S \cap V(T')| < \gamma_{2,2}(T')$, and so $S \cap V(T')$ is not a $(2,2)$-dominating set of T'. Hence $|S \cap N_2[y', T']| = 1$, furthermore, $S \cap N_2[y', T'] = \{y'\}$. Hence we can check easily that $(S \cap V(T')) \cup \{x\}$ and $(S \cap V(T')) \cup \{y\}$ are two different $\gamma_{2,2}$-sets of T', which contradicts with $A(T')$ is the unique $\gamma_{2,2}$-set of T'. So $S \cap V(T')$ is the unique $\gamma_{2,2}$-set $A(T')$ of T' and $|S \cap V(H)| = 2$. It is easy to check that $S \cap V(H) = \{u, v\}$. Hence $S = (S \cap V(T')) \cup (S \cap V(H)) = A(T') \cup \{u, v\} = A(T)$. By Lemma 5, $\gamma_{2,2}(T) = |S| = \gamma_{2,2}(T') + 2 = \frac{2(\gamma(T')+1)}{3} + 2 = \frac{2(\gamma(T)+1)}{3}$. ■

Lemma 7. *Let $T \in \mathcal{T}$ and c be a vertex in T such that c is not in any γ-set of T. Then c is a leaf of T and the support vertex of c is adjacent with at least two leaves in T.*

Proof. Suppose T is obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees such that $T_1 = P_4$, $T = T_k$ and, for $k \geq 2$, T_{i+1} $(1 \leq i < k)$ is obtained from T_i by Operation $\mathcal{O}_j$ $(j = 1, 2$ or $3)$. Let D be a γ-set of T containing all the support vertices. D exists by Lemma 4.

First we show that, for any vertex $x \notin L(T) \cup S(T)$, there exists a γ-set of T containing x. Since $x \notin L(T) \cup S(T)$, by the definition of the operations, there is some i $(2 \leq i < k)$ such that T_{i+1} is obtained from T_i by Operation $\mathcal{O}_3$ by joining $w' \in V(H)$ to a leaf y of T_i and $x = y, w'$ or w. Clearly, each of y, w' and w has degree two in T. To dominate w', one of $\{y, w', w\}$ must be contained in D. Since y and w are dominated by $S(T) \subseteq D$, we can choose one of $\{y, w', w\}$ arbitrarily such that it belongs to D and dominates w'. Thus we can choose D containing x.

Since c is not in any γ-set of T, c is a leaf of T. Let y be the support vertex of c in T. Suppose that y has a unique leaf c in T. Choose a γ-set D of T such that D contains all the support vertices of T and the number of private neighbors of y with respect to D is minimal (A vertex u is called a private neighbor of a vertex v with respect to a dominating set D if $N(u) \cap D = \{v\}$). We claim that c is a unique private neighbor of y with respect to D. Otherwise, let x be another private neighbor of y with respect to D. Then $x \notin L(T) \cup S(T)$. By the above proof, there exists a γ-set D' of T with $x \in D'$ such that D' contains all the support vertices of T, but the number of private neighbors of y with respect to D' is less than the number of private neighbors of y with respect to D, a contradiction with the choice of D. Hence c is the unique private neighbor of y in D. Thus we can replace y by c in D and get a γ-set of T containing c, a contradiction. ■

Theorem 8. *Let T be a nontrivial tree, then*

$$\gamma_{2,2}(T) \geq 2(\gamma(T)+1)/3$$

with equality if and only if $T \in \mathcal{T}$.

Proof. Let T be a tree of order n. We proceed by induction on n. If $1 < n \leq 4$, then we can check that $\gamma_{2,2}(T) \geq 2(\gamma(T)+1)/3$ with equality if and only if $T = P_4 \in \mathcal{T}$. This establishes the base cases. Assume that the result holds for every tree T' of order $4 \leq |V(T')| = n' < n$. If $d(T) = 2$, then T is a star. Hence $\gamma_{2,2}(T) = 2$ and $\gamma(T) = 1$. So we have $\gamma_{2,2}(T) > 2(\gamma(T)+1)/3$. If $d(T) = 3$, then T can be seen as a tree constructed from P_4 by a sequence of operations $\mathcal{O}_1$. Hence $T \in \mathcal{T}$. By Lemma 6, $\gamma_{2,2}(T) = 2(\gamma(T)+1)/3$. So in the following we will assume that $d(T) \geq 4$. Let $P = uvwxyz \cdots r$ be a longest path in T. We root T at r.

Case 1. If $\deg(v) \geq 3$, then there exists another leaf v' adjacent to v. Let $T' = T - v'$. By Lemma 4, we have $\gamma(T) = \gamma(T')$. By Lemma 1, we can choose a $\gamma_{2,2}$-set S of T such that S does not contain v'. Thus S is a $(2,2)$-dominating set of T', too. By the induction hypothesis on T', we have

$$\gamma_{2,2}(T) = |S| \geq \gamma_{2,2}(T') \geq \frac{2}{3}(\gamma(T')+1) = \frac{2}{3}(\gamma(T)+1).$$

Further if $\gamma_{2,2}(T) = 2(\gamma(T)+1)/3$, then $\gamma_{2,2}(T') = 2(\gamma(T')+1)/3$. By the inductive hypothesis on T', $T' \in \mathcal{T}$. Since v is a support vertex of T', T is obtained from T' by Operation $\mathcal{O}_1$. Hence $T \in \mathcal{T}$.

In the following, without loss of generality, we will assume that $\deg(v) = 2$ and each support vertex of T is exactly adjacent with one leaf.

Case 2. If $\deg(w) = 2$, then $T - \{wx\}$ has a component $P_3 = uvw$. Let T' be the subtree of $T - \{wx\}$ containing x and D' be a γ-set of T'. Since $D' \cup \{v\}$ is a dominating set of T, $\gamma(T') \geq \gamma(T) - 1$. We choose S as a $\gamma_{2,2}$-set of T such that S contains as few vertices as possible of $\{u,v,w\}$. We claim that S can be chosen such that $u \in S$. Otherwise $\{v,w\} \subseteq S$. If $x \in S$, we replace v by u and obtain a $\gamma_{2,2}$-set of T containing u. If $x \notin S$, we replace v,w by u,x and obtain a $\gamma_{2,2}$-set of T containing fewer vertices of $\{u, v, w\}$ than S, a contradiction. Hence $S \cap \{u, v, w, x\} = \{u, x\}$, and so $S \cap V(T')$ is a $(2, 2)$-dominating set of T'. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = |S \cap V(T')| + 1 \geq \gamma_{2,2}(T') + 1 \geq \frac{2(\gamma(T') + 1)}{3} + 1 > \frac{2(\gamma(T) + 1)}{3}.$$

Case 3. If $\deg(w) \geq 3$, then the subgraph induced by $D(w)$ consists of i isolated vertices and j copies of P_2, where $i \in \{0, 1\}$ and $j \geq 1$. We first show the following claim.

Claim 1. If there is a vertex c such that $T - c$ contains at least two components P_2, then $\gamma_{2,2}(T) > 2(\gamma(T) + 1)/3$.

***The proof of Claim* 1.** Let ab and $a'b'$ be two components P_2 in $T - c$ with $bc \in E(T)$ and $b'c \in E(T)$. Let $T' = T - \{a, b\}$ and D' be a γ-set of T'. Since $D' \cup \{b\}$ is a dominating set of T, $\gamma(T') \geq \gamma(T) - 1$. Let S be a $\gamma_{2,2}$-set of T containing leaves of T as few as possible. Then $S \cap \{a, b, c\} = \{a\}$ or $\{b, c\}$. We now prove that $S \cap V(T')$ is a $(2, 2)$-dominating set of T'. If $S \cap \{a, b, c\} = \{a\}$, then, to $(2, 2)$-dominate a' and b, $a' \in S$ and there exists at least one neighbor of c in S. Hence $S \cap V(T') = S \setminus \{a\}$ is a $(2, 2)$-dominating set of T'. If $S \cap \{a, b, c\} = \{b, c\}$, then $b' \in S$ and $a' \notin S$ by the choice of S. Hence $S \cap V(T') = S \setminus \{b\}$ is a $(2, 2)$-dominating set of T'. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 1 + |S \cap V(T')| \geq 1 + \gamma_{2,2}(T') \geq 1 + \frac{2(\gamma(T') + 1)}{3} > \frac{2(\gamma(T) + 1)}{3}.$$

This completes the proof of Claim 1.

By Claim 1, in the following, we assume $j = 1$ and complete the proof according to the degree of x. Since $\deg(w) \geq 3$, we have $i = 1$ and $\deg(w) = 3$. Let w' be the unique leaf of w in T. Since $d(T) \geq 4$, $\deg(x) \geq 2$.

Case 3.1. $\deg(x) = 2$.
Let S be a $\gamma_{2,2}$-set of T containing leaves of T and vertices in $D[x]$ as few as possible. Then, by Lemma 2 (2), $S \cap D[x] = \{v, w\}$. If $\deg(y) = 1$, then we can easily prove that $\gamma_{2,2}(T) > 2(\gamma(T) + 1)/3$. In the following, we assume $\deg(y) \geq 2$.

If $y \in S$ or $y \notin S$ and $|N_2(y) \cap S| \geq 3$, then we let $T' = T - D[x]$ and D' be a γ-set of T'. Clearly, $S \cap V(T')$ is a $(2, 2)$-dominating set of T'. Since

$D' \cup \{v, w\}$ is a dominating set of T, $\gamma(T') \geq \gamma(T) - 2$. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 2 + |S \cap V(T')| \geq 2 + \gamma_{2,2}(T') \geq 2 + \frac{2(\gamma(T') + 1)}{3} > \frac{2(\gamma(T) + 1)}{3}.$$

Now, we consider the case $y \notin S$ and $|N_2(y) \cap S| = 2$.

Case 3.1.1. $\deg(y) = 2$.

Let $T' = T - D[y]$ and D' be a γ-set of T'. Since $D' \cup \{v, w, x\}$ is a dominating set of T, $\gamma(T') \geq \gamma(T) - 3$. Since $y \notin S$, $S \cap D[y] = S \cap D[x] = \{v, w\}$. Hence $S \cap V(T')$ is a $(2,2)$-dominating set of T'. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 2 + |S \cap V(T')| \geq 2 + \gamma_{2,2}(T') \geq 2 + \frac{2(\gamma(T') + 1)}{3} \geq \frac{2(\gamma(T) + 1)}{3}.$$

Further, if $\gamma_{2,2}(T) = \frac{2}{3}(\gamma(T) + 1)$, then we have $\gamma_{2,2}(T') = \frac{2}{3}(\gamma(T') + 1)$ and $\gamma(T') = \gamma(T) - 3$. By the inductive hypothesis on T', $T' \in \mathcal{T}$. If $T' = P_4$, one can easily check that $\gamma(T) = 4$. This is a contradiction with $\gamma(T') = \gamma(T) - 3$. Hence $T' \neq P_4$. By Lemma 6, $S \cap V(T') = A(T')$. Since $\gamma(T') = \gamma(T) - 3$, z cannot be contained in any γ-set of T'. By Lemma 7, z is a leaf of T' and the support vertex, say a, of z has at least two leaves in T'. By Lemma 3 (1), $a \in A(T')$ since $S \cap V(T') = A(T')$ and $|N_2(y) \cap S| = 2$. Therefore, T is obtained from T' by Operation $\mathcal{O}_3$, and so $T \in \mathcal{T}$.

Case 3.1.2. $\deg(y) \geq 3$.

Let I be the subgraph induced by $\{u, v, w, w', x\}$ in T. Let J be the subgraph induced by $D(y)$. After proving the above cases, we only need consider the cases that every component of J is isomorphic to I or an isolated vertex by $|N_2(y) \cap S| = 2$ and $w \in N_2(y) \cap S$.

If y is a support vertex of T, let y' denote the unique leaf of y (since we assume that each support vertex of T has a unique leaf). To $(2,2)$-dominate y', $y' \in S$. Hence J has only one component which is isomorphic to I and $S \cap D[y] = \{v, w, y'\}$. Let $T' = T - D[y]$ and D' be a γ-set of T'. Since $D' \cup \{v, w, y\}$ is a dominating set of T, $\gamma(T') \geq \gamma(T) - 3$. Clearly, $(S \cap V(T')) \cup \{z\}$ is a $(2,2)$-dominating set of T'. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 3 + |S \cap V(T')| \geq 2 + \gamma_{2,2}(T') \geq 2 + \frac{2(\gamma(T') + 1)}{3} \geq \frac{2(\gamma(T) + 1)}{3}.$$

We claim that the equality is not true in this case. If $\gamma_{2,2}(T)=\frac{2}{3}(\gamma(T)+1)$, then $\gamma_{2,2}(T')=\frac{2}{3}(\gamma(T')+1)$ and $(S\cap V(T'))\cup\{z\}$ is a $\gamma_{2,2}$-set of T'. By the inductive hypothesis on T', $T'\in\mathcal{T}$. If $T'=P_4$, one can easily check that $\gamma(T)=4<2+3=\gamma(T')+3$, a contradiction. Hence $T'\neq P_4$. By Lemma 6, $(S\cap V(T'))\cup\{z\}=A(T')$. Hence $z\in A(T')$. By Lemma 3 (3), there is another vertex z' in $A(T')$ which is adjacent to z, which contradicts to $|N_2(y)\cap S|=2$.

If y is not a support vertex of T, then there are exactly two components of J which are isomorphic to I (since $|N_2(y)\cap S|=2$). Let I_1 be another component of J with $V(I_1)=\{u_1,v_1,w_1,w_1',x_1\}$ and edge set $E(I_1)=\{u_1v_1,v_1w_1,w_1x_1,w_1w_1'\}$. Let $T'=T-D(y)$ and D' be a γ-set of T'. Since $D'\cup\{v,w,v_1,w_1\}$ is a dominating set of T, $\gamma(T')\geq\gamma(T)-4$. By Lemma 2 (2) and the choice of S, $S\cap D[y]=\{v,w,v_1,w_1\}$. Thus $(S\cap V(T'))\cup\{y\}$ is a $(2,2)$-dominating set of T'. Apply the inductive hypothesis on T',

$$\gamma_{2,2}(T)=|S|=4+|S\cap V(T')|\geq 3+\gamma_{2,2}(T')\geq 3+\frac{2(\gamma(T')+1)}{3}>\frac{2(\gamma(T)+1)}{3}.$$

Case 3.2. $\deg(x)\geq 3$.
Let J denote the subgraph induced by $D(x)$. From the proofs of the above cases, we only need to consider the case that every component of J is isomorphic to a path P_4, a path P_2, or an isolated vertex. Let s,t and h denote the number of components of P_4, P_2 and isolated vertices in J, respectively. Then $s\geq 1$ and $h\in\{0,1\}$. By Claim 1, we can assume that J has at most one component which is isomorphic to P_2, that is $t\in\{0,1\}$. Let S be a $\gamma_{2,2}$-set of T containing leaves and the vertices of $D[x]$ as few as possible. Then, by Lemma 2 (2), $S\cap\{u,v,w,w'\}=\{v,w\}$.

If $|N[x]\cap S|\geq 3$, let T' be the subgraph of $T-\{wx\}$ containing x and D' be a γ-set of T'. Then $S\cap V(T')$ is a $(2,2)$-dominating set of T'. Since $D'\cup\{v,w\}$ is a dominating set of T, $\gamma(T')\geq\gamma(T)-2$. By the induction hypothesis on T',

$$\gamma_{2,2}(T)=|S|=2+|S\cap V(T')|\geq 2+\gamma_{2,2}(T')\geq 2+\frac{2(\gamma(T')+1)}{3}>\frac{2(\gamma(T)+1)}{3}.$$

If $|N[x]\cap S|=1$, then, by $deg(x)\geq 3$ and $|N_2(y)\cap S|=2$, we have $s=1$, $t=1$ and $h=0$. Denote the component of J which is isomorphic to P_2 by ab with $xb\in E(T)$. Let $T'=T-D(x)$. Since any dominating set of T' combined with $\{v,w,b\}$ is a dominating set of T, $\gamma(T')\geq\gamma(T)-3$. To $(2,2)$-dominate a, $a\in S$. By the choice of S, $S\cap D(x)=\{v,w,a\}$.

Hence $(S \cap V(T')) \cup \{x\}$ is a $(2,2)$-dominating set of T'. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 3 + |S \cap V(T')| \geq 2 + \gamma_{2,2}(T') \geq 2 + \frac{2(\gamma(T')+1)}{3} \geq \frac{2(\gamma(T)+1)}{3}.$$

We claim that the equality is not true in this case. If $\gamma_{2,2}(T) = \frac{2}{3}(\gamma(T)+1)$, then $\gamma_{2,2}(T') = \frac{2}{3}(\gamma(T')+1)$ and $(S \cap V(T')) \cup \{x\}$ is a $\gamma_{2,2}$-set of T'. By the inductive hypothesis on T', $T' \in \mathcal{T}$. If $T' = P_4$, one can easily check that $\gamma(T) = 4 < \gamma(T') + 3$, a contradiction. Hence $T' \neq P_4$. By Lemma 6, $(S \cap V(T')) \cup \{z\} = A(T')$ contains a leaf x of T', a contradiction to Lemma 3 (1).

In the following, we assume that $|N[x] \cap S| = 2$. By $|N[x] \cap S| = 2$ and the choice of S, the number of components which are isomorphic to P_4 in J is at most two, that is, $s \in \{1,2\}$. Now we will complete our proof according to the choices of s, t and h.

Case 3.2.1. $s = 1$.
If $t = 1$, denote the component of J which is isomorphic to P_2 by ab with $xb \in E(T)$. Let $T' = T - \{a,b\}$. Clearly, $\gamma(T') \geq \gamma(T) - 1$. Note that $w \in N[x] \cap S$ and $|N[x] \cap S| = 2$. To $(2,2)$-dominate a, $S \cap \{a,b,x\} = \{a\}$ by the choice of S. So $S \cap V(T')$ is a $(2,2)$-dominating set of T'. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 1 + |S \cap V(T')| \geq 1 + \gamma_{2,2}(T') \geq 1 + \frac{2(\gamma(T')+1)}{3} > \frac{2(\gamma(T)+1)}{3}.$$

If $t = 0$, then $h = 1$ since $\deg(x) \geq 3$. Denote the isolated vertex of J by a. Let $T' = T - D[x]$. Clearly, $\gamma(T') \geq \gamma(T) - 3$. By the choice of S and $|N[x] \cap S| = 2$, $y \in S$. Then $S \cap V(T')$ is a $(2,2)$-dominating set of T'. By the induction on T',

$$\gamma_{2,2}(T) = |S| = 2 + |S \cap V(T')| \geq 2 + \gamma_{2,2}(T') \geq 2 + \frac{2(\gamma(T')+1)}{3} \geq \frac{2(\gamma(T)+1)}{3}.$$

Further if $\gamma_{2,2}(T) = \frac{2}{3}(\gamma(T)+1)$, then $\gamma_{2,2}(T') = \frac{2}{3}(\gamma(T')+1)$ and $S \cap V(T')$ is a $\gamma_{2,2}$-set of T'. By the induction hypothesis on T', $T' \in \mathcal{T}$. Note that the subgraph induced by $D[x]$ is isomorphic to H. If $T' = P_4$, then it can be easily checked that T is obtained from P_4 by Operation $\mathcal{O}_2$ if $y \in A(T')$, or $T \notin \mathcal{T}$ if $y \notin A(T')$. If $T' \neq P_4$, then, by Lemma 6, $S \cap V(T') = A(T')$. Hence $y \in A(T')$ and T is obtained from T' by Operation $\mathcal{O}_2$. So $T \in \mathcal{T}$.

Case 3.2.2. $s = 2$.
Let $u_1v_1w_1w_1'$ be another component which is isomorphic to P_4 of J, where w_1 is adjacent to x. By the choice of S, $v_1, w_1 \in S$. Then $N[x] \cap S = \{w, w_1\}$.

Case 3.2.2.1. $t = 1$.
Denote the component P_2 by ab with $bx \in E(T)$. Since $|N[x] \cap S| = 2$, $b \notin S$ and so $a \in S$. Let $T' = T - \{a, b\}$, then $S \cap V(T')$ is a $(2,2)$-dominating set of T'. Clearly, $\gamma(T') \geq \gamma(T) - 1$. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 1 + |S \cap V(T')| \geq 1 + \gamma_{2,2}(T') \geq 1 + \frac{2(\gamma(T') + 1)}{3} > \frac{2(\gamma(T) + 1)}{3}.$$

Case 3.2.2.2. $t = 0$ and $\deg(y) = 2$.
Let $T' = T - D[y]$ and D' be a γ-set of T'. Then $D' \cup \{v, w, v_1, w_1, x\}$ is a dominating set of T and so $\gamma(T') \geq \gamma(T) - 5$. Since $S \cap D[y] = \{v, w, v_1, w_1\}$, $S \cap V(T')$ is a $(2,2)$-dominating set of T'. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 4 + |S \cap V(T')| \geq 4 + \gamma_{2,2}(T') \geq 4 + \frac{2(\gamma(T') + 1)}{3} > \frac{2(\gamma(T) + 1)}{3}.$$

Case 3.2.2.3. $t = 0$ and $\deg(y) \geq 3$.
If $|N_2(y) \cap S| \geq 4$, let $T' = T - D[x]$. Clearly, $S \cap V(T')$ is a $(2,2)$-dominating set of T'. Let D' be a γ-set of T', then $D' \cup \{v, w, v_1, w_1, x\}$ is a dominating set of T. Hence $\gamma(T') \geq \gamma(T) - 5$. By the inductive hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 4 + |S \cap V(T')| \geq 4 + \gamma_{2,2}(T') \geq 4 + \frac{2(\gamma(T') + 1)}{3} > \frac{2(\gamma(T) + 1)}{3}.$$

If $|N_2(y) \cap S| \leq 3$, then, by the proofs of the above cases, we only need to consider the case that the components of $T[D(y)]$ are isomorphic to $T[D[x]]$ or an isolated vertex. Since $\{w, w_1\} \subseteq N_2(y) \cap S$ and $\deg(y) \geq 3$, $T[D(y)]$ has only one $T[D[x]]$ and an isolated vertex, say a. That is $\deg(y) = 3$ and y is a support vertex of T. Let $T' = T - D[y]$ and D' be a γ-set of T'. Since $D' \cup \{v, w, v_1, w_1, x, y\}$ is a dominating set of T, $\gamma(T') \geq \gamma(T) - 6$. Since $N[x] \cap S = \{w, w_1\}$ and $|N_2(y) \cap S| \leq 3$, $S \cap D[y] = \{v, w, v_1, w_1, a\}$. Hence $(S \cap V(T')) \cup \{z\}$ is a $(2,2)$-dominating set of T'. By the induction hypothesis on T',

$$\gamma_{2,2}(T) = |S| = 5 + |S \cap V(T')| \geq 4 + \gamma_{2,2}(T') \geq 4 + \frac{2(\gamma(T') + 1)}{3} \geq \frac{2(\gamma(T) + 1)}{3}.$$

We claim that the equality is not true in this case. If not, then $\gamma_{2,2}(T') = \frac{2}{3}(\gamma(T') + 1)$ and $(S \cap V(T')) \cup \{z\}$ is a $\gamma_{2,2}$-set of T'. By the inductive hypothesis on T', $T' \in \mathcal{T}$. If $T' = P_4$, we can easily check that the equality does not hold. If $T' \neq P_4$. By Lemma 6, $(S \cap V(T')) \cup \{z\} = A(T')$. By Lemma 3 (3), z has a neighbor z' in $A(T')$. So $\{w, w_1, a, z'\} \subseteq N_2(y) \cap S$, which contradicts $|N_2(y) \cap S| \leq 3$. ■

3. The Upper Bound

In this section, we give a trivial upper bound of $\gamma_{2,2}(G)$ in terms of $\gamma(G)$ for any connected graph, and characterize all the trees achieving the equality.

Proposition 9. *If G is a connected graph, then $\gamma_{2,2}(G) \leq \gamma_t(G) \leq 2\gamma(G)$.*

Proof. Let S be a minimum total dominating set of G. Then the subgraph induced by S contains no isolated vertex. Hence, for any $v \in V(G) - S$, $|N_2(v) \cap S| \geq 2$. That is, S is a $(2,2)$-dominating set of G. Hence $\gamma_{2,2}(G) \leq |S| = \gamma_t(G) \leq 2\gamma(G)$. ■

In the following, we will use the result given by Henning [4] to characterize the trees T with $\gamma_{2,2}(T) = 2\gamma(T)$. Let G be a graph and $S \subseteq V(G)$. S is called a *packing* of G if for any two distinct vertices u and v in S, $N_G[u] \cap N_G[v] = \emptyset$.

Lemma 10 [4]. *A tree T of order at least 3 satisfies $\gamma_t(T) = 2\gamma(T)$ if and only if the following three conditions hold:*

(i) *T has a unique γ-set D,*

(ii) *every vertex of D is a support vertex of T, and*

(iii) *D is a packing in T.*

Theorem 11. *Let T be a tree with order at least three. Then $\gamma_{2,2}(T) = 2\gamma(T)$ if and only if T satisfies the following three conditions:*

(1) *T has a unique γ-set D,*

(2) *each vertex of D is adjacent with at least two leaves of T, and*

(3) *D is a packing in T.*

Proof. Let T be a tree with order at least three and $\gamma_{2,2}(T) = 2\gamma(T)$. Then, by Proposition 9, $\gamma_{2,2}(T) \leq \gamma_t(T) \leq 2\gamma(T)$. Hence $\gamma_t(T) = 2\gamma(T)$. By Lemma 10, T satisfies three conditions: (1) T has a unique γ-set D, (2) D is a packing of T, and, (3) each vertex of D is adjacent with at least one leaf of T. So, in the following, we will prove that each vertex of D is adjacent with at least two leaves of T.

If there is a vertex $v \in D$ which is adjacent with only one leaf, say u, we will construct a $(2,2)$-dominating set S of T with $|S| \leq 2\gamma(T) - 1$. Since T is a tree with order at least 3, $N(v) \setminus \{u\} \neq \emptyset$. Let $N(v) \setminus \{u\} = \{w_1, \ldots, w_t\}$ $(t \geq 1)$. For $1 \leq i \leq t$, $N(w_i) \setminus \{v\} \neq \emptyset$ since w_i is not a leaf of T. So we can choose x_i from $N(w_i) \setminus \{v\}$. Since T is a tree, v does not dominate x_i. Hence there exists a vertex $y_i \in D \setminus \{v\}$ such that y_i dominates x_i. Clearly, $|\{v, y_1, \ldots, y_t\}| = t + 1$.

For each $z \in D \setminus \{v, y_1, \ldots, y_t\}$, we choose a neighbor of it. Let S_1 be the set of these neighbors. Let

$$S = (D \setminus \{v\}) \cup \{u, x_1, \ldots, x_t\} \cup S_1.$$

Clearly, $S \setminus \{u\}$ is a total dominating set of $T - \{v, u\}$. By the proof of Proposition 9, $S \setminus \{u\}$ is a $(2,2)$-dominating set of $T - \{v, u\}$. Since $\{u, x_1\} \subseteq N_2(v, T) \cap S$, S is a $(2,2)$-dominating set of T with

$$|S| \leq (\gamma(T) - 1) + (t + 1) + [\gamma(T) - (t + 1)] = 2\gamma(T) - 1,$$

which contradicts $\gamma_{2,2}(T) = 2\gamma(T)$.

Conversely, assume a tree T satisfies the conditions (1), (2) and (3). Let $D = \{x_1, x_2, \ldots, x_{\gamma(T)}\}$ be the unique dominating set of T. Since D is a packing of T, $N[x_1], N[x_2], \ldots, N[x_{\gamma(T)}]$ is a partition of $V(T)$. Let S be a $\gamma_{2,2}$-set of T. For $1 \leq i \leq \gamma(T)$, by Lemma 1, $|N[x_i] \cap S| \geq 2$. So

$$\begin{aligned} \gamma_{2,2}(T) &= |S| = |S \cap V(T)| = |S \cap (\cup_{i=1}^{\gamma(T)} N[x_i])| \\ &= |\cup_{i=1}^{\gamma(T)} (S \cap N[x_i])| = \sum_{i=1}^{\gamma(T)} |S \cap N[x_i]| \geq 2\gamma(T). \end{aligned}$$

By Proposition 9, $\gamma_{2,2}(T) = 2\gamma(T)$. ∎

Remark. By the proof of Proposition 9, $\gamma_{2,2}(G) \leq \gamma_t(G) \leq 2\gamma(G)$. In this section, we give a characterization of trees T with $\gamma_{2,2}(T) = 2\gamma(T)$ by a characterization of trees T with $\gamma_t(T) = 2\gamma(T)$ given by Henning [4]. The characterization of trees T with $\gamma_{2,2}(T) = \gamma_t(T)$ seems a little more difficult. We leave it as an open problem.

Acknowledgements

The authors thank the anonymous referee for the kind comments and helpful suggestions on improvement of the paper.

References

[1] T.J. Bean, M.A. Henning and H.C. Swart, *On the integrity of distance domination in graphs*, Australas. J. Combin. **10** (1994) 29–43.

[2] G. Chartrant and L. Lesniak, Graphs & Digraphs (third ed., Chapman & Hall, London, 1996).

[3] M. Fischermann and L. Volkmann, *A remark on a conjecture for the* (k,p)*-domination number*, Utilitas Math. **67** (2005) 223–227.

[4] M.A. Henning, *Trees with large total domination number*, Utilitas Math. **60** (2001) 99–106.

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (New York, Marcel Deliker, 1998).

[6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics (New York, Marcel Deliker, 1998).

[7] T. Korneffel, D. Meierling and L. Volkmann, *A remark on the* $(2,2)$*-domination number* Discuss. Math. Graph Theory **28** (2008) 361–366.

Received 19 September 2008
Revised 4 June 2009
Accepted 4 June 2009