# Problems Column 

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# A CONJECTURE ON THE PREVALENCE OF CUBIC BRIDGE GRAPHS 

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#### Abstract

Almost all $d$-regular graphs are Hamiltonian, for $d \geq 3$ [8]. In this note we conjecture that in a similar, yet somewhat different, sense almost all cubic non-Hamiltonian graphs are bridge graphs, and present supporting empirical results for this prevalence of the latter among all connected cubic non-Hamiltonian graphs.


Keywords: Hamiltonian graph, non-Hamiltonian graph, cubic bridge graph.
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## 1. A Conjecture

In 1994, Robinson and Wormald [8] proved a striking result, which states that almost all $d$-regular graphs are Hamiltonian, for $d \geq 3$. The interested reader is referred to [1] for an excellent discussion on Hamiltonian cycles in regular graphs.

All graphs in this note are connected and undirected. A cubic, or 3regular, graph is a graph where every vertex is connected to exactly three other vertices. More generally, in a $k$-regular graph, every vertex is connected to exactly $k$ other vertices. A Hamiltonian cycle is a simple cycle
that goes through every vertex in the graph exactly once. A graph is Hamiltonian if it possesses at least one Hamiltonian cycle, and non-Hamiltonian otherwise. One statement of the famous NP-complete Hamiltonian cycle problem (HCP) is: given a graph, determine whether it is Hamiltonian.

Given a graph, a bridge is an edge the removal of which disconnects the graph. A bridge graph is a graph that contains at least one bridge. Bridge graphs are non-Hamiltonian [4]. Moreover, it is straightforward that we can detect bridge graphs in polynomial time. In this note, we consider two exhaustive and mutually exclusive subsets of non-Hamiltonian graphs: bridge graphs, to which we refer as easy non-Hamiltonian graphs, and nonHamiltonian graphs that are not bridge graphs, are hard non-Hamiltonian graphs.

From numerical experiments using GENREG software [6] and the cubhamg utility in the package nauty [5] on cubic graphs of various orders, we observe that bridge graphs constitute the majority of non-Hamiltonian graphs. Moreover, as the graph order $N$ increases, so does the ratio of cubic bridge graphs over all cubic non-Hamiltonian graphs of the same order. This can be seen from Table 1.

Table 1. Ratio of cubic bridge graphs over cubic non-Hamiltonian graphs, of order 10 to 22 .

| Graph Order <br> N | Number of <br> Cubic Graphs | Number of Cubic <br> Non-H Graphs | Number of Cubic <br> Bridge Graphs | Ratio of <br> Bridge/Non-H |
| :---: | :---: | :---: | :---: | :---: |
| 10 | 19 | 2 | 1 | 0.5000 |
| 12 | 85 | 5 | 4 | 0.8000 |
| 14 | 509 | 35 | 29 | 0.8286 |
| 16 | 4060 | 219 | 186 | 0.8493 |
| 18 | 41301 | 1666 | 1435 | 0.8613 |
| 20 | 510489 | 14498 | 12671 | 0.8740 |
| 22 | 7319447 | 148790 | 131820 | 0.8859 |
| 24 | 117940535 | 1768732 | 1590900 | 0.8995 |

For cubic graphs of order 40 and 50, we consider a 1000000-graph sample for each order. The observed ratios of cubic bridge graphs to cubic nonHamiltonian graphs in Table 2 are even closer to 1. This naturally gives rise to a conjecture on the prevalence of cubic bridge graphs.

Table 2. Ratio of cubic bridge graphs over cubic non-Hamiltonian graphs, of order 40 and 50 .

| Graph Order <br> N | Number of <br> Cubic Graphs | Number of Cubic <br> Non-H Graphs | Number of Cubic <br> Bridge Graphs | Ratio of <br> Bridge/Non-H |
| :---: | :---: | :---: | :---: | :---: |
| 40 | 1000000 | 912 | 855 | 0.9375 |
| 50 | 1000000 | 549 | 530 | 0.9650 |

Conjecture 1. Consider cubic graphs of order $N$.

$$
\lim _{N \rightarrow \infty} \frac{\text { \#cubic bridge graphs }}{\# \text { cubic non-Hamiltonian graphs }}=
$$

$$
\lim _{N \rightarrow \infty} \frac{\text { \#cubic easy non-Hamiltonian graphs }}{\#[\text { cubic easy non-Hamiltonian graphs + cubic hard non-Hamiltonian graphs }]}=1 .
$$

## 2. Discussion

If the above conjecture were, indeed, true it would be possible to argue that the difficulty of the NP-completeness of the HCP for cubic graphs is even more of an anomaly than indicated by the result of [8]. In particular, if an arbitrary cubic graph is considered, a polynomial algorithm can tell us whether or not it is a bridge graph. If it is, then it is non-Hamiltonian and if not, then it is even more likely to be Hamiltonian than might have been expected on the basis of [8] alone.

Furthermore, it is reasonable to assume that if the conjecture holds for cubic graphs then its obvious extension to all $d$-regular graphs (with $d \geq 3$ ) will also hold. The underlying intuition is that, somehow, the "easiest" way to create non-Hamiltonian, $d$-regular, graphs with $N$ vertices is to join via bridges graphs with fewer than $N$ vertices. Regrettably, we do not know how to prove the stated conjecture. An approach, based on recursive counting arguments, along those outlined in Chapter 5 of Nguyen [7] may be worth pursuing. Another, approach could, perhaps, be based on the location of bridge graphs in the 2-dimensional multifilar structure introduced in [2] and [3].

We include an adaptation of Figure 5.1 from [3], but here we only distinguish bridge graphs, represented by crosses, from the rest. Given a graph of order $N$, let $\lambda_{i}$ be eigenvalues of the adjacency matrix $A$. Define the expected value function $\mu(A, t)$ of $\left(1-t \lambda_{i}\right)^{-1}$ to be $\frac{1}{N} \sum_{i}\left(1-t \lambda_{i}\right)^{-1}$, and the
variance function $\sigma^{2}(A, t)$ to be $\frac{1}{N} \sum_{i}\left(1-t \lambda_{i}\right)^{-2}-\mu^{2}(A, t)$. Let $t=1 / 9$ and plot the mean-variance coordinates $\left(\mu(A, t), \sigma^{2}(A, t)\right)$ across all cubic graphs of order 14 in Figure 1. We obtain a self-similar multifilar structure, zooming into each large and approximately linear cluster reveals smaller, also approximately linear, clusters with different slopes and between-distances [3]. Figure 1 indicates that bridge graphs are at, or near, the top of their clusters.


Figure 1. Mean-variance plot for all cubic graphs of order 14.

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## References

[1] B. Bollobas, Random Graphs (Cambridge University Press, 2001).
[2] V. Ejov, J.A. Filar S.K. Lucas and P. Zograf, Clustering of spectra and fractals of regular graphs, J. Math. Anal. and Appl. 333 (2007) 236-246.
[3] V. Ejov, S. Friedland and G.T. Nguyen, A note on the graph's resolvent and the multifilar structure, Linear Algebra and Its Application 431 (2009) 1367-1379.
[4] A.S. Lague, Les reseaux (ou graphes), Memorial des sciences math. 18 (1926).
[5] B.D. McKay, website for nauty: http://cs.anu.edu.au/ bdm/nauty/.
[6] M. Meringer, Fast generation of regular graphs and construction of cages, J. Graph Ttheory 30 (1999) 137-146.
[7] G.T. Nguyen, Hamiltonian cycle problem, Markov decision processes and graph spectra, PhD Thesis (University of South Australia, 2009).
[8] R. Robinson and N. Wormald, Almost all regular graphs are Hamiltonian, Random Structures and Algorithms 5 (1994) 363-374.

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