# ON EDGE DETOUR GRAPHS 

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#### Abstract

For two vertices $u$ and $v$ in a graph $G=(V, E)$, the detour distance $D(u, v)$ is the length of a longest $u-v$ path in $G$. A $u-v$ path of length $D(u, v)$ is called a $u-v$ detour. A set $S \subseteq V$ is called an edge detour set if every edge in $G$ lies on a detour joining a pair of vertices of $S$. The edge detour number $d n_{1}(G)$ of $G$ is the minimum order of its edge detour sets and any edge detour set of order $d n_{1}(G)$ is an edge detour basis of $G$. A connected graph $G$ is called an edge detour graph if it has an edge detour set. It is proved that for any non-trivial tree $T$ of order $p$ and detour diameter $D, d n_{1}(T) \leq p-D+1$ and $d n_{1}(T)=p-D+1$ if and only if $T$ is a caterpillar. We show that for each triple $D, k$, $p$ of integers with $3 \leq k \leq p-D+1$ and $D \geq 4$, there is an edge detour graph $G$ of order $p$ with detour diameter $D$ and $d n_{1}(G)=k$. We also show that for any three positive integers $R, D, k$ with $k \geq 3$ and $R<D \leq 2 R$, there is an edge detour graph $G$ with detour radius $R$, detour diameter $D$ and $d n_{1}(G)=k$. Edge detour graphs $G$ with detour diameter $D \leq 4$ are characterized when $d n_{1}(G)=p-2$ or $d n_{1}(G)=p-1$.


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## 1. Introduction

By a graph $G=(V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies, we refer to $[1,5]$.

For vertices $u$ and $v$ in a connected graph $G$, the detour distance $D(u, v)$ is the length of a longest $u-v$ path in $G$. A $u-v$ path of length $D(u, v)$ is called a $u-v$ detour. It is known that the detour distance is a metric on the vertex set $V$. The detour eccentricity $e_{D}(v)$ of a vertex $v$ in $G$ is the maximum detour distance from $v$ to a vertex of $G$. The detour radius, $\operatorname{rad}_{D} G$ of $G$ is the minimum detour eccentricity among the vertices of $G$, while the detour diameter, $\operatorname{diam}_{D} G$ of $G$ is the maximum detour eccentricity among the vertices of $G$. These concepts were studied by Chartrand et al. [2].

A vertex $x$ is said to lie on a $u-v$ detour $P$ if $x$ is a vertex of $P$ including the vertices $u$ and $v$. A set $S \subseteq V$ is called a detour set if every vertex $v$ in $G$ lies on a detour joining a pair of vertices of $S$. The detour number $d n(G)$ of $G$ is the minimum order of a detour set and any detour set of order $d n(G)$ is called a detour basis of $G$. These concepts were studied by Chartrand et al. $[3,4]$.

An edge $e$ of $G$ is said to lie on a $u-v$ detour $P$ if $e$ is an edge of the detour $P$. In general, there are graphs $G$ for which there exist edges which do not lie on a detour joining any pair of vertices of $V$. For the graph $G$ given in Figure 1.1, the edge $v_{1} v_{2}$ does not lie on a detour joining any pair of vertices of $V$. This motivated us to introduce the concepts of weak edge detour set of a graph [6] and edge detour graphs [7].


Figure 1.1. $G$
A set $S \subseteq V$ is called a weak edge detour set of $G$ if every edge in $G$ has both its ends in $S$ or it lies on a detour joining a pair of vertices of $S$. The weak edge detour number $d n_{w}(G)$ of $G$ is the minimum order of its weak edge detour sets and any weak edge detour set of order $d n_{w}(G)$ is called a
weak edge detour basis of $G$. Weak edge detour sets and weak edge detour number of a graph were introduced and studied by Santhakumaran and Athisayanathan in [6].

A set $S \subseteq V$ is called an edge detour set of $G$ if every edge in $G$ lies on a detour joining a pair of vertices of $S$. The edge detour number $d n_{1}(G)$ of $G$ is the minimum order of its edge detour sets and any edge detour set of order $d n_{1}(G)$ is an edge detour basis of $G$. A graph $G$ is called an edge detour graph if it has an edge detour set. Edge detour graphs were introduced and studied by Santhakumaran and Athisayanathan in [7]. It is proved in [7] that every edge detour set of an edge detour graph contains its end-vertices and no edge detour basis contains its cut vertices.

For the graph $G$ given in Figure 1.2(a), the sets $S_{1}=\{u, x\}, S_{2}=$ $\{u, w, x\}$ and $S_{3}=\{u, v, x, y\}$ are a detour basis, weak edge detour basis and edge detour basis of $G$ respectively and hence $d n(G)=2, d n_{w}(G)=3$ and $d n_{1}(G)=4$. For the graph $G$ given in Figure 1.2(b), the set $S=\left\{u_{1}\right.$, $\left.u_{2}\right\}$ is a detour basis, weak edge detour basis and an edge detour basis so that $d n(G)=d n_{w}(G)=d n_{1}(G)=2$. The graphs $G$ given in Figure 1.2 are edge detour graphs. For the graph $G$ given in Figure 1.1, the set $S=\left\{v_{1}, v_{2}\right\}$ is a detour basis and also a weak edge detour basis, but it does not contain an edge detour set and so $G$ is not an edge detour graph.


Figure 1.2. $G$
A caterpillar is a tree for which the removal of all end-vertices leaves a path. A wounded spider is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1, t}$ for $t \geq 0$. For a cut-vertex $v$ in a connected graph $G$ and a component $H$ of $G-v$, the subgraph $H$ and the vertex $v$ together with all edges joining $v$ to $V(H)$ is called a branch of $G$ at $v$. An end-block of $G$ is a block containing exactly one cut-vertex of $G$. The following theorems are used in the sequel.

Theorem 1.1 ([7]). For any edge detour graph $G$ of order $p \geq 2,2 \leq$ $d n_{1}(G) \leq p$.

Theorem 1.2 ([7]). Each end-vertex of an edge detour graph $G$ belongs to every edge detour set of $G$. Also if the set $S$ of all end-vertices of $G$ is an edge detour set, then $S$ is the unique edge detour basis for $G$.

Theorem 1.3 ([7]). If $T$ is a tree with $k$ end-vertices, then $d n_{1}(T)=k$.
Theorem 1.4 ([7]). Any cycle $G$ is an edge detour graph and $d n_{1}(G)=2$ if $G$ is an even cycle, and $d n_{1}(G)=3$ if $G$ is an odd cycle.

Theorem $1.5([7])$. Let $G=\left(K_{n_{1}} \cup K_{n_{2}} \cup \cdots \cup K_{n_{r}} \cup k K_{1}\right)+v$ be a block graph of order $p \geq 5$ such that $r \geq 2$, each $n_{i} \geq 2$ and $n_{1}+n_{2}+\cdots+n_{r}+k=$ $p-1$. Then $G$ is an edge detour graph and $d n_{1}(G)=2 r+k$.

Throughout this paper $G$ denotes a connected graph with at least two vertices.

## 2. Edge Detour Number and Detour Diameter of an Edge Detour Graph

The following Theorem gives an upper bound for the detour number of a graph in terms of its order and detour diameter.

Theorem A [3]. If $G$ is a non-trivial connected graph of order $p \geq 3$ and detour diameter $D$, then $d n(G) \leq p-D+1$.

Remark 2.1. Theorem A is not true for the edge detour number $d n_{1}(G)$ of an edge detour graph $G$. There are edge detour graphs $G$ for which $d n_{1}(G)=p-D+1, d n_{1}(G)<p-D+1$ and $d n_{1}(G)>p-D+1$. For an even cycle $C$ of order $p \geq 4, D=p-1$ and by Theorem $1.4, d n_{1}(C)=2$ so that $d n_{1}(C)=p-D+1$. For the graph $G$ in Figure $1.2(\mathrm{~b}), p=6, D=4$ and $d n_{1}(G)=2$ so that $d n_{1}(G)<p-D+1$. For an odd cycle $C$ of order $p \geq 3$, $D=p-1$ and by Theorem $1.4, d n_{1}(C)=3$ so that $d n_{1}(C)>p-D+1$.

Theorem 2.2. If $G$ is an edge detour graph of order $p \geq 2$ with $D=p-1$, then $d n_{1}(G) \geq p-D+1$.

Proof. For any edge detour graph $G, d n_{1}(G) \geq 2$. Since $D=p-1$, we have $p-D+1=2$ and so $d n_{1}(G) \geq p-D+1$.

Remark 2.3. The converse of Theorem 2.2 is not true. For the edge detour graph $G$ given in Figure 2.1, $p=6$ and $D=4$ so that $p-D+1=3$ and $d n_{1}(G)=4$. Thus $d n_{1}(G)>p-D+1$ and $D \neq p-1$.


Figure 2.1. $G$

Theorem 2.4. If $G$ is a non-trivial tree of order $p$, then $d n_{1}(G) \leq p-D+1$.
Proof. Let $u$ and $v$ be the vertices of $G$ for which $D(u, v)=D$ and let $P: u=v_{0}, v_{1}, \ldots, v_{D-1}, v_{D}=v$ be $u-v$ detour of length $D$. Let $S=$ $V(G)-\left\{v_{1}, v_{2}, \ldots, v_{D-1}\right\}$. It is clear that $S$ is an edge detour set of $G$ and so $d n_{1}(G) \leq|S|=p-D+1$.

We give below a characterization theorem for trees.
Theorem 2.5. For every non-trivial tree $T$ of order $p, d n_{1}(T)=p-D+1$ if only if $T$ is a caterpillar.

Proof. Let $T$ be any non-trivial tree. Let $D=D(u, v)$ and $P: u=$ $v_{0}, v_{1}, \ldots, v_{D-1}, v_{D}=v$ be a detour diameteral path. Let $k$ be the number of end-vertices of $T$ and $l$ be the number of internal vertices of $T$ other than $v_{1}, v_{2}, \ldots, v_{D-1}$. Then $D-1+l+k=p$. By Theorem $1.3, d n_{1}(T)=k=$ $p-D-l+1$. Hence $d n_{1}(T)=p-D+1$ if and only if $l=0$, if and only if all the internal vertices of $T$ lie on the detour diameteral path $P$, if and only if $T$ is a caterpillar.

Corollary 2.6. For a wounded spider $T$ of order $p, d n_{1}(T)=p-D+1$ if and only if $T$ is obtained from $K_{1, t}(t \geq 1)$ by subdividing at most two of its edges.

Proof. It is clear that a wounded spider $T$ is a caterpillar if and only if $T$ is obtained from $K_{1, t}(t \geq 1)$ by subdividing at most two of its edges. Then the result follows from Theorem 2.5.

The following two theorems give realization results under certain conditions.
Theorem 2.7. For each triple $D, k, p$ of integers with $3 \leq k \leq p-D+1$ and $D \geq 4$, there exists an edge detour graph $G$ of order $p$ with detour diameter $D$ and $d n_{1}(G)=k$.

Proof. Case 1. When $D$ is even, let $G$ be the graph obtained from the cycle $C_{D}: u_{1}, u_{2}, \ldots, u_{D}, u_{1}$ of order $D$ by adding $k-1$ new vertices $v_{1}$, $v_{2}, \ldots, v_{k-1}$ and joining each vertex $v_{i}(1 \leq i \leq k-1)$ to $u_{1}$ and adding $p-D-k+1$ new vertices $w_{1}, w_{2}, \ldots, w_{p-D-k+1}$ and joining each vertex $w_{i}$ $(1 \leq i \leq p-D-k+1)$ to both $u_{1}$ and $u_{3}$. The graph $G$ is connected of order $p$ and detour diameter $D$ and is shown in Figure 2.2(a).

Now, we show that $d n_{1}(G)=k$. Let $S=\left\{v_{1}, v_{2}, \ldots, v_{k-1}\right\}$ be the set of all end-vertices of $G$. No edge of $G$ other than the edges $u_{1} v_{i}(1 \leq i \leq k-1)$ lies on a detour joining a pair of vertices of $S$ and so $S$ is not an edge detour set of $G$. Let $T=S \cup\{v\}$, where $v$ is the antipodal vertex of $u_{1}$ in $C_{D}$. Then every edge of $G$ lies on a detour joining a vertex $v_{i}(1 \leq i \leq k-1)$ and $v$ so that $T$ is an edge detour set of $G$. Now, it follows from Theorem 1.2 that $T$ is an edge detour basis of $G$ and so $d n_{1}(G)=k$.

Case 2. When $D$ is odd, let $G$ be the graph obtained from the cycle $C_{D}$ : $u_{1}, u_{2}, \ldots, u_{D}, u_{1}$ of order $D$ by adding $k-2$ new vertices $v_{1}, v_{2}, \ldots, v_{k-2}$ and joining each vertex $v_{i}(1 \leq i \leq k-2)$ to $u_{1}$ and adding $p-D-k+2$ new vertices $w_{1}, w_{2}, \ldots, w_{p-D-k+2}$ and joining each vertex $w_{i}(1 \leq i \leq$ $p-D-k+2)$ to both $u_{1}$ and $u_{3}$. The graph $G$ is connected of order $p$ and detour diameter $D$ and is shown in Figure 2.2(b).

Now, we show that $d n_{1}(G)=k$. Let $S=\left\{v_{1}, v_{2}, \ldots, v_{k-2}\right\}$ be the set of all end-vertices of $G$. As in Case 1, $S$ is not an edge detour set of $G$. Let $S_{1}=S \cup\{v\}$, where $v$ is any vertex of $G$ such that $v \neq v_{i}(1 \leq i \leq k-2)$. It is easy to see that $S_{1}$ is not an edge detour set of $G$. Now, the set $T=S \cup\left\{u_{2}, u_{D}\right\}$ is clearly an edge detour set of $G$. Hence it follows from Theorem 1.2 that $T$ is an edge detour basis of $G$ and so $d n_{1}(G)=k$.

Chartrand et al. [2] proved that the detour radius and detour diameter of a connected graph $G$ satisfy $\operatorname{rad}_{D} G \leq \operatorname{diam}_{D} G \leq 2 \operatorname{rad}_{D} G$. They also proved that every pair $a, b$ of positive integers can be realized as the detour
radius and detour diameter respectively of some connected graph provided $a \leq b \leq 2 a$. We extend this theorem so that the edge detour number can be prescribed as well when $a<b \leq 2 a$.


Figure 2.2. $G$
Theorem 2.8. Let $R, D, k$ be three positive integers such that $k \geq 3$ and $R<D \leq 2 R$. Then there exists an edge detour graph $G$ such that $\operatorname{rad}_{D} G=$ $R, \operatorname{diam}_{D} G=D$ and $d n_{1}(G)=k$.

Proof. Case 1. Let $R$ be an odd integer. When $R=1$, let $G=K_{1, k}$. Clearly, $\operatorname{rad}_{D} G=1, \operatorname{diam}_{D} G=2$ and by Theorem 1.3, $d n_{1}(G)=k$. When $R \geq 3$ and $R<D \leq 2 R$, we construct a graph $G$ with the desired properties as follows: Let $C_{R+1}: v_{0}, v_{1}, \ldots, v_{R}, v_{0}$ be a cycle of order $R+1$ and let $P_{D-R+1}: u_{0}, u_{1}, \ldots, u_{D-R}$ be a path of order $D-R+1$. Let $H$ be the graph obtained from $C_{R+1}$ and $P_{D-R+1}$ by identifying $v_{0}$ of $C_{R+1}$ with $u_{0}$ of $P_{D-R+1}$. The required graph $G$ is obtained from $H$ by adding $k-2$ new
vertices $w_{1}, w_{2}, \ldots, w_{k-2}$ to $H$ and joining each $w_{i}(1 \leq i \leq k-2)$ to the vertex $u_{D-R-1}$ and is shown in Figure 2.3(a). It is clear that $G$ is connected with $\operatorname{rad}_{D} G=R$ and $\operatorname{diam}_{D} G=D$.

Now, we show that $d n_{1}(G)=k$. Let $S=\left\{w_{1}, w_{2}, \ldots, w_{k-2}, u_{D-R}\right\}$ be the set of all end-vertices of $G$. No edge of $G$ other than the edges $w_{i} u_{D-R-1}$ $(1 \leq i \leq k-2)$ and the edge $u_{D-R} u_{D-R-1}$ lies on a detour joining a pair of vertices of $S$ and so $S$ is not an edge detour set of $G$. Let $T=S \cup\{v\}$, where $v$ is the antipodal vertex of $v_{0}$ in $C_{R+1}$. Then $T$ is an edge detour set of $G$ and hence it follows from Theorem 1.2 that $T$ is an edge detour basis of $G$ so that $d n_{1}(G)=k$.

Case 2. Let $R$ be an even integer. Construct the graph $H$ as in Case 1. Then $G$ is obtained from $H$ by adding $k-3$ new vertices $w_{1}, w_{2}, \ldots, w_{k-3}$ to $H$ and joining each $w_{i}(1 \leq i \leq k-3)$ to the vertex $u_{D-R-1}$ and is shown in Figure 2.3(b). It is clear that $G$ is connected with $\operatorname{rad}_{D} G=R$ and $\operatorname{diam}_{D} G=D$.

Now, we show that $d n_{1}(G)=k$. Let $S=\left\{w_{1}, w_{2}, \ldots, w_{k-3}, u_{D-R}\right\}$ be the set of all end-vertices of $G$. As in Case 1, $S$ is not an edge detour set of $G$. Let $S_{1}=S \cup\{v\}$, where $v$ is any vertex of $G$ such that $v \notin S$. It is easy to see that $S_{1}$ is not an edge detour set of $G$. Now the set $T=S \cup\left\{v_{1}, v_{R}\right\}$ is clearly an edge detour set of $G$. Hence it follows from Theorem 1.2 that $T$ is an edge detour basis of $G$ and so $d n_{1}(G)=k$.

(a)


Figure 2.3. $G$

## 3. Edge Detour Graphs with Detour Diameter $D \leq 4$

It is proved in [7] (see Theorem 1.1) that for any edge detour graph $G$ of order $p \geq 2,2 \leq d n_{1}(G) \leq p$. The bounds in this inequality are sharp. For the complete graph $K_{p}(p=2$ or 3$), d n_{1}\left(K_{p}\right)=p$. The set of two end-vertices of a path $P_{n}(n \geq 2)$ is its unique edge detour set so that $d n_{1}\left(P_{n}\right)=2$. Thus the complete graph $K_{p}(p=2$ or 3$)$ has the largest possible edge detour number $p$ and the non-trivial paths have the smallest edge detour number 2 .

The following problem seems to be a difficult one and we leave it open.
Problem 3.1. Does there exist a graph $G$ of order $p \geq 4$ for which $d n_{1}(G)=p$ ?

In this section we characterize edge detour graphs $G$ with detour diameter $D \leq 4$ for which $d n_{1}(G)=p-2$ or $d n_{1}(G)=p-1$. First, we characterize graphs $G$ with detour diameter $D \leq 4$ for which $d n_{1}(G)=p-2$. For this purpose we introduce the collection $\mathscr{K}$ of graphs given in Figure 3.1.

Theorem 3.2. Let $G$ be an edge detour graph of order $p \geq 5$ with detour diameter $D \leq 4$. Then $d n_{1}(G)=p-2$ if and only if $G$ is a double star or $G \in \mathscr{K}$.

Proof. It is straightforward to verify that if $G$ is a double star or $G \in \mathscr{K}$, then $d n_{1}(G)=p-2$. For the converse, let $G$ be an edge detour graph of order $p \geq 5, D \leq 4$ and $d n_{1}(G)=p-2$.
If $\boldsymbol{D} \leq \mathbf{2}$, then it is clear that there are no graphs $G$ for which $d n_{1}(G)=$ $p-2$.
Suppose $\boldsymbol{D}=\mathbf{3}$. If $G$ is a tree, then $G$ is a double star and the result follows from Theorem 1.3. Assume that $G$ is not a tree. Let $c(G)$ denote the length of a longest cycle in $G$. Since $D=3$, it follows that $c(G) \leq 4$. We consider two cases.

Case 1. Let $c(G)=4$. Let $C: v_{1}, v_{2}, v_{3}, v_{4}, v_{1}$ be a 4 -cycle in $G$. Since $p \geq 5$ and $G$ is connected there exists a vertex $x$ not on $C$ such that it is adjacent to some vertex, say $v_{1}$ of $C$. Then $x, v_{1}, v_{2}, v_{3}, v_{4}$ is a path of length 4 in $G$ so that $D \geq 4$, which is a contradiction.

Case 2. Let $c(G)=3$. If $G$ contains two or more triangles, then $c(G)=4$ or $D \geq 4$, which is a contradiction. Hence $G$ contains a unique triangle


Figure 3.1. Graphs in family $\mathscr{K}$.
$C_{3}: v_{1}, v_{2}, v_{3}, v_{1}$. Now, if there are two or more vertices of $C_{3}$ having degree 3 or more, then $D \geq 4$, which is a contradiction. Thus exactly one vertex in $C_{3}$ has degree 3 or more. Since $D=3$, it follows that $G=K_{1, p-1}+e$ and
so $d n_{1}\left(K_{1, p-1}+e\right)=p-1$, which is a contradiction. Thus it follows that $G$ is a double star.
Suppose $\boldsymbol{D}=4$. If $G$ is a tree, then there exists a path of length 4 so that there are at least 3 internal vertices of $G$. Hence there are at most $(p-3)$ end-vertices of $G$, so that by Theorem $1.3, d n_{1}(G) \leq p-3$, which is a contradiction. So, assume that $G$ is not a tree. Let $c(G)$ denote the length of a longest cycle in $G$. Since $D=4$, it follows that $c(G) \leq 5$. We consider three cases.

Case 1. Let $c(G)=5$. Then, since $D=4$, it is clear that $G$ has exactly five vertices. Now, it is easily verified that the graphs $G_{1}, G_{2}, G_{3}, G_{4}$ and $G_{5} \in \mathscr{K}$ given in Figure 3.1 are the only graphs with $d n_{1}\left(G_{i}\right)=p-2$ $(1 \leq i \leq 5)$ among all graphs on five vertices having a largest cycle of length 5.

Case 2. Let $c(G)=4$. Suppose that $G$ contains $K_{4}$ as an induced subgraph. Since $p \geq 5, D=4$ and $c(G)=4$, every vertex not on $K_{4}$ is pendant and adjacent to exactly one vertex of $K_{4}$. Thus the graph reduces to the graph $G_{6} \in \mathscr{K}$ given in Figure 3.1. Also since $d n_{1}\left(G_{6}\right)=p-2, G_{6}$ is the only graph in this case satisfying the requirements of the theorem.

Now, suppose that $G$ does not contain $K_{4}$ as an induced subgraph. We claim that $G$ contains exactly one 4 -cycle $C_{4}$. Suppose that $G$ contains two or more 4 -cycles. If two 4 -cycles in $G$ have no edges in common, then it is clear that $D \geq 5$, which is a contradiction. If two 4 -cycles in $G$ have exactly one edge in common, then $G$ must contain the graphs given in Figure 3.2 as subgraphs or induced subgraphs. In any case $D \geq 5$ or $c(G) \geq 5$, which is a contradiction.


Figure 3.2. G
If two 4 -cycles in $G$ have exactly two edges in common, then $G$ must contain the graphs given in Figure 3.3 as subgraphs. It is easily verified that all other
subgraphs having two edges in common will have cycles of length $\geq 5$, which is a contradiction.


Figure 3.3. $G$
Now, if $G=H_{1}$, then $d n_{1}(G)=p-3$, which is a contradiction. Assume first that $G$ contains $H_{1}$ as a proper subgraph. Then there is a vertex $x$ such that $x \notin V\left(H_{1}\right)$ and $x$ is adjacent to at least one vertex of $H_{1}$. If $x$ is adjacent to $v_{1}$, we get a path $x, v_{1}, v_{2}, v_{3}, v_{4}, v_{5}$ of length 5 so that $D \geq$ 5 , which is a contradiction. Hence $x$ cannot be adjacent to $v_{1}$. Similarly $x$ cannot be adjacent to $v_{3}$ and $v_{5}$. Thus $x$ is adjacent to $v_{2}$ or $v_{4}$ or both. If $x$ is adjacent only to $v_{2}$, then $x$ must be a pendant vertex of $G$, for otherwise, we get a path of length 5 so that $D \geq 5$, which is a contradiction. Thus in this case, the graph $G$ reduces to the one given in Figure 3.4.


Figure 3.4. $G$
However, for this graph $G$, it follows from Theorem 1.2 that the set $\left\{v_{4}, v_{6}\right.$, $\left.v_{7}, \ldots, v_{p}\right\}$ is an edge detour basis so that $d n_{1}(G)=p-4$, which is a contradiction. So, in this case there are no graphs satisfying the requirements of
the theorem. If $x$ is adjacent only to $v_{4}$, then we get a graph $G$ isomorphic to the one given in Figure 3.4 and hence in this case also there are no graphs satisfying the requirements of the theorem. If $x$ is adjacent to both $v_{2}$ and $v_{4}$, then the graph reduces to the one given in Figure 3.5.


Figure 3.5. G
However, for this graph, $\left\{x, v_{3}\right\}$ is an edge detour basis so that $d n_{1}(G)=2$ and hence $d n_{1}(G) \leq p-4$, which is a contradiction. Thus a vertex not in $H_{1}$ cannot be adjacent to both $v_{2}$ and $v_{4}$.

Next, if a vertex $x$ not on $H_{1}$ is adjacent only to $v_{2}$ and a vertex $y$ not on $H_{1}$ is adjacent only to $v_{4}$, then $x$ and $y$ must be pendant vertices of $G$, for otherwise, we get either a path or a cycle of length $\geq 5$ so that $D \geq 5$, which is a contradiction. Thus in this case, the graph reduces to the one given in Figure 3.6.


Figure 3.6. G
For this graph $G$, it follows from Theorem 1.2 that the set of all end-vertices is an edge detour basis so that $d n_{1}(G)=p-5$. So, in this case also there are no graphs satisfying the requirements of the theorem. Thus we conclude that in this case there are no graphs $G$ with $H_{1}$ as proper subgraph.

Next, if $G=H_{2}$, then the edge $v_{2} v_{4}$ does not lie on any detour joining a pair of vertices of $G$ so that $G$ is not an edge detour graph. If $G$ contains $H_{2}$ as a proper subgraph, then as in the case of $H_{1}$, it is easily seen that the graph reduces to any one of the graphs given in Figure 3.7.


Figure 3.7. $G$
Since the edge $v_{2} v_{4}$ of $G_{i}(1 \leq i \leq 3)$ in Figure 3.7 does not lie on a detour joining any pair of vertices of $G_{i}$, these graphs are not edge detour graphs. Thus in this case there are no edge detour graphs $G$ with $H_{2}$ as proper subgraph satisfying the requirements of the theorem. Thus we conclude that, if $G$ does not contain $K_{4}$ as an induced subgraph, then $G$ has a unique 4 -cycle. Now we consider two subcases.

Subcase 1. The unique cycle $C_{4}: v_{1}, v_{2}, v_{3}, v_{4}, v_{1}$ contains exactly one chord $v_{2} v_{4}$. Since $p \geq 5, D=4$ and $G$ is connected, any vertex $x$ not on $C_{4}$ is pendant and is adjacent to at least one vertex of $C_{4}$. The vertex $x$ cannot be adjacent to both $v_{1}$ and $v_{3}$, for in this case we get $c(G)=5$, which is a
contradiction. Suppose that $x$ is adjacent to $v_{1}$ or $v_{3}$, say $v_{1}$. Also if $y$ is a vertex such that $y \neq x, v_{1}, v_{2}, v_{3}, v_{4}$, then $y$ cannot be adjacent to $v_{2}$ or $v_{3}$ or $v_{4}$, for in each case $D \geq 5$, which is a contradiction. Hence $y$ is a pendant vertex and cannot be adjacent to $x$ or $v_{2}$ or $v_{3}$ or $v_{4}$ so that in this case the graph $G$ reduces to the one given in Figure 3.8.


Figure 3.8. G
It follows from Theorem 1.2 that the set of all end vertices together with the vertex $v_{3}$ forms an edge detour basis for this graph $G$ so that $d n_{1}(G)=p-3$. Similarly, if $x$ is adjacent to $v_{3}$, we get a contradiction.

Now, if $x$ is adjacent to both $v_{2}$ and $v_{4}$, we get the graph $H$ given in Figure 3.9 as a subgraph which is isomorphic to the graph $H_{2}$ given in Figure 3.3. Then as in the first part of case 2, we see that there are no graphs which satisfy the requirements of the theorem.


Figure 3.9. $H$
Thus $x$ is adjacent to exactly one of $v_{2}$ or $v_{4}$, say $v_{2}$. Also if $y$ is a vertex such that $y \neq x, v_{1}, v_{2}, v_{3}, v_{4}$, then $y$ cannot be adjacent to $x$ or $v_{1}$ or $v_{3}$, for in each case $D \geq 5$, which is a contradiction. If $y$ is adjacent to $v_{2}$ and $v_{4}$, then we get the graph $H$ given in Figure 3.10 as a subgraph. Then exactly as in the first part of case 2 it can be seen that there are no graphs satisfying the requirements of the theorem.


Figure 3.10. $H$
Thus $y$ must be adjacent to $v_{2}$ or $v_{4}$ only. Hence we conclude that in either case the graph $G$ must reduce to the graph $G_{7}$ or $G_{8} \in \mathscr{K}$ as given in Figure 3.1. Similarly, if $x$ is adjacent to $v_{4}$, then the graph $G$ reduces to the graph $G_{7}$ or $G_{8} \in \mathscr{K}$ as given in Figure 3.1. It is clear that $d n_{1}(G)=p-2$ for these two classes of graphs $G$. Thus these two classes of graphs satisfy the requirements of the theorem. It is to be noted that the graph $G_{7}$ is nothing but $K_{1, p-1}+e+f$ where $e$ and $f$ are adjacent edges.

Subcase 2. The unique cycle $C_{4}: v_{1}, v_{2}, v_{3}, v_{4}, v_{1}$ has no chord. In this case we claim that $G$ contains no triangle. Suppose that $G$ contains a triangle $C_{3}$. If $C_{3}$ has no vertex in common with $C_{4}$ or exactly one vertex in common with $C_{4}$, we get a path of length at least 5 so that $D \geq 5$. If $C_{3}$ has exactly two vertices in common with $C_{4}$, we get a cycle of length 5 . Thus, in all cases, we have a contradiction and hence it follows that $G$ contains a unique chordless cycle $C_{4}$ with no triangles. Since $p \geq 5, D=4, c(G)=4$ and $G$ is connected, any vertex $x$ not on $C_{4}$ is pendant and is adjacent to exactly one vertex of $C_{4}$, say $v_{1}$. Also if $y$ is a vertex such that $y \neq x, v_{1}$, $v_{2}, v_{3}, v_{4}$, then $y$ cannot be adjacent to $v_{2}$ or $v_{4}$, for in this case $D \geq 5$, which is a contradiction. Thus $y$ must be adjacent to $v_{3}$ only. Hence we conclude that in either case $G$ must reduce to the graphs $H_{1}$ or $H_{2}$ as given in Figure 3.11.

For these graphs $H_{1}$ and $H_{2}$ in Figure 3.11, it follows from Theorem 1.2 that $d n_{1}\left(H_{1}\right)=p-3$ and $d n_{1}\left(H_{2}\right)=p-4$. Hence there are no graphs satisfying the requirements of the theorem. Thus when $D=4$ and $c(G)=$ 4, the graphs satisfying the requirements of the theorem are $G_{6}, G_{7}$, and $G_{8} \in \mathscr{K}$ as in Figure 3.1.

Case 3. Let $c(G)=3$.


Figure 3.11. $G$
Case 3a. $G$ contains exactly one triangle $C_{3}: v_{1}, v_{2}, v_{3}, v_{1}$. Since $p \geq 5$, there are vertices not on $C_{3}$. If all the vertices of $C_{3}$ have degree three or more, then since $D=4$, the graph $G$ must reduce to the one given in Figure 3.12.


Figure 3.12. $G$
But, in this case $d n_{1}(G)=p-3$, which is a contradiction. Hence we conclude that at most two vertices of $C_{3}$ have degree $\geq 3$.

Subcase 1. Exactly two vertices of $C_{3}$ have degree 3 or more. Let $d e g_{G} v_{3}=2$. Now, since $p \geq 5, D=4, c(G)=3$ and $G$ is connected, we see that the graph reduces to the graph $G_{9} \in \mathscr{K}$ as given in Figure 3.1, for which $d n_{1}(G)=p-2$. Thus in this case the graph $G_{9} \in \mathscr{K}$ satisfies the requirements of the theorem.

Subcase 2. Exactly one vertex $v_{1}$ of $C_{3}$ has degree 3 or more. Since $G$ is connected, $p \geq 5, D=4$ and $c(G)=3$, the graph reduces to the one given in Figure 3.13.


Figure 3.13. $G$
Now, we claim that exactly one neighbor of $v_{1}$ other than $v_{2}$ and $v_{3}$ has degree $\geq 2$. If the claim is not true, then more than one neighbor of $v_{1}$ other than $v_{2}$ and $v_{3}$ has degree $\geq 2$ and so the set of all end-vertices together with $v_{2}$ and $v_{3}$ forms an edge detour set of $G$. Hence $d n_{1}(G) \leq p-3$, which is a contradiction. Thus in this case the graph reduces to the graph $G_{10} \in \mathscr{K}$ as in Figure 3.1, which satisfies the requirements of the theorem.

Case 3b. $G$ contains more than one triangle. Since $D=4$ and $c(G)=3$, it is clear that all the triangles must have a vertex $v$ in common. Now, if two triangles have two vertices in common then it is clear that $c(G) \geq 4$. Hence all triangles must have exactly one vertex in common. Since $p \geq 5$, $D=4, c(G)=3$ and $G$ is connected, all the vertices of all the triangles are of degree 2 except $v$. Thus the graph reduces to the graphs given in Figure 3.14.


Figure 3.14. $G$

If $G=H_{1}$, then by Theorem 1.5, $d n_{1}(G)=p-1$, which is a contradiction. If $G=H_{2}$, then we claim that exactly one neighbor of $v$ not on the triangles has degree $\geq 2$. If the claim is not true, then more than one neighbor of $v$ not on the triangles has degree $\geq 2$ and so the set of all end-vertices together with all the vertices of all triangles except $v$ forms an edge detour set of $G$. Hence $d n_{1}(G) \leq p-3$, which is a contradiction. Thus in this case the graph reduces to the graph $G_{11} \in \mathscr{K}$ as in Figure 3.1, which satisfies the requirements of the theorem. This completes the proof of the theorem.

Remark 3.3. For $p=4$, the graphs are $G=P_{4}, C_{4}$ and $K_{4}-e$ and $d n_{1}(G)=p-2$. For $p=2$ and 3 , there are no graphs $G$ for which $d n_{1}(G)=$ $p-2$.

In view of Theorem 3.2 we leave the following problem as an open question.
Problem 3.4. Characterize edge detour graphs $G$ with detour diameter $D \geq 5$ for which $d n_{1}(G)=p-2$.

The following theorem characterizes trees $T$ for which $d n_{1}(T)=p-2$.
Theorem 3.5. For any tree $T$ of order $p \geq 5, d n_{1}(T)=p-2$ if and only if $T$ is a double star.

Proof. If $T$ is a double star, then by Theorem 1.3, $d n_{1}(T)=p-2$. Conversely, assume that $d n_{1}(T)=p-2$. If $D \leq 2$, then it is proved in Theorem 3.2 that there are no graphs $G$ for which $d n_{1}(G)=p-2$. If $D=3$, then it is proved in Theorem 3.2 that $T$ is a double star. If $D \geq 4$, then there exist at least three internal vertices of $T$ so that there are at most $p-3$ end-vertices of $T$ so that by Theorem 1.3, $d n_{1}(T) \leq p-3$, which is a contradiction. This completes the proof.

The next theorem characterizes graphs $G$ with detour diameter $D \leq 4$ for which $d n_{1}(G)=p-1$. The proof is similar to that of Theorem 3.2 and hence we omit it.

Theorem 3.6. Let $G$ be an edge detour graph of order $p \geq 3$ with detour diameter $D \leq 4$. Then $d n_{1}(G)=p-1$ if and only if $G$ is $K_{4}$ or $K_{1, p-1}$ or $K_{1, p-1}+e_{1}+e_{2}+\cdots+e_{t}(t \geq 1)$, where the edges $e_{i}(1 \leq i \leq t)$ are mutually nonadjacent.

In view of Theorem 3.6 we leave the following problem as an open question.
Problem 3.7. Characterize edge detour graphs $G$ with detour diameter $D \geq 5$ for which $d n_{1}(G)=p-1$.

The following theorem characterizes trees $T$ for which $d n_{1}(T)=p-1$.
Theorem 3.8. For any tree $T$ of order $p \geq 3, d n_{1}(T)=p-1$ if and only if $T$ is the star $K_{1, p-1}$.

Proof. The proof is similar to that of Theorem 3.5 and follows from Theorems 1.3 and 3.6.

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