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ON MULTISET COLORINGS OF GRAPHS

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Abstract

A vertex coloring of a graph G is a multiset coloring if the multisets of colors of the neighbors of every two adjacent vertices are different. The minimum k for which G has a multiset k-coloring is the multiset chromatic number $\chi_m(G)$ of G. For every graph G, $\chi_m(G)$ is bounded above by its chromatic number $\chi(G)$. The multiset chromatic numbers of regular graphs are investigated. It is shown that for every pair k, r of integers with $2 \leq k \leq r - 1$, there exists an r-regular graph with multiset chromatic number k. It is also shown that for every positive integer N, there is an r-regular graph G such that $\chi(G) - \chi_m(G) = N$. In particular, it is shown that $\chi_m(K_n \times K_2)$ is asymptotically $\sqrt{n}$. In fact, $\chi_m(K_n \times K_2) = \chi_m(\mathrm{cor}(K_{n+1}))$. The corona $\mathrm{cor}(G)$ of a graph G is the graph obtained from G by adding, for each vertex v in G, a new vertex v' and the edge vv'. It is shown that $\chi_m(\mathrm{cor}(G)) \leq \chi_m(G)$ for every nontrivial connected graph G. The multiset chromatic numbers of the corona of all complete graphs are determined.

From this, it follows that for every positive integer N, there exists a graph G such that $\chi_m(G) - \chi_m(\mathrm{cor}(G)) \geq N$. The result obtained on the multiset chromatic number of the corona of complete graphs is then extended to the corona of all regular complete multipartite graphs.

Keywords: vertex coloring, multiset coloring, neighbor-distinguishing coloring.

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1. Introduction

A primary goal of vertex colorings of a graph G is distinguishing the two vertices in each pair of adjacent vertices of G using as few colors as possible. This can be accomplished by proper colorings, of course, where adjacent vertices are required to be assigned distinct colors. A variety of methods in graph theory have been introduced to distinguish all of the vertices of a graph or to distinguish every two adjacent vertices in a graph. A coloring c of the vertices of a graph G (where adjacent vertices may be assigned the same color) is called *neighbor-distinguishing* if every two adjacent vertices of G are distinguished from each other in some manner by the coloring c. In a proper vertex coloring of a graph G, every two adjacent vertices of G are assigned distinct colors and so such a coloring is neighbor-distinguishing. The minimum number of colors required of a proper coloring of G is the well-known *chromatic number* $\chi(G)$.

As described in [7], edge colorings of graphs, whether proper or not, have been introduced that use the multisets of colors of the incident edges of each vertex in a graph G for the purpose of distinguishing all vertices of G or of distinguishing every two adjacent vertices of G. The papers by Burris [4] and Chartrand, Escuadro, Okamoto, and Zhang [5] deal with the former (vertex-distinguishing edge colorings), while the papers by Addario-Berry, Aldred, Dalal, and Reed [1], Karoński, Łuczak, and Thomason [10], and Escuadro, Okamoto, and Zhang [9] deal with the latter. Furthermore, vertex colorings (proper or not) of a graph G have been introduced that use the multisets of colors of the neighboring vertices of each vertex for the purpose of distinguishing all vertices of G. These concepts have been studied by Chartrand, Lesniak, VanderJagt, and Zhang [6], Radcliffe and Zhang [11], and Anderson, Barrientos, Brigham, Carrington, Kronman, Vitray, and Yellen [2]. In [7] a new neighbor-distinguishing vertex coloring of a graph G was introduced that never requires more than $\chi(G)$ colors.

For a connected graph G, a *multiset k-coloring* c of G is a vertex coloring $c : V(G) \to \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, where adjacent vertices may be colored the same and such that the multisets of colors of the neighbors of every two adjacent vertices are different. The minimum k for which G has a multiset k-coloring is the *multiset chromatic number* $\chi_m(G)$. Since every proper coloring is a multiset coloring, it follows that $\chi_m(G) \le \chi(G)$. For example, the Petersen graph P has chromatic number 3 and multiset chromatic number 2. A multiset 2-coloring of P is shown in Figure 1.

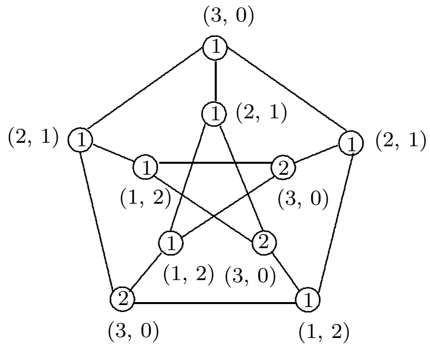

Figure 1. A multiset 2-coloring of the Petersen graph.

Let c be a multiset k-coloring of a graph G. For a vertex v of G, let $M(v)$ denote the multiset of colors of the neighbors of v. The multiset $M(v)$ can also be represented by the k-vector

$$\text{code}(v) = (a_1, a_2, \ldots, a_k) = a_1 a_2 \cdots a_k,$$

where a_i denotes the number of neighbors of v colored i for $1 \le i \le k$ (see Figure 1). Thus $\sum_{i=1}^{k} a_i = \deg v$. The k-vector code(v) is called the *multiset color code* of v. Since $M(u) \ne M(v)$ if $\deg u \ne \deg v$, determining $\chi_m(G)$ is most challenging when G contains many pairs of adjacent vertices having the same degree. It is well-known if H is any subgraph of G, then $\chi(H) \le \chi(G)$. This, however, is not the case for the multiset chromatic number. In fact, it is possible that H is a subgraph of G but $\chi_m(H) > \chi_m(G)$.

The following observations and results were presented in [7]. We refer to the book [8] for graph theory notation and terminology not described in this paper.

Observation 1.1. *The multiset chromatic number of a graph G is* 1 *if and only if every two adjacent vertices of G have distinct degrees.*

Observation 1.2. *If u and v are two adjacent vertices in a graph G such that $N(u) - \{v\} = N(v) - \{u\}$, then $c(u) \neq c(v)$ for every multiset coloring c of G.*

For positive integers ℓ and n,

$$f(\ell, n) = \binom{n + \ell - 1}{n}$$

is the number of n-element multisets of an ℓ-element set.

Theorem 1.3. *For positive integers k and n, the multiset chromatic number of the regular complete k-partite graph $K_{k(n)}$ containing n vertices in each partite set is the unique positive integer ℓ for which*

$$f(\ell - 1, n) < k \leq f(\ell, n).$$

We denote the complete multipartite graph containing k_i partite sets of cardinality n_i for $1 \leq i \leq t$ by $K_{k_1(n_1),k_2(n_2),\ldots,k_t(n_t)}$.

Theorem 1.4. *Let $G = K_{k_1(n_1),k_2(n_2),\ldots,k_t(n_t)}$, where $n_1, n_2, \ldots, n_t$ are t distinct positive integers. Then*

$$\chi_m(G) = \max\{\chi_m(K_{k_i(n_i)}) : \ 1 \leq i \leq t\}.$$

Proposition 1.5. *For each pair a, b of positive integers with $a \leq b$, there exists a connected graph G such that $\chi_m(G) = a$ and $\chi(G) = b$.*

For a connected graph G of order n and an integer k with $1 \leq k < n$, the k^{th} *power* G^k of G is that graph with $V(G^k) = V(G)$ such that $uv \in E(G^k)$ if and only if $1 \leq d_G(u, v) \leq k$. Thus $G^1 = G$ and $G^k = K_n$ if $k \geq \text{diam}(G)$.

Proposition 1.6. *For each integer $n \geq 3$, $\chi_m(C_n) = \chi(C_n)$.*

Since $C_{2k}^{k-1} = K_{k(2)}$, we have the following by Theorem 1.3.

Proposition 1.7. *For each integer $k \geq 2$,*

$$\chi_m(C_{2k}^{k-1}) = \left\lceil \frac{-1 + \sqrt{8k+1}}{2} \right\rceil.$$

2. On Multiset Colorings of Regular Graphs

We have seen that $\chi_m(G) \le \chi(G)$ for every connected graph G. Also, $\chi(G) \le \Delta(G) + 1$ and by a theorem of Brooks [3], $\chi(G) = \Delta(G) + 1$ if and only if G is an odd cycle or a complete graph. Since $\chi_m(C_n) = 3 = \Delta(C_n)+1$ if n is odd and $\chi_m(K_n) = n = \Delta(K_n)+1$, it follows that $\chi_m(G) = \Delta(G)+1$ if and only if G is an odd cycle or a complete graph as well. In particular, if G is an r-regular graph that is not K_{r+1}, then $2 \le \chi_m(G) \le r$. We next investigate those pairs k, r of integers with $2 \le k \le r$ for which there exists an r-regular graph G with $\chi_m(G) = k$. For a positive integer k, let $\mathbb{N}_k = \{1, 2, \ldots, k\}$.

Theorem 2.1. *For each pair k, r of integers with $2 \le k \le r-1$, there exists an r-regular graph G with $\chi_m(G) = k$.*

Proof. For $k = 2$, consider $G = K_{r,r}$. For $3 \le k \le r-1$, let G be the graph with $V(G) = U \cup W \cup X \cup Y$, where

$$U = \{u_1, u_2, \ldots, u_k\}, \qquad W = \{w_1, w_2, \ldots, w_k\},$$
$$X = \{x_1, x_2, \ldots, x_{r-k+1}\}, \quad Y = \{y_1, y_2, \ldots, y_{r-k+1}\},$$

and

$$N(u_i) = [U \cup X] - \{u_i\}, \quad N(w_i) = [W \cup Y] - \{w_i\},$$
$$N(x_j) = [U \cup Y] - \{y_j\}, \quad N(y_j) = [W \cup X] - \{x_j\}$$

for $1 \le i \le k$ and $1 \le j \le r-k+1$. Therefore, G is r-regular.

By Observation 1.2, $\chi_m(G) \ge k$ since $N(u) - \{u'\} = N(u') - \{u\}$ for every two vertices u and u' in U. On the other hand, consider the k-coloring $c : V(G) \to \mathbb{N}_k$ defined by $c(u_i) = c(w_i) = i$ for $1 \le i \le k$ and $c(x_j) = 1$ and $c(y_j) = k$ for $1 \le j \le r-k+1$. Then c is a multiset coloring of G, implying that $\chi_m(G) \le k$. Therefore, $\chi_m(G) = k$. ■

Observe that for $r \in \{2, 3\}$, there is an infinite class of connected r-regular graphs with multiset chromatic number r. For $r = 2$, the family of even cycles has the desired property. To see that there are infinitely many connected cubic graphs having multiset chromatic number 3, let $k \ge 2$ be an integer and consider the cubic graph G_k of order $4k$ whose vertex set is

$$\{u_1, u_2, \ldots, u_k\} \cup \{w_1, w_2, \ldots, w_k\} \cup \{x_1, x_2, \ldots, x_k\} \cup \{y_1, y_2, \ldots, y_k\},$$

and

$$N(v) = \begin{cases} \{u_i, w_i, x_i, y_i\} - \{v\} & \text{if } v \in \{u_i, w_i\}, \\ \{u_i, w_i, y_{i-1}\} & \text{if } v = x_i, \\ \{u_i, w_i, x_{i+1}\} & \text{if } v = y_i \end{cases}$$

for $1 \le i \le k$, where $x_{k+1} = x_1$ and $y_0 = y_k$. (Note that $\chi(G_k) = \omega(G_k) = 3$.)

We show that $\chi_m(G_k) = 2$ if k is even while $\chi_m(G_k) = 3$ if k is odd. If k is even, then consider a 2-coloring c of G_k defined by $c(u_i) = 1$, $c(w_i) = 2$, and

$$c(x_i) = c(y_i) = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

for $1 \le i \le k$ and observe that c is a multiset coloring. Therefore, $\chi_m(G_k) = 2$ if k is even.

If k is odd, assume, to the contrary, that there exists a multiset 2-coloring c' of G_k using the colors 1 and 2. By Observation 1.2, $c'(u_i) \neq c'(w_i)$ and so $\{\text{code}(x_i), \text{code}(y_i)\} \subseteq \{(2,1), (1,2)\}$ for $1 \le i \le k$. If $c'(x_i) \neq c'(y_i)$, then $\{\text{code}(u_i), \text{code}(w_i)\} = \{(2,1), (1,2)\}$, which is impossible. Therefore,

$$c'(x_i) = c'(y_i) \quad \text{for } 1 \le i \le k. \tag{1}$$

On the other hand, since $\text{code}(x_{i+1}) \neq \text{code}(y_i)$,

$$c'(x_{i+1}) \neq c'(y_i) \quad \text{for } 1 \le i \le k. \tag{2}$$

By (1) and (2), it follows that $c'(x_i) \neq c'(x_{i+1})$ for $1 \le i \le k$, which is impossible since k is odd. Therefore, $\chi_m(G_k) = 3$ if k is odd.

For integers k and r with $3 \le k \le r - 1$, the graph G constructed in the proof of Theorem 2.1 is r-regular and has multiset chromatic number k. In addition, this graph G has chromatic number $k + 1$. Therefore, there exists an infinite class $\mathcal{G}$ of regular graphs G such that $\chi(G) - \chi_m(G) = 1$ for each $G \in \mathcal{G}$. Actually, there exists a class of regular graphs with an even stronger property.

Theorem 2.2. *For every positive integer k, there is a regular graph G such that*

$$\chi(G) - \chi_m(G) = k.$$

Proof. Let $G = C_4[K_{k+1}]$ with $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ such that for each integer i with $1 \leq i \leq 4$, the subgraph induced by V_i is the complete graph K_{k+1} and each vertex in V_i is adjacent to each vertex in $V_{\alpha(i)}$ where α is the cyclic permutation (1234). Thus G is $(3k+2)$-regular. We first show that $\chi(G) = 2k+2$. Since the clique number of G is $\omega(G) = 2k+2$, it follows that $\chi(G) \geq 2k+2$. On the other hand, the coloring of G that assigns (i) the colors $1, 2, \ldots, k+1$ to the vertices of V_1 and V_3 and (ii) the colors $k+2, k+3, \ldots, 2k+2$ to the vertices of V_2 and V_4 is a proper coloring of G using $2k+2$ colors. Thus $\chi(G) = 2k+2$.

Next we show that $\chi_m(G) = k+2$. Let c be a multiset coloring of G. By Observation 1.2, no two vertices in V_i $(1 \leq i \leq 4)$ can be colored the same by c, implying that $\chi_m(G) \geq k+1$. If $\chi_m(G) = k+1$, then the $k+1$ vertices in each set V_i must be colored by $1, 2, \ldots, k+1$ by a multiset coloring c. Let $u \in V_1$ and $v \in V_2$ such that $c(u) = c(v)$. Then $M(u) = M(v)$, which is impossible. Thus $\chi_m(G) \geq k+2$. The coloring of G that assigns (i) the colors $1, 2, \ldots, k+1$ to the vertices of V_1 and V_3 and (ii) the colors $2, 3, \ldots, k+2$ to the vertices of V_2 and V_4 is a multiset coloring of G using $k+2$ colors. Thus $\chi_m(G) = k+2$. Therefore, $\chi(G) - \chi_m(G) = k$. ■

We are now left with the following problem.

Problem 2.3. Is there an r-regular graph, where $r \geq 4$, whose multiset chromatic number equals r ?

For a given positive integer k and the r-regular graph $G = C_4[K_{k+1}]$, it follows from the proof of Theorem 2.2 that

$$r - \chi(G) = \chi(G) - \chi_m(G) = k$$

since $r = 3k+2$. There is, in fact, a class of r-regular graphs that produce a conclusion similar to that in Theorem 2.2 but for which $\chi(G) = r$.

Let G be a graph and consider a subset X of $V(G)$. For a (not necessarily proper) vertex coloring c of G, let $M_c(X)$ (or simply $M(X)$) be the multiset of colors of the vertices in X.

Theorem 2.4. *For each positive integer r,*

$$\chi_m(K_r \times K_2) = \left\lceil \frac{1 + \sqrt{4r+1}}{2} \right\rceil .$$

Proof. The result is immediate for $r \in \{1, 2\}$ since the graph is regular and bipartite. Hence, suppose that $r \geq 3$.

We first show that if $\chi_m(G) = k$, then $r \leq k(k-1)$. Construct G from two disjoint copies H_1 and H_2 of K_r with $V(H_1) = U = \{u_1, u_2, \ldots, u_r\}$ and $V(H_2) = W = \{w_1, w_2, \ldots, w_r\}$ by joining u_i and w_i for $1 \leq i \leq r$. Let c be a multiset k-coloring of G using the colors $1, 2, \ldots, k$. Hence no more than k vertices in U can be assigned the same color by c. If $r > k(k-1)$, then U contains k vertices that are assigned the same color, say $c(u_i) = 1$ for $1 \leq i \leq k$. Since c is a multiset coloring, this implies that no two vertices in the set $\{w_1, w_2, \ldots, w_k\} \subseteq W$ can be assigned the same color. Thus, we may assume that $c(w_i) = i$ for $1 \leq i \leq k$.

If $c(u_i) = c(w_i)$ for some i with $k+1 \leq i \leq r$, say $c(u_{k+1}) = c(w_{k+1})$, then $M(u_1) = M(u_{k+1})$, which is impossible. Therefore, $c(u_i) = c(w_i)$ if and only if $i = 1$. Furthermore, this implies that U contains at most $k-1$ vertices that are assigned the color j for $2 \leq j \leq k$. Since $r > k(k-1)$, it then follows that $r = k(k-1)+1$, where U contains k vertices that are assigned the color 1 and $k-1$ vertices that are assigned the color j for $2 \leq j \leq k$. Let j be an integer with $2 \leq j \leq k$ and suppose that $c(u_{i_1}) = c(u_{i_2}) = \cdots = c(u_{i_{k-1}}) = j$. Then $\{c(w_{i_1}), c(w_{i_2}), \ldots, c(w_{i_{k-1}})\} = \mathbb{N}_k - \{j\}$. Since this holds for every j with $2 \leq j \leq k$, it follows that W also contains k vertices that are assigned the color 1 and $k-1$ vertices that are assigned the color j for $2 \leq j \leq k$, that is, $M(U) = M(W)$. However then, $M(u_1) = M(U) = M(W) = M(w_1)$, which is another contradiction. Therefore, $r \leq k(k-1)$, which then implies that

$$\chi_m(G) = k \geq \left\lceil \frac{1+\sqrt{4r+1}}{2} \right\rceil.$$

We next show that $\chi_m(G) \leq \left\lceil \frac{1+\sqrt{4r+1}}{2} \right\rceil$ by defining a multiset k-coloring of G, where $k = \left\lceil \frac{1+\sqrt{4r+1}}{2} \right\rceil \geq 3$. Note that if $k = \left\lceil \frac{1+\sqrt{4r+1}}{2} \right\rceil$, then $(k-1)(k-2) < r \leq k(k-1)$. We consider three cases.

Case 1. $r = k(k-1)$.

We first construct $G^* = K_r \times K_2$ from two disjoint copies H_1 and H_2 of K_r with $V(H_1) = U^* = U_1 \cup U_2 \cup \cdots \cup U_k$ and $V(H_2) = W^* = W_1 \cup W_2 \cup \cdots \cup W_k$, where $U_i = \{u_{1,i}, u_{2,i}, \ldots, u_{k-1,i}\}$ and $W_i = \{w_{1,i}, w_{2,i}, \ldots, w_{k-1,i}\}$ for $1 \leq i \leq k$, by joining $u_{j,i}$ to $w_{j,i}$ for $1 \leq i \leq k$ and $1 \leq j \leq k-1$. Define a

k-coloring c^* of G^* by

$$c^*(v) = \begin{cases} i & \text{if } v = u_{j,i} \;\; (1 \le i \le k,\, 1 \le j \le k-1), \\ j & \text{if } v = w_{j,i} \;\; (1 \le i \le k,\, 1 \le j \le k-1,\, i \ne j), \\ k & \text{if } v = w_{i,i} \;\; (1 \le i \le k-1). \end{cases}$$

Thus, $M(U^*) = M(W^*) = M$.

We verify that c^* is a multiset coloring of G^*. Suppose that x and y are adjacent in G^*. If $x, y \in U^*$, then let $w_1, w_2 \in W^*$ such that $xw_1, yw_2 \in E(G^*)$. Then since

(i) $c^*(x) \ne c^*(y)$ or $c^*(w_1) \ne c^*(w_2)$ and

(ii) $c^*(x) \ne c^*(w_1)$ and $c^*(y) \ne c^*(w_2)$, it follows that

$$M(x) = [M \cup \{c^*(w_1)\}] - \{c^*(x)\} \ne [M \cup \{c^*(w_2)\}] - \{c^*(y)\} = M(y).$$

Similarly, $M(x) \ne M(y)$ if $x, y \in W^*$. Also, if $x \in U^*$ and $y \in W^*$, then $c^*(x) \ne c^*(y)$ and so

$$M(x) = [M \cup \{c^*(y)\}] - \{c^*(x)\} \ne [M \cup \{c^*(x)\}] - \{c^*(y)\} = M(y).$$

Thus, c^* is a multiset k-coloring of G^*.

Case 2. $(k-1)^2 \le r \le k(k-1) - 1$.
Then $r = k(k-1) - p$ for some p with $1 \le p \le k-1$. Construct $G' = K_r \times K_2$ from G^* in Case 1 by deleting the $2p$ vertices in the set $\{u_{i,k} :\ k-p \le i \le k-1\} \cup \{w_{i,k} :\ k-p \le i \le k-1\}$. Let $U' = U^* \cap V(G')$ and $W' = W^* \cap V(G')$ and consider the coloring c' of G', which is the restriction of c^* to $V(G')$. Then $M_{c'}(U') = M_{c^*}(U^*) - A$ and $M_{c'}(W') = M_{c^*}(W^*) - B$, where A is a multiset containing p elements all of which equal k and $B = \{k-1, k-2, \ldots, k-p\}$. Therefore, $M_{c'}(U')$ contains $k-1$ elements that equal $k-1$, while $M_{c'}(W')$ contains $k-2$ elements that equal $k-1$.

Suppose that x and y are adjacent in G'. If $x, y \in U'$, then

$$M_{c'}(x) = M_{c^*}(x) - A \ne M_{c^*}(y) - A = M_{c'}(y).$$

Similarly, $M_{c'}(x) \ne M_{c'}(y)$ if $x, y \in W'$. If $x \in U'$ and $y \in W'$, then

$$M_{c'}(x) \cup M_{c'}(y) = M_{c'}(U') \cup M_{c'}(W').$$

Since $M_{c'}(U') \cup M_{c'}(W')$ contains $2k-3$ elements that equal $k-1$, it follows that the two multisets $M_{c'}(x)$ and $M_{c'}(y)$ contain unequal numbers of elements that equal $k-1$. Therefore, $M_{c'}(x) \neq M_{c'}(y)$. This verifies that c' is a multiset k-coloring of G'.

Case 3. $(k-1)(k-2)+1 \leq r \leq (k-1)^2-1$.
Then $r=(k-1)^2-p$ for some p with $1 \leq p \leq k-2$. Construct $G'' = K_r \times K_2$ from G^* in Case 1 by deleting the $2(k-1+p)$ vertices in the set

$$U_k \cup W_k \cup \{u_{i,k-1} :\ k-p \leq i \leq k-1\} \cup \{w_{i,k-1} :\ k-p \leq i \leq k-1\}.$$

Let $U'' = U^* \cap V(G'')$ and $W'' = W^* \cap V(G'')$ and consider the coloring c'' of G'', which is the restriction of c^* to $V(G'')$. Then $M_{c''}(U'') = M_{c^*}(G) - [A_1 \cup A_2]$ and $M_{c''}(W'') = M_{c^*}(G) - [B_1 \cup B_2]$, where A_1 is a multiset containing $k-1$ elements all of which equal k, A_2 is a multiset containing p elements all of which equal $k-1$, $B_1 = \mathbb{N}_{k-1}$, and $B_2 = \{k\}$ if $p=1$ and $B_2 = \{k, k-2, k-3, \ldots, k-p\}$ if $p \geq 2$. Therefore, $M_{c''}(U'')$ contains $k-1$ elements that equal 1 and $M_{c''}(W'')$ contains $k-2$ elements that equal 1.

Suppose that x and y are adjacent in G''. If $x, y \in U''$, then

$$M_{c''}(x) = M_{c^*}(x) - [A_1 \cup A_2] \neq M_{c^*}(y) - [A_1 \cup A_2] = M_{c''}(y).$$

Similarly, $M_{c''}(x) \neq M_{c''}(y)$ if $x, y \in W''$. If $x \in U''$ and $y \in W''$, on the other hand, then

$$M_{c''}(x) \cup M_{c''}(y) = M_{c''}(U'') \cup M_{c''}(W'').$$

Since $M_{c''}(U'') \cup M_{c''}(W'')$ contains $2k-3$ elements that equal 1, it follows that the two multisets $M_{c''}(x)$ and $M_{c''}(y)$ contain unequal numbers of elements that equal 1. Therefore, $M_{c''}(x) \neq M_{c''}(y)$. Hence, c'' is a multiset k-coloring of G''. ∎

Corollary 2.5. *For each positive integer k, there exists a positive integer r for which there is an r-regular, r-chromatic graph G with $\chi(G) - \chi_m(G) \geq k$.*

3. On Multiset Colorings of Coronas of Graphs

There are many classes $\mathcal{G}$ of graphs and operations f on graphs for which $\chi(f(G)) = \chi(G)$ for every $G \in \mathcal{G}$. Our goal here is to consider one such class and one such operation for which this is the case and determine what occurs when chromatic number is replaced by multiset chromatic number. In this paper we consider the problem of studying the multiset chromatic number of the corona of a regular graph. The *corona* $\mathrm{cor}(G)$ of a graph G is the graph obtained from G by adding, for each vertex v in G, a new vertex v' and the edge vv'. While the chromatic numbers of a nontrivial connected graph and its corona are the same, such is not necessarily the case for the multiset chromatic number.

If c is a multiset k-coloring of a nontrivial connected graph G using the colors $1, 2, \ldots, k$, then a multiset k-coloring c^* of $\mathrm{cor}(G)$ can be defined by $c^*(x) = c(x)$ if $x \in V(G)$ and $c^*(x) = 1$ if $x \in V(\mathrm{cor}(G)) - V(G)$. Thus we have the following observation.

Observation 3.1. *For every nontrivial connected graph G,*

$$\chi_m(\mathrm{cor}(G)) \leq \chi_m(G).$$

If G is a graph with $\chi_m(G) = 2$, then G contains two adjacent vertices having the same degree. Thus $\mathrm{cor}(G)$ also contains two adjacent vertices having the same degree and so $\chi_m(\mathrm{cor}(G)) = 2$ by Observations 1.1 and 3.1. For example, if $n \geq 4$ is an even integer, then $\chi_m(\mathrm{cor}(C_n)) = \chi_m(C_n) = 2$. If $\chi_m(G) \geq 3$, however, then it is possible that $\chi_m(\mathrm{cor}(G)) < \chi_m(G)$, as we see next.

Proposition 3.2. *For each odd integer $n \geq 3$, $\chi_m(\mathrm{cor}(C_n)) = 2$.*

Proof. Since $G = \mathrm{cor}(C_n)$ contains pairs of adjacent vertices having the same degree, $\chi_m(G) \geq 2$. We construct G from $C_n : v_1, v_2, \ldots, v_n, v_1$ by adding n new vertices $u_1, u_2, \ldots, u_n$ and joining each u_i to v_i $(1 \leq i \leq n)$. If $n = 3$, the coloring of the vertices with the colors 1 and 2 such that v_3 and u_2 are the only vertices colored 2 is a multiset coloring. For $n \geq 5$, consider the 2-coloring $c : V(G) \to \{1, 2\}$ such that $c(x) = 1$ if and only if

(i) $x = v_i$ and $i \neq 3$ or

(ii) $x = u_i$ where i is odd and $5 \leq i \leq n$.

Then observe that

$$\operatorname{code}(v_i) = \begin{cases} (3,0) & \text{if } 5 \le i \le n \text{ and } i \text{ is odd,} \\ (1,2) & \text{if } i = 2,4, \\ (2,1) & \text{otherwise} \end{cases}$$

and so c has the desired property. ■

For the graphs G we have seen thus far, either $\chi_m(\operatorname{cor}(G)) = \chi_m(G)$ or $\chi_m(\operatorname{cor}(G)) = \chi_m(G) - 1$. The difference between $\chi_m(G)$ and $\chi_m(\operatorname{cor}(G))$ can be arbitrarily large, however. To show this, we determine the multiset chromatic number of a complete graph.

Theorem 3.3. *For every integer $n \ge 2$,*

$$\chi_m(\operatorname{cor}(K_n)) = \left\lceil \frac{1+\sqrt{4n-3}}{2} \right\rceil.$$

Proof. Suppose that the vertex set of K_n is $\{v_1, v_2, \ldots, v_n\}$. Then the graph $G = \operatorname{cor}(K_n)$ can be constructed from K_n by adding n new vertices to K_n, say $u_1, u_2, \ldots, u_n$, and joining u_i to v_i for each i $(1 \le i \le n)$.

Let $k = \left\lceil \frac{1+\sqrt{4n-3}}{2} \right\rceil$. Then k is the unique integer such that

$$2\binom{k-1}{2} + 1 < n \le 2\binom{k}{2} + 1. \tag{3}$$

We show that $\chi_m(G) = k$. First we verify $\chi_m(G) \ge k$. Assume, to the contrary, that there exists a multiset $(k-1)$-coloring c of G. If $c(v_i) = c(v_j)$ for some $i \ne j$, then $c(u_i) \ne c(u_j)$ or otherwise $\operatorname{code}(v_i) = \operatorname{code}(v_j)$. Thus each of the $k-1$ colors can be assigned to at most $k-1$ vertices of the n vertices $v_1, v_2, \ldots, v_n$. Since $n \ge (k-1)(k-2)+2$, there are two colors, say 1 and 2, such that $k-1$ of the n vertices $v_1, v_2, \ldots, v_n$ are assigned the color i for $i = 1,2$. Without loss of generality, suppose that $c(v_i) = 1$ for $1 \le i \le k-1$ and $c(v_i) = 2$ for $k \le i \le 2k-2$. Then the $k-1$ vertices $u_1, u_2, \ldots, u_{k-1}$ must be colored differently so that the vertices $v_1, v_2, \ldots, v_{k-1}$ have distinct color codes. Similarly, the $k-1$ vertices $u_k, u_{k+1}, \ldots, u_{2k-2}$ are colored differently. Since there are $k-1$ colors available, one of the vertices of $u_1, u_2, \ldots, u_{k-1}$ must be colored 1, say $c(u_1) = 1$. Similarly, we may assume that $c(u_k) = 2$. However then, $\operatorname{code}(v_1) = \operatorname{code}(v_k)$, which is a contradiction.

We now show that $\chi_m(G) \le k$. Let $k \ge 2$ be an integer and $n_k = 2\binom{k}{2} + 1$. We first consider the case where $n = n_k$. Let $H = \mathrm{cor}(K_{n_k})$ and $c : V(H) \to \{1, 2, \ldots, k\}$ such that

$$c(v_i) = \begin{cases} 1 & \text{if } 1 \le i \le k-1 \text{ or } i = n_k, \\ j & \text{if } 2 \le j \le k \text{ and } (j-1)(k-1)+1 \le i \le j(k-1). \end{cases}$$

Furthermore, each of the remaining n_k vertices $u_1, u_2, \ldots, u_{n_k}$ is assigned a color as follows. For $1 \le i \le k-1$ and $i = n_k$,

$$c(u_i) = \begin{cases} i & \text{if } 1 \le i \le k-1, \\ k & \text{if } i = n_k. \end{cases}$$

For $k \le i \le n_k - 1$, write $i = (j-1)(k-1) + \ell$, where j and ℓ are integers with $2 \le j \le k$ and $1 \le \ell \le k-1$ and let

$$c(u_i) = \begin{cases} \ell & \text{if } \ell \ne j, \\ k & \text{if } \ell = j. \end{cases}$$

It is then not difficult to show that this coloring c has the desired property.

For $n_{k-1} + 1 \le n \le n_k - 1$, construct $G = \mathrm{cor}(K_n)$ from K_n with the vertices $v_1, v_2, \ldots, v_{n-1}, v_{n_k}$ by adding n new vertices $u_1, u_2, \ldots, u_{n-1}, u_{n_k}$ and joining u_i to v_i for each i $(1 \le i \le n-1,\ i = n_k)$. Then the coloring $c' : V(G) \to \{1, 2, \ldots, k\}$ defined by $c' = c\,|_{V(G)}$ has the desired property. ■

For each positive integer k, if $n = n_{k+1} = k^2 + k + 1$, then $\chi_m(\mathrm{cor}(K_n)) = k + 1$. Thus

$$\chi_m(K_n) - \chi_m(\mathrm{cor}(K_n)) = k^2.$$

Observe as well that

$$\lim_{n\to\infty} \frac{\chi_m(\mathrm{cor}(K_n))}{\sqrt{\chi(\mathrm{cor}(K_n))}} = 1.$$

We next determine the multiset chromatic number of the corona of the connected regular complete k-partite graph $K_{k(n)}$, each of whose partite sets contains $n \ge 2$ vertices. In order to do this, we first present a useful lemma, whose proof is straightforward and is therefore omitted. For positive integers ℓ and n, define

$$g(\ell, n) = \binom{\ell + n - 2}{n - 1} + \ell\binom{\ell + n - 2}{n}.$$

Lemma 3.4. *For integers $n, k \geq 2$, there exists a unique integer $\ell \geq 2$ such that*

$$g(\ell - 1, n) < k \leq g(\ell, n).$$

For s-vectors α and β, we write $\alpha - \beta$ to denote the s-vector obtained from α and β by coordinate-wise subtraction.

Theorem 3.5. *For integers $n, k \geq 2$, the multiset chromatic number of* $\mathrm{cor}(K_{k(n)})$ *is the unique positive integer ℓ such that*

$$g(\ell - 1, n) < k \leq g(\ell, n).$$

Proof. By Lemma 3.4, such an integer $\ell \geq 2$ exists. Let $G = K_{k(n)}$ with $V(G) = U = U_1 \cup U_2 \cup \cdots \cup U_k$, where $U_i = \{u_{1,i}, u_{2,i}, \ldots, u_{n,i}\}$ is a partite set for $1 \leq i \leq k$. We obtain the corona of G by adding a set $W = W_1 \cup W_2 \cup \cdots \cup W_k$ of nk new vertices, where $W_i = \{w_{1,i}, w_{2,i}, \ldots, w_{n,i}\}$, and joining $w_{j,i}$ to $u_{j,i}$ for $1 \leq j \leq n$ and $1 \leq i \leq k$.

We first show that $\chi_m(\mathrm{cor}(G)) \geq \ell$. Suppose that c is a multiset s-coloring of $\mathrm{cor}(G)$ using the colors $1, 2, \ldots, s$. For $1 \leq i \leq s$, let t_i be the number of vertices in U that are assigned the color i. Then $\sum_{i=1}^{s} t_i = nk$. Consider an arbitrary vertex u in U and let w be the end-vertex adjacent to u. Also, let $\mathcal{C}$ be the set of colors of the n vertices in the partite set to which u belongs, say partite set U_1. We consider two cases, according to whether $c(w) \in \mathcal{C}$ or $c(w) \notin \mathcal{C}$.

Case 1. $c(w) \in \mathcal{C}$.
Let i be an integer where $1 \leq i \leq s$. If $i \neq c(w)$, let a_i be the number of vertices in U_1 that are colored i; while if $i = c(w)$, let a_i be the number that is one less than the number of vertices in U_1 that are colored i. Thus $a_i \geq 0$ for $1 \leq i \leq s$ and $\sum_{i=1}^{s} a_i = n - 1$. Furthermore,

$$\mathrm{code}(u) = (t_1, t_2, \ldots, t_s) - (a_1, a_2, \ldots, a_s).$$

Then the number of possible codes for u is

$$\binom{(s-1) + (n-1)}{n-1} = \binom{s+n-2}{n-1}.$$

Case 2. $c(w) \notin \mathcal{C}$.
Let i be an integer where $1 \le i \le s$. If $i \ne c(w)$, let b_i be the number of vertices in U_1 that are colored i; while if $i = c(w)$, let $b_i = -1$. Thus $b_i \ge 0$ if $1 \le i \le s$ and $i \ne c(w)$ and $\sum_{i=1}^{s} b_i = n-1$. Moreover,

$$\mathrm{code}(u) = (t_1, t_2, \ldots, t_s) - (b_1, b_2, \ldots, b_s).$$

The number of possible codes for u is

$$s\binom{(s-2)+n}{n} = s\binom{s+n-2}{n}.$$

Therefore, the number of distinct codes for the vertices in U is at most

$$\binom{s+n-2}{n-1} + s\binom{s+n-2}{n} = g(s,n).$$

Since $\mathrm{cor}(G)$ contains K_k as a subgraph, the number of distinct codes for the vertices in U is at least k, implying that $k \le g(s,n)$. Since $k > g(\ell-1,n)$, it follows that $\chi_m(\mathrm{cor}(G)) > \ell - 1$ and so $\chi_m(\mathrm{cor}(G)) \ge \ell$.

Next we show that $\chi_m(\mathrm{cor}(G)) \le \ell$ by finding a multiset ℓ-coloring of $\mathrm{cor}(G)$. Let $\mathcal{A}$ be the set of ℓ-tuples $\vec{a} = (a_1, a_2, \ldots, a_\ell)$ such that (i) each of the ℓ coordinates is a nonnegative integer and (ii) $\sum_{i=1}^{\ell} a_i = n-1$. Also, let $\mathcal{B}$ be the set of ℓ-tuples $\vec{b} = (b_1, b_2, \ldots, b_\ell)$ such that (i) exactly one of the ℓ coordinates equals -1 while each of the remaining $\ell-1$ coordinates is a nonnegative integer and (ii) $\sum_{i=1}^{\ell} b_i = n-1$. Observe that

$$|\mathcal{A}| = \binom{\ell+n-2}{n-1} \quad \text{and} \quad |\mathcal{B}| = \ell\binom{\ell+n-2}{n}.$$

Since $k \le g(\ell,n) = |\mathcal{A}| + |\mathcal{B}| = |\mathcal{A} \cup \mathcal{B}|$, we may choose k distinct ℓ-tuples $\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_k$ from the set $\mathcal{A} \cup \mathcal{B}$. We now define an ℓ-coloring c of $\mathrm{cor}(G)$ as follows. If $\vec{\alpha}_i = (a_1, a_2, \ldots, a_\ell) \in \mathcal{A}$ where $1 \le i \le k$, then let $\{a_{i_1}, a_{i_2}, \ldots, a_{i_{\ell'}}\}$ be the set of positive coordinates of $\vec{\alpha}_i$. Hence $\sum_{j=1}^{\ell'} a_{i_j} = n-1$. Then let $c(w) = 1$ for every vertex w in W_i, $c(u_{n,i}) = 1$, and assign colors to the $n-1$ vertices in $U_i - \{u_{n,i}\}$ so that there are a_{i_j} vertices that are assigned the color i_j for $1 \le j \le \ell'$.

If $\vec{\alpha}_i = (b_1, b_2, \ldots, b_\ell) \in \mathcal{B}$, then let $\{b_{i_1}, b_{i_2}, \ldots, b_{i_{\ell''}}\}$ be the set of positive coordinates of $\vec{\alpha}_i$ and $b_{i_{\ell''+1}} = -1$. Hence $\sum_{j=1}^{\ell''} b_{i_j} = n$. Then let $c(w) = i_{\ell''+1}$ for every vertex w in W_i and assign colors to the n vertices in U_i so that there are b_{i_j} vertices that are assigned the color i_j for $1 \le j \le \ell''$.

We show that c is a multiset coloring of $\mathrm{cor}(G)$. Since $\deg x = 1 < \deg y$ if $x \in W$ and $y \in U$, we need only verify that $\mathrm{code}(x) \neq \mathrm{code}(y)$ for arbitrary two vertices $x \in U_p$ and $y \in U_q$, where $1 \leq p, q \leq k$ and $p \neq q$. Note that $\vec{\alpha}_p \neq \vec{\alpha}_q$. For $1 \leq i \leq \ell$, let t_i be the number of vertices in U that are assigned the color i by c. Then

$$\mathrm{code}(x) = (t_1, t_2, \ldots, t_\ell) - \vec{\alpha}_p \neq (t_1, t_2, \ldots, t_\ell) - \vec{\alpha}_q = \mathrm{code}(y).$$

Therefore, c is a multiset ℓ-coloring of $\mathrm{cor}(G)$, implying that $\chi_m(\mathrm{cor}(G)) \leq \ell$ and so $\chi_m(\mathrm{cor}(G)) = \ell$. ■

If we let $\ell = \left\lceil \frac{1+\sqrt{4k-3}}{2} \right\rceil$, then

$$\ell = \chi_m(\mathrm{cor}(K_k)) = \chi_m(\mathrm{cor}(K_{k(1)}))$$

by Theorem 3.3. Since ℓ is the unique integer such that (3) holds, it follows that Theorem 3.5 is true for $n = 1$ as well.

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